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The variational iteration method for solving second order linear non-homogeneous differential equations with constant coefficients

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Abstract
In this work, we present a different method for calculating ordinary linear differential equations by considering Variational Iteration Method (VIM).In this method, the problems are initially approximated by imposing the initial conditions. Then a correction functional for the differential equation is precisely generated by a general Lagrange Multiplier, which can be specified optimally via variational theory. By means of our procedure, The method is very effective and able to reduce the size of the calculations and treat linear equations in a direct manner.

Keywords: *Variational Iteration Method; Linear Differential Equations; Correction Functional; Lagrange Multiplier.*

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1. Introduction

The variational iteration method is one of the estimation methods that can be easily applied to a lot of linear and nonlinear problems and can reduce the size of the calculations.The authors have spent many years developing the variational iteration method [5] [6]. It has grown into a complete theory thanks to the efforts of several authors, including Moghimi and Wazwaz [3] [10]. The variational iteration method (VIM) [5][6][8][10] is suitable for dealing with a big class of linear and nonlinear differential problems. This method is good for applications in the sciences [11][7][12]. A.M. Wazwaz used the (VIM) to solve the nonlinear and linear and ODE with variable coefficients [7]. Lan Xu and Eric W.M. Lee also applied this method to solve the boundary layer equations of magnetohydrodynamic flow [1]. Kıymaz and Cetinkaya [4] applied variational iteration method for solving nonlinear differential equations. Xu and Lee [2] has performed the variational iteration method to solve Helmholtz equation. Ji-Huan [8] used (VIM) to give an approximate solution for some nonlinear problems. In [9] N.H. Sweilam., used the variational iteration method (VIM) for solving fourth order integro-differential equation. This method does not need much effort and time when applied to the computer.

In this paper, we have obtained the general solution of the second order non-homogeneous differential equations by using the variational iteration method (VIM), which mainly depends on lagrange multiplier.

2. Variational Iteration Method

To represent the technique's core concept, we consider the following general differential equation :

$$\mathtt{L}[\zeta(x)] + \mathtt{N}[\zeta(x)] = \psi(x) \tag{2.1}$$

Where $\mathtt{N}, \mathtt{L}$ is the non-linear and linear operator respectively, and $\psi(x)$ is a given continuous function.
The method main advantage is that it creates a correction function for equation (2.1), which can be written as:

$$\zeta_{m+1}(x) = \zeta_m(x) + \int_0^x \lambda(x,\rho)[\mathtt{L}\zeta_m(\rho) + \mathtt{N}\hat{\zeta}_m(\rho) - \psi(\rho)]d\rho \tag{2.2}$$

Where ζ_m is the m^{th} approximation solution, λ is a General lagrange multiplier that may be ideally found using variational theory and $\hat{\zeta}_m$ is restricted variation.

The first stages of the variational iteration method need to determine the optimum Lagrangian multiplier λ. After finding the Lagrangian multiplier, the successive estimation $\zeta_{m+1}, m \geq 0$, of the solution ζ will be easily obtained upon using any selective function ζ_0. As a result, the solution will be as follows

$$\zeta = \lim_{m\to\infty} \zeta_m.$$

3. Finding the Lagrange Multiplier for 2^{nd} order linear non-homogeneous differential equations with constant coefficients

The linear non-homogeneous differential equation with constant coefficients of second order

$$a\zeta'' + b\zeta' + c\zeta = \beta(x) \tag{3.1}$$

The correction functional can be written as follows

$$\zeta_{m+1}(x) = \zeta_m(x) + \int_0^x \lambda(x,\rho)[a\zeta_m''(\rho) + b\zeta_m'(\rho) + c\zeta_m(\rho) - \beta(\rho)]d\rho \tag{3.2}$$

Making the above correction functional stationary with respect to ζ_m, noticing that $\delta\zeta_m(0) = 0$, get

$$\begin{aligned}\delta\zeta_{m+1}(x) &= \delta\zeta_m(x) + \delta\int_0^x \lambda(x,\rho)[a\zeta_m''(\rho) + b\zeta_m'(\rho) + c\zeta_m(\rho) - \beta(\rho)]d\rho \\ &= \delta\zeta_m(x) + (a\lambda(x,\rho)\delta\zeta_m'(\rho) - a\lambda'(x,\rho)\delta\zeta_m(\rho) + b\lambda(x,\rho)\delta\zeta_m(\rho))|_{\rho=x} \\ &\quad + \int_0^x [(a\lambda''(x,\rho) - b\lambda'(x,\rho) + c\lambda(x,\rho))\delta\zeta_m(\rho)]d\rho\end{aligned}$$

