RMM - Cyclic Inequalities Marathon 1101 - 1200

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1101. *If $a,b,c\geq 0$ such that $a+b+c=1$ and $0\leq\lambda\leq 1$ then :*

$$\frac{1}{a^2+\lambda}+\frac{1}{b^2+\lambda}+\frac{1}{c^2+\lambda}\geq\frac{3\lambda+7}{(1+\lambda)^2}$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ will\ prove\ that:\ \ If\ 0\leq a,\lambda\leq 1\ then:\ \frac{1}{a^2+\lambda}\geq\frac{\lambda+3-2a}{(1+\lambda)^2}.$$

$$We\ have:\ \frac{1}{a^2+\lambda}\geq\frac{\lambda+3-2a}{(1+\lambda)^2}\Leftrightarrow\ (1+\lambda)^2\geq(a^2+\lambda)(\lambda+3-2a)$$

$$\Leftrightarrow 2a^3-(\lambda+3)a^2+2\lambda a+1-\lambda\geq 0\ \Leftrightarrow\ (a-1)^2[2a+(1-\lambda)]\geq 0$$

$$Which\ is\ true\ because:\ a,\ \ 1-\lambda\geq 0.$$

$$Then:\ \frac{1}{a^2+\lambda}\geq\frac{\lambda+3-2a}{(1+\lambda)^2}\ \ (and\ analogs)$$

$$Therefore,\ \ \frac{1}{a^2+\lambda}+\frac{1}{b^2+\lambda}+\frac{1}{c^2+\lambda}\geq\frac{3\lambda+9-2(a+b+c)}{(1+\lambda)^2}=\frac{3\lambda+7}{(1+\lambda)^2}.$$

$$Equality\ holds\ iff\ \lambda=1\ and\ (a=1,b=c=0)\ and\ permutation.$$

1102. *If $a,b,c>0$, then*

$$\sum_{cyc}\frac{b+c}{\sqrt{a}}-\sum_{cyc}\sqrt{2(b+c)}\geq\frac{1}{4(a+b+c)\sqrt{abc}}\sum_{cyc}c(a-b)^2$$

Proposed by Neculai Stanciu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\ \sum_{cyc}\frac{b+c}{\sqrt{a}}-\sum_{cyc}\sqrt{2(b+c)}\overset{AM-GM}{\geq}\sum_{cyc}\frac{b+c}{\sqrt{a}}-\sum_{cyc}\left(\frac{b+c}{2\sqrt{a}}+\sqrt{a}\right)=$$

$$=\sum_{cyc}\left(\frac{b+c}{2\sqrt{a}}-\sqrt{a}\right)=\sum_{cyc}\left(\frac{c-a}{2\sqrt{a}}-\frac{a-b}{2\sqrt{a}}\right)=\sum_{cyc}\left(\frac{a-b}{2\sqrt{b}}-\frac{a-b}{2\sqrt{a}}\right)=$$

$$\sum_{cyc}\frac{(a-b)(\sqrt{a}-\sqrt{b})}{2\sqrt{ab}}=\frac{1}{\sqrt{abc}}\sum_{cyc}\frac{c(a-b)^2}{2\sqrt{c}(\sqrt{a}+\sqrt{b})}\geq$$

$$\overset{AM-GM}{\geq}\frac{1}{\sqrt{abc}}\sum_{cyc}\frac{c(a-b)^2}{(c+a)+(c+b)}\geq\frac{1}{4(a+b+c)\sqrt{abc}}\sum_{cyc}c(a-b)^2,\ \textit{as desired.}$$

Equality holds iff $a=b=c$.

