A SIMPLE PROOF FOR KLAMKIN'S INEQUALITY AND APPLICATIONS

DANIEL SITARU - ROMANIA

Abstract. In this paper it is proved the famous Klamkin's inequality and are given a few applications.

KLAMKIN'S INEQUALITY

If $-1 < x, y, z < 1$ then:

$$\frac{1}{(1-x)(1-y)(1-z)} + \frac{1}{(1+x)(1+y)(1+z)} \geq 2 \tag{1}$$

Proof.
Denote $\alpha = x + y + z, \alpha \in (-3, 3)$. Let be

$$f : (-3,3) \to \mathbb{R}, f(\alpha) = \frac{1}{(9-\alpha^2)^{\frac{3}{2}}}, f'(\alpha) = \frac{2\alpha}{(9-\alpha^2)^3}$$

$$f'(\alpha) = 0 \Rightarrow \alpha = 0 \Rightarrow \min f = f(0) = \frac{1}{27}$$

$$\frac{1}{(9-\alpha^2)^{\frac{3}{2}}} \geq \frac{1}{27}, \forall \alpha \in (-3,3) \tag{2}$$

$$\frac{1}{(1-x)(1-y)(1-z)} + \frac{1}{(1+x)(1+y)(1+z)} \overset{\text{AM-GM}}{\geq}$$

$$\geq \frac{1}{(\frac{1-x+1-y+1-z}{3})^3} + \frac{1}{(\frac{1+x+1+y+1+z}{3})^3} = \frac{27}{(3-\alpha)^3} + \frac{27}{(3+\alpha)^3} \overset{\text{AM-GM}}{\geq}$$

$$\geq 2 \cdot \sqrt{\frac{27}{(3-\alpha)^3} \cdot \frac{27}{(3+\alpha)^3}} = 54 \cdot \frac{1}{(9-\alpha^2)^{\frac{3}{2}}} \overset{(2)}{\geq} 54 \cdot \frac{1}{27} = 2$$

Equality holds for $x + y + z = 0$. □

Application 1: If $-1 < a, b < 1$ then:

$$\log^3\left(\frac{1-a}{1-b}\right) + \log^3\left(\frac{1+a}{1+b}\right) \geq 2(b-a)^3$$

Proof.
By integrating *LHS* of (1) we obtain:

$$\int_a^b \int_a^b \int_a^b \frac{dxdydz}{(1-x)(1-y)(1-z)} + \int_a^b \int_a^b \int_a^b \frac{dxdydz}{(1+x)(1+y)(1+z)} =$$

$$= \left(\int_a^b \frac{dx}{1-x}\right)^3 + \left(\int_a^b \frac{dx}{1+x}\right)^3 = \log^3\left(\frac{1-a}{1-b}\right) + \log^3\left(\frac{1+a}{1+b}\right) \tag{3}$$

By integration RHS of (1) we obtain:

$$\int_a^b \int_a^b \int_a^b 2dxdydz = 2(b-a)^3 \tag{4}$$

By (3), (4):

$$\log^3\left(\frac{1-a}{1-b}\right) + \log^3\left(\frac{1+a}{1+b}\right) \geq 2(b-a)^3$$

Equality holds for $a = b$. □

Application 2: If $-\frac{\pi}{3} < a, b, c < \frac{\pi}{4}$ then:

$$\cos^2 a \cdot \cos^2 b \cdot \cos^2 c \cdot \left(1 + \frac{1}{\cos 2a \cdot \cos 2b \cdot \cos 2c}\right) \geq 2$$

Proof.
We take in (1) : $x = \tan^2 a, y = \tan^2 b, z = \tan^2 c, x, y, z \in [0, 1)$.

$$\prod_{cyc} \frac{1}{(1-\tan^2 a)} + \prod_{cyc} \frac{1}{(1+\tan^2 a)} \geq 2$$

$$\prod_{cyc} \frac{\cos^2 a}{(\cos^2 a - \sin^2 a)} + \prod_{cyc} \frac{\cos^2 a}{(\cos^2 a + \sin^2 a)} \geq 2$$

$$\prod_{cyc} \frac{\cos^2 a}{\cos(2a)} + \prod_{cyc} \cos^2 a \geq 2$$

$$\cos^2 a \cdot \cos^2 b \cdot \cos^2 c \cdot \left(1 + \frac{1}{\cos 2a \cdot \cos 2b \cdot \cos 2c}\right) \geq 2$$

Equality holds for $a = b = c = 0$. □

Application 3: If $0 < a, b, c < 2$ then:

$$\frac{1}{abc} + \frac{1}{(2-a)(2-b)(2-c)} \geq 2$$

Proof.
We take in (1) : $a = x + 1, b = y + 1, c = z + 1$

$$x = a - 1, y = b - 1, z = c - 1$$

$$-1 < a - 1, b - 1, c - 1 < 1$$

$$0 < a, b, c < 2$$

$$\frac{1}{(1-a+1)(1-b+1)(1-c+1)} + \frac{1}{abc} \geq 2$$

$$\frac{1}{abc} + \frac{1}{(2-a)(2-b)(2-c)} \geq 2$$

Equality holds for $a = b = c = 1$. □

Application 4: If $\frac{1}{e} < a, b, c < e$ then:

$$\frac{1}{\log(\frac{e}{a}) \cdot \log(\frac{e}{b}) \cdot \log(\frac{e}{c})} + \frac{1}{\log(ea) \cdot \log(eb) \cdot \log(ec)} \geq 2$$

Proof.
We take in (1) : $x = \log a, y = \log b, z = \log c$.

$$-1 < \log a, \log b, \log c < 1$$

$$\log \frac{1}{e} < \log a, \log b, \log c < \log e$$

$$\frac{1}{e} < a, b, c < e$$

$$\prod_{cyc} \frac{1}{\log e - \log a} + \prod_{cyc} \frac{1}{\log e + \log a} \geq 2$$

$$\frac{1}{\log(\frac{e}{a}) \cdot \log(\frac{e}{b}) \cdot \log(\frac{e}{c})} + \frac{1}{\log(ea) \cdot \log(eb) \cdot \log(ec)} \geq 2$$

Equality holds for $a = b = c = 1$. $\square$

References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

Mathematics Department, National Economic College "Theodor Costescu", Drobeta Turnu - Severin, Romania
Email address: dansitaru63@yahoo.com