A SIMPLE PROOF FOR SIEFFERT'S INEQUALITY AND APPLICATIONS

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Abstract. In this paper it is proved the famous Sieffert's inequality and are given a few applications.

SIEFFERT'S INEQUALITY

If $x, y \in \mathbb{R}, xy > 0$ then:

$$\frac{2xy}{x+y} + \sqrt{\frac{x^2+y^2}{2}} \geq \sqrt{xy} + \frac{x+y}{2} \tag{1}$$

Proof.

Denote:

$$A = \sqrt{\frac{x^2+y^2}{2}} \Rightarrow A^2 = \frac{x^2+y^2}{2} \Rightarrow 2A^2 = x^2+y^2$$

$$B = \sqrt{xy} \Rightarrow B^2 = xy$$

$$(x+y)^2 = x^2+y^2+2xy = 2(A^2+B^2) \Rightarrow x+y = \sqrt{2(A^2+B^2)}$$

The inequality (1) becomes:

$$\frac{2B^2}{\sqrt{2(A^2+B^2)}} + A \geq B + \frac{\sqrt{2(A^2+B^2)}}{2}$$

$$A - B \geq \frac{\sqrt{2(A^2+B^2)}}{2} - \frac{2B^2}{\sqrt{2(A^2+B^2)}}$$

$$(A-B)^2 \geq \left(\frac{\sqrt{2(A^2+B^2)}}{2} - \frac{2B^2}{\sqrt{2(A^2+B^2)}}\right)^2$$

$$(A-B)^2 \geq \frac{2(A^2+B^2)}{4} + \frac{4B^4}{2(A^2+B^2)} - 2\cdot\frac{\sqrt{2(A^2+B^2)}}{2}\cdot\frac{2B^2}{\sqrt{2(A^2+B^2)}}$$

$$(A-B)^2 \geq \frac{A^2+B^2}{2} + \frac{2B^4}{A^2+B^2} - 2B^2$$

$$2(A^2+B^2)(A-B)^2 \geq (A^2-B^2)^2$$

$$(A-B)^2(2(A^2+B^2)-(A+B)^2) \geq 0$$

$$(A-B)^4 \geq 0$$

Equality holds for $A = B \Rightarrow x = y$.

□

Application 1: If $x > 0$ then:

$$\frac{2x}{x+1} + \sqrt{\frac{x^2+1}{2}} \geq \sqrt{x} + \frac{x+1}{2}$$

Proof.
We take in $(1): y = 1$

$$\frac{2x\cdot 1}{x+1}+\sqrt{\frac{x^2+1^2}{2}}\ge\sqrt{x\cdot 1}+\frac{x+1}{2}$$

$$\frac{2x}{x+1}+\sqrt{\frac{x^2+1}{2}}\ge\sqrt{x}+\frac{x+1}{2}$$

Equality holds for: $x = 1$. □

Application 2: If $x, y \in \mathbb{R}, xy > 0, x + y = 2$ then:

$$xy+\sqrt{\frac{x^2+y^2}{2}}\ge\sqrt{xy}+1$$

Proof.
We take in $(1): x + y = 2$.

$$\frac{2xy}{2}+\sqrt{\frac{x^2+y^2}{2}}\ge\sqrt{xy}+\frac{2}{2}$$

$$xy+\sqrt{\frac{x^2+y^2}{2}}\ge\sqrt{xy}+1$$

Equality holds for: $x = y = 1$. □

Application 3: If $a \in (0, \frac{\pi}{2})$ then:

$$\frac{\sin 2a}{\sin a+\cos a}+\frac{1}{\sqrt{2}}\ge\sqrt{\frac{\sin 2a}{2}}+\frac{\sin a+\cos a}{2}$$

Proof.
We take in $(1): x = \sin a, y = \cos a$.

$$\frac{2\sin a\cos a}{\sin a+\cos a}+\sqrt{\frac{\sin^2 a+\cos^2 a}{2}}\ge\sqrt{\sin a\cos a}+\frac{\sin a+\cos a}{2}$$

$$\frac{\sin 2a}{\sin a+\cos a}+\frac{1}{\sqrt{2}}\ge\sqrt{\frac{\sin 2a}{2}}+\frac{\sin a+\cos a}{2}$$

Equality holds for: $a = \frac{\pi}{4}$. □

Application 4: If $a, b \in \mathbb{R}, ab > 0$ then:

$$\frac{2}{a+b}+\frac{1}{ab}\sqrt{\frac{a^2+b^2}{2}}\ge\frac{1}{\sqrt{ab}}+\frac{a+b}{2ab}$$

Proof.
We take in $(1): x = \frac{1}{a}, y = \frac{1}{b}$.

$$\frac{2\cdot\frac{1}{a}\cdot\frac{1}{b}}{\frac{1}{a}+\frac{1}{b}}+\sqrt{\frac{\frac{1}{a^2}+\frac{1}{b^2}}{2}}\ge\sqrt{\frac{1}{a}\cdot\frac{1}{b}}+\frac{\frac{1}{a}+\frac{1}{b}}{2}$$

$$\frac{2}{a+b}+\frac{1}{ab}\sqrt{\frac{a^2+b^2}{2}}\ge\frac{1}{\sqrt{ab}}+\frac{a+b}{2ab}$$

Equality holds for: $a = b$. □

References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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