Solution 2 by Soumava Chakraborty-Kolkata-India

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y$ $\Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say) yielding $2\sum_{cyc}a=\sum_{cyc}x=2s\Rightarrow\boxed{\sum_{cyc}a\overset{(\bullet)}{=}s}\Rightarrow a=s-x, b=s-y, c=s-z\Rightarrow\boxed{abc\overset{(\bullet\bullet)}{=}r^2s}$

Via aforementioned substitutions, $\sum_{cyc}ab=\sum_{cyc}(s-x)(s-y)=4Rr+r^2$

$$\Rightarrow\boxed{\sum_{cyc}ab\overset{(\bullet\bullet\bullet)}{=}4Rr+r^2}$$

$$\sum_{cyc}\frac{b+c}{\sqrt{a}}-\frac{1}{4(a+b+c).\sqrt{abc}}\sum_{cyc}c(a-b)^2$$

$$=\frac{1}{\sqrt{abc}}\sum_{cyc}\left((b+c)\sqrt{bc}\right)$$

$$-\frac{1}{4(a+b+c).\sqrt{abc}}\left(\sum_{cyc}\left(ab\left(\sum_{cyc}a-c\right)\right)-6abc\right)$$

$$\overset{G-H}{\geq}\frac{4\sum_{cyc}a}{4(a+b+c).\sqrt{abc}}\sum_{cyc}\left((b+c)\left(\frac{2bc}{b+c}\right)\right)$$

$$-\frac{1}{4(a+b+c).\sqrt{abc}}\cdot\left(\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-9abc\right)$$

$$=\frac{8(\sum_{cyc}a)(\sum_{cyc}ab)-(\sum_{cyc}a)(\sum_{cyc}ab)+9abc}{4(\sum_{cyc}a).\sqrt{abc}}$$

$$\therefore\boxed{\sum_{cyc}\frac{b+c}{\sqrt{a}}-\frac{1}{4(a+b+c).\sqrt{abc}}\sum_{cyc}c(a-b)^2\overset{(*)}{\geq}\frac{7(\sum_{cyc}a)(\sum_{cyc}ab)+9abc}{4(\sum_{cyc}a).\sqrt{abc}}}\ \text{and}\ \sum_{cyc}\sqrt{2(b+c)}\overset{CBS}{\leq}\sqrt{6}.\sqrt{\sum_{cyc}(b+c)}$$

$$\therefore\boxed{\sum_{cyc}\sqrt{2(b+c)}\overset{(**)}{\leq}\sqrt{12\sum_{cyc}a}}$$

$$\therefore (*),(**) \Rightarrow \text{in order to prove : } \sum_{cyc}\frac{b+c}{\sqrt{a}} - \frac{1}{4(a+b+c).\sqrt{abc}}\sum_{cyc} c(a-b)^2$$

$$\geq \sum_{cyc}\sqrt{2(b+c)}, \text{it suffices to prove : } \frac{7(\sum_{cyc}a)(\sum_{cyc}ab)+9abc}{4(\sum_{cyc}a).\sqrt{abc}} \geq \sqrt{12\sum_{cyc}a}$$

$$\Leftrightarrow \frac{\left(7(\sum_{cyc}a)(\sum_{cyc}ab)+9abc\right)^2}{16abc(\sum_{cyc}a)^2} \geq 12\sum_{cyc}a \overset{\text{via }(\bullet),(\bullet\bullet),(\bullet\bullet\bullet)}{\Leftrightarrow} \left(7s(4Rr+r^2)+9r^2s\right)^2$$

$$\geq 192r^2s.s^3 \Leftrightarrow 12s^2 \overset{(i)}{\leq} 49R^2+56Rr+16r^2$$

$$\text{Now, } 12s^2 \overset{\text{Gerretsen}}{\leq} 48R^2+48Rr+36r^2 \overset{?}{\leq} 49R^2+56Rr+16r^2 \Leftrightarrow R^2+8Rr-20r^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow (R-2r)(R+10r) \overset{?}{\geq} 0 \to \text{true} \because R \overset{\text{Euler}}{\geq} 2r \Rightarrow (i) \text{ is true}$$

$$\therefore \sum_{cyc}\frac{b+c}{\sqrt{a}} - \frac{1}{4(a+b+c).\sqrt{abc}}\sum_{cyc}c(a-b)^2 \geq \sum_{cyc}\sqrt{2(b+c)} \Rightarrow \forall\, a,b,c$$

$$> 0, \sum_{cyc}\frac{b+c}{\sqrt{a}} - \sum_{cyc}\sqrt{2(b+c)}$$

$$\geq \frac{1}{4(a+b+c).\sqrt{abc}}\sum_{cyc}c(a-b)^2, \text{equality iff } a=b=c \text{ (QED)}$$