As a result, we obtain the following stationary conditions:

$$a\frac{\partial^2\lambda(x,\rho)}{\partial\rho^2} - b\frac{\partial\lambda(x,\rho)}{\partial\rho} + c\lambda(x,\rho) = 0 \tag{3.3}$$

$$\lambda'(x,x) = \frac{1}{a}, \lambda(x,x) = 0. \tag{3.4}$$

The characteristic equation of the differential equation (3.1) is

$$ar^2 + br + c = 0 \tag{3.5}$$

According to the characteristic equation here, three cases will be examined.

case 1 If the discriminant of a quadratic equation (3.5) greater than zero ($\triangle > 0$) then the solution of the differential equation (3.3) is

$$\lambda(x,\rho) = f_1(x)e^{-r_1\rho} + f_2(x)e^{-r_2\rho}. \tag{3.6}$$

Substituting (3.4) into equation (3.6) results in the following system:

$$f_1(x)e^{-r_1x} + f_2(x)e^{-r_2x} = 0$$

$$-r_1f_1(x)e^{-r_1x} - r_2f_2(x)e^{-r_2x} = \frac{1}{a}.$$

From solving this system we get

$$f_1(x) = \frac{e^{r_1 x}}{a(r_2 - r_1)}$$

and

$$f_2(x) = -\frac{e^{r_2 x}}{a(r_2 - r_1)}.$$

Therefore, the lagrange multiplier in this case can be defined as:

$$\lambda(x, \rho) = \frac{e^{r_2(x-\rho)}}{a(r_1 - r_2)} - \frac{e^{r_1(x-\rho)}}{a(r_1 - r_2)}$$

case 2 If the discriminant of a quadratic equation (3.5) is zero ($\triangle = 0$) then the solution of the differential equation (3.3) is

$$\lambda(x, \rho) = (f_1(x) + f_2(x)\rho)e^{-r\rho}. \tag{3.7}$$

Substituting (3.4) into equation (3.7) results in the following system:

$$f_1(x)e^{-rx} + xf_2(x)e^{-rx} = 0$$

$$-rf_1(x)e^{-rx} + (1 - rx)f_2(x)e^{-rx} = \frac{1}{a}.$$

From solving this system we get

$$f_1(x) = \frac{-xe^{rx}}{a}$$

and

$$f_2(x) = \frac{e^{rx}}{a}.$$

Therefore, the lagrange multiplier in this case can be defined as:

$$\lambda(x, \rho) = \frac{(\rho - x)e^{r(x-\rho)}}{a}.$$

case 3 If the discriminant of a quadratic equation (3.5) less than zero ($\triangle < 0$) then the solution of the differential equation (3.3) is

$$\lambda(x, \rho) = [f_1(x)cosn\rho + f_2(x)sinn\rho]e^{-m\rho}. \tag{3.8}$$

Substituting (3.4) into equation (3.8) results in the following system:

$$[f_1(x)cosnx + f_2(x)sinnx]e^{-mx} = 0,$$

$$[f_1(x)cosnx + f_2(x)sinnx](-m)e^{-mx} + [-nf_1(x)sinnx + nf_2(x)cosnx]e^{-mx} = \frac{1}{a}.$$

From solving this system we get

$$f_1(x) = \frac{-e^{mx}sin(nx)}{na}$$

and

$$f_2(x) = \frac{e^{mx}cos(nx)}{na}.$$

Therefore, the lagrange multiplier in this case can be defined as:

$$\lambda(x, \rho) = \frac{sinn(\rho - x)e^{m(x-\rho)}}{na}$$

4. Some examples

In fact, because the Lagrangian multiplier can be exactly determined, the exact solution to linear problems by using the variational iteration method can be obtained by only one iteration step and we will show this by solving some examples.

