**Solution to a Logarithmic Integral**

In this short note, we prove a logarithmic integral using the notion of the derivatives of the incomplete beta function.

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# 1 Proposed Problem

If $G$ denotes the Catalan's constant defined by $\sum_{n=1}^{\infty}(-1)^{n+1}/(2n-1)^2$, $\mathrm{Li}_3(z)$ is the trilogarithm function, and $\Im(.)$ is the imaginary part, then prove that

$$\int_0^1 \frac{\log^2(1+x)}{(1+x)(1+x^2)}\,dx = \frac{7}{48}\log^3(2) + \frac{3\pi}{32}\log^2(2) - \frac{\pi^2}{192}\log(2) + \frac{7\pi^3}{128} - G\log(2) - 2\Im\left(\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right).$$

*Problem proposed by Narendra Bhandari, Bajura, Nepal*

# 2 Proposed Solution

*Solution proposed by Lĕ Thù, Hanoi, Vietnam*

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This problem was proposed by Narendra Bhandari, which appeared in *romanian Mathematical Magazine*, 2021. The closed form of the problem has been proved by Singhania [2] via the complex analysis method, and Duc Nam [1] by the real method. We encourage curious readers looking to the references [2, 1]. In this recent note, we provide a new approach to the problem by employing the techniques of the derivatives of the incomplete beta function. We begin as follows:

Let the integral be denoted by $\Omega$, then it is easy to see that

$$\Omega = \Im\left(\int_0^1 \frac{\log^2(1+x)}{(1+x)(x-i)}\,dx\right) = \Im\left(\frac{\partial^2}{\partial y^2}\int_0^1 \frac{(1+x)^y}{(x-i)}\Bigg|_{y=-1}\right). \tag{1}$$

Let the latter integral be denoted by $I(y)$. We evaluate $I(y)$ by employing the notion of incomplete beta function. Enforcing the substitution $x \to x(i+1)-1$, we get

$$\begin{aligned} I(y) &= (1+i)^y \int_{1-i}^{\frac{1-i}{2}} \frac{x^y}{1-x}\,dx = (1+i)^y \left(\int_0^{\frac{1-i}{2}} - \int_0^{1-i}\right)\frac{x^y}{1-x}\,dx \\ &= (1+i)^y \left(B_{\frac{1-i}{2}}(y+1,0) - B_{1-i}(y+1,0)\right). \end{aligned} \tag{2}$$

Replace $y+1$ with $y$ leads to

$$I(y-1) = (1+i)^y \left( B_{\frac{1-i}{2}}(y,0) - B_{1-i}(y,0) \right).$$

The last expression follows by using the definition of the incomplete beta function. To evaluate (2), we invoke the following result.

$$\frac{\partial^m}{\partial y^m} B_z\left(y=0,0\right) = \frac{\ln^{m+1}(z)}{m+1} + \sum_{j=1}^{\infty}\sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^k k!}{j^{k+1}} z^j \ln^{m-k}(z) \tag{3}$$

Noting that $\frac{\partial^2}{\partial y^2}(uv) = u_{yy}v + 2u_y v_y + uv_{yy}$. Using the noted result in (3), we obtain

$$\begin{aligned}
\frac{\partial^2}{\partial y^2} B_{\frac{1-i}{2}}\left(y=0,0\right) &= -\frac{\ln^3(1+i)}{3} + \ln^2\left(\frac{1-i}{2}\right)\sum_{j=1}^{\infty}\frac{1}{j(1+i)^j} - 2\ln\left(\frac{1-i}{2}\right)\sum_{j=1}^{\infty}\frac{1}{j^2(1+i)^j} \\
&\quad + 2\sum_{j=1}^{\infty}\frac{1}{j^3(1+i)^j} \\
&= -\frac{\ln^3(1+i)}{3} + \ln^2\left(1+i\right)\ln(1-i) + 2\ln(1+i)\operatorname{Li}_2\left(\frac{1-i}{2}\right) \\
&\quad + 2\operatorname{Li}_3\left(\frac{1-i}{2}\right).
\end{aligned} \tag{4}$$

In a similar fashion, we can easily obtain that

$$\frac{\partial^2}{\partial y^2} B_{1-i}\left(y=0,0\right) = \frac{\ln^3(1-i)}{3} - \ln^2(1-i)\ln(i) - 2\ln(1-i)\operatorname{Li}_2(1-i) + 2\operatorname{Li}_3(1-i) \tag{5}$$

