TRIGONOMETRY INEQUAITLY

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Prove that

$$-\sqrt{2} \leq \sin x + \cos x \leq \sqrt{2}, \quad \forall x \in \mathbb{R}$$

Proof: Consider $f(t) = t^2 + t(\sin x + \cos x) + \frac{1}{2}$, we have discriminant is:

$$\Delta = (\sin x + \cos x)^2 - 4.\frac{1}{2} = \sin^2 x + \cos^2 x + 2\sin x \cos x - 2$$
$$= 2\sin x \cos x - 1 = \sin 2x - 1 \leq 0$$

Thus

$$f(t) = t^2 + t(\sin x + \cos x) + \frac{1}{2} \geq 0, \quad \forall t \in \mathbb{R}$$

Choose $t = -\frac{\sqrt{2}}{2}$, we have:

$$f\left(-\frac{\sqrt{2}}{2}\right) = \left(-\frac{\sqrt{2}}{2}\right)^2 - \frac{\sqrt{2}}{2}.(\sin x + \cos x) + \frac{1}{2} \geq 0$$
$$\iff \quad -\frac{\sqrt{2}}{2}.(\sin x + \cos x) + 1 \geq 0$$
$$\iff \quad \sin x + \cos x \leq \sqrt{2}$$

Choose $t = \frac{\sqrt{2}}{2}$, we have:

$$f\left(\frac{\sqrt{2}}{2}\right) = \left(\frac{\sqrt{2}}{2}\right)^2 + \frac{\sqrt{2}}{2}.(\sin x + \cos x) + \frac{1}{2} \geq 0$$
$$\iff \quad \frac{\sqrt{2}}{2}.(\sin x + \cos x) + 1 \geq 0$$
$$\iff \quad -\sqrt{2} \leq \sin x + \cos x$$