1103. *If $a,b,c>0$ then :*

$$1 \overset{(1)}{<} \sqrt{\frac{a}{a+2b+2c}} + \sqrt{\frac{b}{b+2c+2a}} + \sqrt{\frac{c}{c+2a+2b}} \overset{(2)}{\leq} \frac{3}{\sqrt{5}}.$$

Proposed by Vasile Mircea Popa-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{We have : } \sqrt{\frac{a}{a+2b+2c}} = \frac{2a}{2\sqrt{a(a+2b+2c)}} \overset{AM-GM}{\geq} \frac{2a}{a+(a+2b+2c)} = \frac{a}{a+b+c}$$

Equality holds for $a = a+2b+2c$ *which is not possible* $\therefore b,c>0$.

$$\text{Then : } \sqrt{\frac{a}{a+2b+2c}} > \frac{a}{a+b+c} \quad (\text{and analogs})$$

$$\text{Therefore, } \sqrt{\frac{a}{a+2b+2c}} + \sqrt{\frac{b}{b+2c+2a}} + \sqrt{\frac{c}{c+2a+2b}} > \frac{a+b+c}{a+b+c} = 1.$$

WLOG, we assume that $a+b+c=3$ *and* $c=\max\{a,b,c\}$. *We have :*

$$a,b<\frac{a+b+c}{2}=\frac{3}{2}.$$

$$\textit{The inequality } (2) \textit{ becomes}: \ f(a)+f(b)+f(c)\le\frac{3}{\sqrt{5}}, \ \textit{where}$$

$$f(x)=\sqrt{\frac{x}{6-x}}, \ \ x\in(0,3).$$

$$\textit{We have}: \ f'(x)=\frac{3}{\sqrt{x(6-x)^3}} \textit{ and } f''(x)=-\frac{3(3-2x)}{\sqrt{x^3(6-x)^5}}<0, \ \ x\in\left(0,\frac{3}{2}\right).$$

$$\textit{Then } f \textit{ is concave on } \left(0,\frac{3}{2}\right) \textit{ and by Jensen's inequality we have}:$$

$$f(a)+f(b)\le 2f\left(\frac{a+b}{2}\right)=2f\left(\frac{3-c}{2}\right) \ \ \therefore 0<a,b<\frac{3}{2}.$$

$$\textit{So it suffices to prove}: \ 2f\left(\frac{3-c}{2}\right)+f(c)\le\frac{3}{\sqrt{5}} \Leftrightarrow 2\sqrt{\frac{3-c}{9+c}}+\sqrt{\frac{c}{6-c}}\le\frac{3}{\sqrt{5}}$$

$$\overset{Squaring}{\Leftrightarrow} \ 4\sqrt{\frac{(3-c)c}{(9+c)(6-c)}}\le\frac{9}{5}-\left(\frac{4(3-c)}{9+c}+\frac{c}{6-c}\right)=\frac{2(63+54c-17c^2)}{5(9+c)(6-c)}$$

$$\overset{Squaring}{\Leftrightarrow} \ 4(3-c)c\times 25(9+c)(6-c)\le(63+54c-17c^2)^2$$

$$\Leftrightarrow 7c^4-68c^3+262c^2-348c+147\ge 0 \ \Leftrightarrow \ (c-1)^2\left[7\left(c-\frac{27}{7}\right)^2+\frac{300}{7}\right]\ge 0.$$

$$\textit{Which is true and the proof of } (2) \textit{ is complete. Equality holds iff } a=b=c.$$

1104. *Let* $a,b,c\ge 0: \ a^2+b^2+c^2=1.$ *Prove that*:

$$\frac{a^4-a^2}{bc-1}+\frac{b^4-b^2}{ca-1}+\frac{c^4-c^2}{ab-1}\le ab+bc+ca.$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have:

$$\frac{a^4-a^2}{bc-1}=\frac{a^2(b^2+c^2)}{a^2+b^2+c^2-bc}\overset{AM-GM}{\leq}\frac{a^2(b^2+c^2)}{2a\sqrt{b^2+c^2-bc}}=$$

$$=\frac{a(b+c)(b^2+c^2)}{2\sqrt{(b+c)(b^3+c^3)}}\overset{CBS}{\leq}\frac{a(b+c)(b^2+c^2)}{2(b^2+c^2)}=\frac{ca+ab}{2}.$$

Similarly we have :

$$\frac{b^4-b^2}{ca-1}\leq\frac{ab+bc}{2}\ \ and\ \ \frac{c^4-c^2}{ab-1}\leq\frac{bc+ca}{2}.$$

Summing up these inequalities yields the desired inequality.