Example 4.1. We consider a linear non-homogeneous second order differential equations

$$\zeta'' - 3\zeta' + 2\zeta = e^{-x} \tag{4.1}$$

with initial conditions $\zeta(0) = \frac{13}{6}, \zeta'(0) = \frac{17}{6}$. The exact solution of this problem is $\zeta(x) = e^{2x} + e^{x} + \frac{1}{6}e^{-x}$, Since $\triangle > 0$, and based on the previous discussion, we find that

$$\lambda(x, \rho) = e^{x-\rho} - e^{2(x-\rho)}.$$

Therefore, we can construct an iteration formula as follows:

$$\zeta_{m+1} = \zeta_m + \int_0^x [e^{x-\rho} - e^{2(x-\rho)}(\zeta_m''(\rho) - 3\zeta_m'(\rho) + 2\zeta_m(\rho) - e^{-\rho})]d\rho \tag{4.2}$$

We begin with initial approximation $\zeta_0(x) = \frac{17x+13}{6}$. By the iteration formula (4.2), we obtain

$$\zeta_1(x) = \frac{17x+13}{6} + \int_0^x [e^{x-\rho} - e^{2(x-\rho)}(-\frac{17}{2} + \frac{17\rho+13}{3} - e^{-\rho})]d\rho = e^{2x} + e^{x} + \frac{1}{6}e^{-x}.$$

Which is the general solution of equation (4.1).

Example 4.2. We consider a linear non-homogeneous second order differential equations

$$\zeta'' - 4\zeta' + 4\zeta = 4e^{2x} \tag{4.3}$$

with initial conditions $\zeta(0) = 5, \zeta'(0) = 11$. The exact solution of this problem is $\zeta(x) = (5 + x + 2x^2)e^{2x}$, Since $\triangle = 0$, and based on the previous discussion, we find that

$$\lambda(x, \rho) = e^{2(x-\rho)}(\rho - x)$$

Therefore, we can construct an iteration formula as follows:

$$\zeta_{m+1} = \zeta_m + \int_0^x [e^{-2(x-\rho)}(\rho - x)(\zeta_m''(\rho) + 4\zeta_m'(\rho) + 4\zeta_m(\rho) - 4e^{2\rho})]d\rho \tag{4.4}$$

We begin with initial approximation $\zeta_0(x) = 11x + 5$. By the iteration formula (4.4), we obtain

$$\zeta_1(x) = 11x + 5 + \int_0^x [e^{2(x-\rho)}(\rho - x)(-24 + 44\rho - 4e^{2\rho})]d\rho = (5 + x + 2x^2)e^{2x}$$

Which is the general solution of equation (4.3).

Example 4.3. We consider a linear non-homogeneous second order differential equations

$$\zeta'' - 2\zeta' + 5\zeta = 2x + 4 \tag{4.5}$$

with initial conditions $\zeta(0) = 2, \zeta'(0) = 6$. The exact solution of this problem is $\zeta(x) = e^{x}(\frac{26}{25}cos2x + \frac{57}{25}sin2x) + \frac{2}{5}x + \frac{24}{25}$, Since $\triangle < 0$, and based on the previous discussion, we find that

$$\lambda(x, \rho) = \frac{e^{x-\rho}sin2(\rho - x)}{2}$$

Therefore, we can construct an iteration formula as follows:

$$\zeta_{m+1} = \zeta_m + \int_0^x [\frac{e^{x-\rho}sin2(\rho - x)}{2}(\zeta_m''(\rho) - 2\zeta_m'(\rho) + 5\zeta_m(\rho) - 2\rho - 4)]d\rho \tag{4.6}$$

We begin with initial approximation $\zeta_0(x) = 6x + 2$. By the iteration formula (4.6), we obtain

$$\zeta_1(x) = 6x + 2 + \int_0^x [\frac{e^{x-\rho}sin2(\rho - x)}{2}(28\rho - 6)]d\rho = e^{x}(\frac{26}{25}cos2x + \frac{57}{25}sin2x) + \frac{2}{5}x + \frac{24}{25}$$

Which is the general solution of equation (4.5).

5. Conclusion

The second order linear non-homogeneous differential equations with constant coefficients have been analyzed using the variational iteration method. All the examples show that the use of the Variational Iteration Method may result in exact solutions by only one iteration step. This is because the Lagrangian multiplier can be exactly determined. It can be concluded that the Variational Iteration Method is a very effective tool for solving linear initial value problems.

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