PROVE THAT A CASE OF THE AM-GM INEQUALITY

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If $a, b > 0$ and $ab = 1$ then $a + b \geq 2$.

Proof: Consider $f : \mathbb{R} \to \mathbb{R}$, $f(t) = -abt^2 + (a+b)t - 1$.
The discriminant is $\Delta = (a+b)^2 - 4.(-ab).(-1) = (a-b)^2 \geq 0$.
If $\Delta = (a-b)^2 = 0$ then:

$$f(t) \geq 0, \quad \forall t \in \mathbb{R}$$

Choose $t = 1$ we have:

$$a + b \geq 2$$

If $\Delta = (a-b)^2 > 0$, we can assume $a > b$, hence $a^2 > ab$, thus $a > 1$. We have

$$t_1 = \frac{-(a+b) + \sqrt{\Delta}}{-2ab} = \frac{-(a+b) + a - b}{-2ab} = \frac{-2b}{-2ab} = \frac{1}{a}$$

$$t_2 = \frac{-(a+b) - \sqrt{\Delta}}{-2ab} = \frac{-(a+b) - (a-b)}{-2ab} = \frac{-2a}{-2ab} = \frac{1}{b}$$

Because of $a > b$ we have $\frac{1}{a} < \frac{1}{b}$. Therefore:

$$f(t) > 0, \quad \forall t \in \left(\frac{1}{a}, \frac{1}{b}\right) = \left(\frac{1}{a}, a\right)$$

Because of $a > 1$ we have $a > 1 > \frac{1}{a}$ i.e $1 \in \left(\frac{1}{a}, a\right)$. Thus:

$$f(1) > 0 \quad \iff \quad -ab + (a+b) - 1 > 0 \quad \iff \quad a + b > 2$$