Equality holds iff (a,b,c)

$$=\left(\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}\right),\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0\right)\ or\ (1,0,0)\ and\ permutations.$$

1105. *Let* $a,b,c\geq 0 : ab+bc+ca=1$. *Prove that :*

$$\frac{a^2(1+bc)}{b^2+c^2}+\frac{b^2(1+ca)}{c^2+a^2}+\frac{c^2(1+ab)}{a^2+b^2}\geq(a-b)^2+(b-c)^2+(c-a)^2$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{a^2(1+bc)}{b^2+c^2}\overset{bc\geq 0}{\geq}\frac{a^2[a(b+c)+2bc]}{(b+c)^2}\overset{abc\geq 0}{\geq}\frac{a^2.a}{b+c}$$

$$=a\left(\frac{a^2}{b+c}+(b+c)\right)-a(b+c)\overset{AM-GM}{\geq}a.2a-a(b+c).$$

$$Then:\ \frac{a^2(1+bc)}{b^2+c^2}\geq 2a^2-a(b+c)\ \ (and\ analogs)$$

Therefore,

$$\sum_{cyc}\frac{a^2(1+bc)}{b^2+c^2}\geq\sum_{cyc}[2a^2-a(b+c)]=(a-b)^2+(b-c)^2+(c-a)^2.$$

Equality holds iff $(a,b,c)=(1,1,0)$ *and permutation.*

1106. *If* $x, y, z > 0$ *and* $0 \le \lambda \le \frac{1}{25}$ *then :*

$$\sum_{cyc} \frac{x}{\sqrt[3]{\lambda y^3 + xyz + \lambda z^3}} \overset{(*)}{\ge} \frac{3}{\sqrt[3]{2\lambda + 1}}$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let\ \ p := x + y + z = 3,\ \ q := xy + yz + zx \le \frac{p^2}{3} = 3,\ \ r := xyz \overset{AM-GM}{\le} \left(\frac{p}{3}\right)^3 = 1.$$

$$By\ H\ddot{o}lder's\ inequality\ we\ have:$$

$$\left(LHS_{(*)}\right)^3 . \sum_{cyc} x(\lambda y^3 + xyz + \lambda z^3) \ge (x + y + z)^4 = 81.$$

$$Also\ we\ have:\ \sum_{cyc} x(y^3 + z^3) = p^2 q - 2q^2 - pr = -2q^2 + 9q - 3r.$$

$$Then:\ \left(LHS_{(*)}\right)^3 \ge \frac{81}{\lambda(-2q^2 + 9q - 3r) + 3r} \overset{?}{\ge} \left(\frac{3}{\sqrt[3]{2\lambda + 1}}\right)^3$$

$$\Leftrightarrow\ 3(2\lambda + 1) \ge \lambda(-2q^2 + 9q - 3r) + 3r \Leftrightarrow 3(1 - r) + \lambda(2q^2 - 9q + 3r + 6) \ge 0\ (1)$$

$$If\ 2q^2 - 9q + 3r + 6 \ge 0\ then:\ LHS_{(*)} \overset{r \le 1\ \&\ \lambda \ge 0}{\ge} 0.$$

$$If\ 2q^2 - 9q + 3r + 6 \le 0\ then:\ LHS_{(*)} \overset{\lambda \le \frac{1}{25}}{\ge} 3(1 - r) + \frac{1}{25}(2q^2 - 9q + 3r + 6) =$$