Equality (5) is obtained by the use of the analytical continuation. Likewise, we can easily construct the following equality by performing the first order partial derivatives of (3), i.e.,

$$\frac{\partial}{\partial y} B_{\frac{1-i}{2}}(y=0,0) = \frac{\ln^2(1+i)}{2} - \frac{\pi^2}{16} - \frac{\ln^2(2)}{4} - \operatorname{Li}_2\left(\frac{1-i}{2}\right), \tag{6}$$

$$\frac{\partial}{\partial y} B_{1-i}(y=0,0) = \frac{\ln^2(1-i)}{2} - \ln(1-i)\ln(i) - \operatorname{Li}_2(1-i). \tag{7}$$

Equality (7) is due to application of the analytical continuation. Moreover, we note that

$$\lim_{y\to 0^+}\left( B_{\frac{1-i}{2}}(y,0) - B_{1-i}(y,0) \right) = -\ln(1-i).$$

In the course of the evaluation, we escape the minor details. We leave details to the reader.

Substituting the values (4), (5), (6), (7) into (1) and extracting the imaginary parts, we have

$$\Omega \overset{3}{=} \frac{\ln^3(2)}{6} - \frac{\pi^2}{16}\ln(2) - \ln(2)G + \Im\left((1-i)\left(\mathrm{Li}_3\left(\frac{1-i}{2}\right) - \mathrm{Li}_3(1-i)\right)\right)$$
$$\overset{4}{=} \frac{7}{48}\ln^3(2) + \frac{3\pi}{32}\ln^2(2) - \frac{\pi^2}{192}\ln(2) + \frac{7\pi^3}{128} - \ln(2)G - 2\Im\left(\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right).$$

This completes the proof of the proposed problem.

**Note:** We always choose the $\arg(z) \in (-\pi, \pi]$ for all $z \in \mathbb{C}$ as the usual branch.

In the course of calculation of the above last two results, we utilize the following identities.

# 3 Dilogarithm Results

The following equalities hold:

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(1-z) = \frac{\pi^2}{6} - \ln(z)\ln(1-z)$$
$$\mathrm{Li}_(i) = -\frac{\pi^2}{48} + iG$$
$$\mathrm{Li}_2(1-i) = -iG + \frac{3\pi^2}{16} - \frac{i\pi}{2}\ln(1-i).$$

# 4 Trilogarithm Results

The following equalities hold:

$$\mathrm{Li}_3(z) = -\mathrm{Li}_3\left(\frac{z}{1-z}\right) - \mathrm{Li}_3(1-z) + \frac{\ln^3(1-z)}{6} - \frac{\ln(z)\ln^2(1-z)}{2} + \frac{\pi^2}{6}\ln(1-z) + \zeta(3)$$
$$\mathrm{Li}_3(x) - \mathrm{Li}_3\left(\frac{1}{x}\right) = -\frac{\ln^3(x)}{6} - \frac{\pi}{2}\frac{\sqrt{-(x-1)^2}}{x-1}\ln^2(x) + \frac{\pi^2}{3}\ln(x).$$

Setting $z = x = \frac{1+i}{2}$, we yield

$$\Im\left(i\,\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right) = \Re\left(\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right)$$
$$= \frac{35}{64}\zeta(3) - \frac{5\pi^2}{192}\ln(2) + \frac{\ln^3(2)}{48},$$

which is a well-known identity (see [4]) in the mathematical literature. The above mentioned all the formulas can be found in [3].

# 5 Appendix

In this section, we highlight about the analytical continuation, which is used in the previous sections. Based on the result stated in [5], we now shed light on few examples as follows:

Example:

$$\int_2^3 \frac{dx}{1-\sqrt{x}} = \sum_{n=0}^{\infty} \int_2^3 x^{3/2}\, dx = \sum_{n=0}^{\infty} \frac{6\cdot 3^{n/2} - 4\cdot 2^{n/2}}{n+2} = 2\left(\sqrt{2}-\sqrt{3}-\ln\left(\sqrt{3}-1\right) - \operatorname{arcsinh}(1)\right).$$

We used the elementary result, $\sum_{n=0}^{\infty} \frac{z^n}{n+2} = -\frac{z+\ln|1-z|}{z^2}$ for all $z \in [-1,1)$. Yet another example, one may consider solving a counter example:

$$\int_0^2 \frac{\log(x)}{1+x}\, dx = \ln(2)\ln(3) + \operatorname{Li}_2(-2).$$

# 6 Acknowledgments

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# References


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[3] L. Lewin, *Polylogarithm and Associated Functions*, North Holland, New York, 1981.

[4] C. I. Vălean, *A special of extracting the real part of the Trilogarithm,* $\operatorname{Li}_3\left(\frac{1\pm i}{2}\right)$. Available at : https://www.researchgate.net/publication/337868999.

[5] Website, *Math Stack Exchange*, Available at: https://math.stackexchange.com/questions/3938688/confusion-on-analytic-continuation .