$$= \frac{2q^2 - 9q - 72r + 81}{25} \overset{9r \le pq = 3q}{\ge} \frac{2q^2 - 9q - 24q + 81}{25} = \frac{(3 - q)(27 - 2q)}{25} \overset{q \le 3}{\ge} 0.$$

$$So\ the\ proof\ is\ complete.\ Equality\ holds\ iff\ x = y = z.$$

1107. *Let* $a, b, c > 0 : a + b + c + 2 = abc.$ *Prove that :*

$$\frac{\sqrt{1 + ab + ac}}{a + 1} + \frac{\sqrt{1 + bc + ba}}{b + 1} + \frac{\sqrt{1 + ca + cb}}{c + 1} \ge 3$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:$$

$$(a+b+c)\sqrt{1+ab+ac}=\sqrt{[(-a+b+c)^2+4a(b+c)][1+a(b+c)]}\ge$$

$$\overset{CBS}{\ge}(-a+b+c)+2a(b+c)=(-a+b+c)(a+1)+a(a+b+c)$$

$$Then: \frac{\sqrt{1+ab+ac}}{a+1}\ge\frac{-a+b+c}{a+b+c}+\frac{a}{a+1}\ (and\ analogs)$$

$$Therefore,\ \sum_{cyc}\frac{\sqrt{1+ab+ac}}{a+1}\ge\sum_{cyc}\left(\frac{-a+b+c}{a+b+c}+1-\frac{1}{a+1}\right)=1+3-\sum_{cyc}\frac{1}{a+1}=$$

$$=4-\frac{2(a+b+c)+(ab+bc+ca)+3}{abc+(a+b+c)+(ab+bc+ca)+1}\overset{abc=a+b+c+2}{=}4-1=3.$$

$$Equality\ holds\ iff\ a=b=c=2.$$

1108. $If\ a,b,c\ge 0: ab+bc+ca>0\ then:$

$$\frac{a}{4a^2+bc}+\frac{b}{4b^2+ca}+\frac{c}{4c^2+ab}\ge\frac{1}{a+b+c}.$$

$$When\ does\ equality\ holds?$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:$$

$$\sqrt{(4a^2+bc)[a(b+c)+bc]}\overset{CBS}{\ge}2a\sqrt{a(b+c)}+bc\overset{GM-HM}{\ge}2a.\frac{2a.(b+c)}{a+(b+c)}+bc=$$

$$=\frac{(b+c)(4a^2+bc)+abc}{a+b+c}\overset{abc\ge 0}{\ge}\frac{(b+c)(4a^2+bc)}{a+b+c}.$$

$$Then: \frac{1}{\sqrt{4a^2+bc}}\ge\frac{b+c}{(a+b+c)\sqrt{ab+bc+ca}}\ (and\ analogs)$$

$$Therefore,\ \sum_{cyc}\frac{a}{4a^2+bc}\ge\sum_{cyc}\frac{a(b+c)^2}{(a+b+c)^2(ab+bc+ca)}=$$

$$=\frac{(a+b+c)(ab+bc+ca)+3abc}{(a+b+c)^2(ab+bc+ca)}\overset{abc\ge 0}{\ge}\frac{1}{a+b+c},\ \textit{as desired.}$$

Equality holds iff $a=b\neq 0$ and $c=0$ and permutation.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

The given inequality is successively equivalent to :

$$(a+b+c).\sum_{cyc}a(4b^2+ca)(4c^2+ab)\ge(4a^2+bc)(4b^2+ca)(4c^2+ab)$$

$$\overset{expanding}{\Leftrightarrow}\quad 4\sum_{sym}a^4b^2+21abc\sum_{sym}a^2b\ge 8\sum_{cyc}(bc)^3+3abc\sum_{cyc}a^3+17(abc)^2$$

$$\Leftrightarrow 4(a-b)^2(b-c)^2(c-a)^2+13abc\sum_{sym}a^2b+5abc\sum_{cyc}a^3+7(abc)^2\ge 0.$$

Which is true.

$$\therefore (a-b)^2(b-c)^2(c-a)^2=\sum_{sym}a^4b^2+2abc\sum_{sym}a^2b-2\sum_{cyc}(bc)^3-2abc\sum_{cyc}a^3-6(abc)^2.$$

Equality holds iff $a=b\neq 0$ and $c=0$ and permutation.

1109. *Let $a,b,c\ge 0$: $ab+bc+ca=1$. Prove that :*

$$\frac{1-a^2}{a^2+\sqrt{bc}}+\frac{1-b^2}{b^2+\sqrt{ca}}+\frac{1-c^2}{c^2+\sqrt{ab}}\ge 1$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have :

$$a^2+b^2+c^2\overset{ab+bc+ca=1}{=}\sqrt{[(-a^2+b^2+c^2)^2+4a^2(b^2+c^2)][bc+a(b+c)]}\overset{CBS}{\ge}$$

$$\ge(-a^2+b^2+c^2)\sqrt{bc}+2a\sqrt{a(b+c)(b^2+c^2)}\overset{\sqrt{ab+bc+ca}\ge\sqrt{a(b+c)}}{\ge}$$

$$\ge(-a^2+b^2+c^2)\sqrt{bc}+\frac{2a^2(b+c)\sqrt{b^2+c^2}}{\sqrt{ab+bc+ca}}\ge$$

$$\overset{b+c\geq\sqrt{b^2+c^2}}{\geq}(-a^2+b^2+c^2)\sqrt{bc}+2a^2(b^2+c^2)$$
$$=(-a^2+b^2+c^2)(a^2+\sqrt{bc})+a^2(a^2+b^2+c^2).$$

$$Then:\ \frac{1-a^2}{a^2+\sqrt{bc}}\geq\frac{-a^2+b^2+c^2}{a^2+b^2+c^2}.\ (and\ analogs)$$

$$Therefore,\ \sum_{cyc}\frac{1-a^2}{a^2+\sqrt{bc}}\geq\sum_{cyc}\frac{-a^2+b^2+c^2}{a^2+b^2+c^2}=1.$$

$$Equality\ holds\ iff\ a=b=1, c=0\ and\ permutation.$$

1110. *If $a,b,c>0$ then:*

$$\sum_{cyc}ln\left(\frac{c+a}{c}\right)ln\left(\frac{c+b}{b}\right)<\sum_{cyc}\left(\frac{a}{b}\right)^{2022}$$

Proposed by Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ know\ that:$$

$$0\leq ln(1+x)\leq x,\ \ \forall x\geq 0\ with\ equality\ when\ x=0.$$

$$Then\ we\ have:$$

$$\sum_{cyc}ln\left(\frac{c+a}{c}\right)ln\left(\frac{c+b}{b}\right)=\sum_{cyc}ln\left(1+\frac{a}{c}\right)ln\left(1+\frac{c}{b}\right)<\sum_{cyc}\frac{a}{c}\cdot\frac{c}{b}=\sum_{cyc}\frac{a}{b}.$$

$$Also,$$

$$\sum_{cyc}\left(\frac{a}{b}\right)^{2022}\overset{H\ddot{o}lder}{\geq}3^{-2021}\cdot\left(\sum_{cyc}\frac{a}{b}\right)^{2022}\overset{AM-GM}{\geq}3^{-2021}\cdot\left(3\sqrt[3]{\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{a}}\right)^{2021}\cdot\sum_{cyc}\frac{a}{b}=\sum_{cyc}\frac{a}{b}.$$

$$Therefore,\ \sum_{cyc}ln\left(\frac{c+a}{c}\right)ln\left(\frac{c+b}{b}\right)<\sum_{cyc}\left(\frac{a}{b}\right)^{2022}.$$

1111. *Let $a,b,c\geq 0$. Prove that :*

$$\frac{15(a^2+b^2+c^2)}{a+b+c}-2(a+b+c)\geq 2\sqrt[5]{81(a^5+b^5+c^5)}+3\sqrt[3]{abc}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Since:\ 3\sqrt[3]{abc}\overset{AM-GM}{\leq} a+b+c,\ it\ suffices\ to\ prove:$$

$$\frac{15(a^2+b^2+c^2)}{a+b+c}-3(a+b+c)\geq 2\sqrt[5]{81(a^5+b^5+c^5)}\ (*)$$

$$WLOG, we\ may\ assume\ that\ a+b+c=1.$$

$$Let\ ab+bc+ca=\frac{1-q^2}{3}(q\in[0,1])\ and\ abc=r.$$

$$Using\ the\ following\ identity:$$

$$a^5+b^5+c^5=(a+b+c)^5-5[(a+b+c)(ab+bc+ca)-abc][(a+b+c)^2-(ab+bc+ca)],$$

$$we\ get:\ a^5+b^5+c^5=\frac{5q^4+5q^2-1}{9}+\frac{5(q^2+2)r}{3}.$$

$$Then:\ (*)\ \Leftrightarrow 1+5q^2\geq\sqrt[5]{9(5q^4+5q^2-1)+135(q^2+2)r}$$

$$\Leftrightarrow r\leq\frac{(1+5q^2)^5-9(5q^4+5q^2-1)}{135(q^2+2)}=\frac{2-4q^2+41q^4+250q^6+625q^8+625q^{10}}{27(q^2+2)}.$$

$$By\ Vo\ Quoc\ Ba\ Can's\ inequality\ we\ have:$$

$$r\leq\frac{(1-q)^2(1+2q)}{27}, so\ it\ suffices\ to\ prove:$$

$$(1-q)^2(1+2q)(q^2+2)\leq 2-4q^2+41q^4+250q^6+625q^8+625q^{10}$$

$$\Leftrightarrow\ q^2(1-2q)^2+39q^4+q^4(1-q)^2+249q^6+625q^8+625q^{10}\geq 0.$$

$$Which\ is\ true\ and\ the\ proof\ is\ complete.\ Equality\ holds\ iff\ q=0\ \Leftrightarrow\ a=b=c.$$

1112. *Prove that :*

$$\sqrt{\frac{a+b+2c}{c+2}}+\sqrt{\frac{b+c+2a}{a+2}}+\sqrt{\frac{c+a+2b}{b+2}}\geq 3\sqrt{\frac{(k+1)abc-2}{kabc-2}}$$

$$holds\ for\ a,b,c\geq 0, ab+bc+ca=1\ and\ k=6\sqrt{3}-9.$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have :

$$\frac{(a+b+c)(a+b+2c)}{(b+c)(c+a)}=(a+b+c)\left(\frac{1}{b+c}+\frac{1}{c+a}\right)=2+\frac{a}{b+c}+\frac{b}{c+a}=$$

$$=\left(a(b+c)+\frac{a}{b+c}\right)+\left(b(c+a)+\frac{b}{c+a}\right)+c(a+b)\overset{AM-GM}{\geq}$$

$$\geq\ 2a+2b+c(a+b)=(a+b)(c+2)$$

$$Then:\ \frac{a+b+2c}{c+2}\geq\frac{(a+b)(b+c)(c+a)}{a+b+c}\quad(1).$$

$$Now\ let's\ prove\ that:\ \frac{(a+b)(b+c)(c+a)}{a+b+c}\geq\frac{(k+1)abc-2}{kabc-2}\quad(2)$$

$$Let\ p:=a+b+c,\ \ q:=ab+bc+ca=1,\ \ r:=abc\overset{AM-GM}{\leq}\sqrt{\left(\frac{q}{3}\right)^3}=\frac{\sqrt{3}}{9}.$$

$$(2)\ \Leftrightarrow\frac{pq-r}{p}\geq\frac{(k+1)r-2}{kr-2}\Leftrightarrow 1-\frac{r}{p}\geq 1-\frac{r}{2-kr}\Leftrightarrow\ p+kr\geq 2.\ \ \therefore 2-kr>0.$$

$$If\ p\geq 2\ the\ inequality\ is\ obvious.\ Assume\ now\ that\ p\leq 2.$$

$$By\ Schur's\ inequality\ we\ have:\ r\geq\frac{4pq-p^3}{9}=\frac{4p-p^3}{9},then:$$

$$p+kr\geq p+k.\frac{4p-p^3}{9}=2+\frac{(2-p)(p-\sqrt{3})\left[(2\sqrt{3}-3)p+\sqrt{3}\right]}{3}\geq 2,$$

$$because\ 2 \geq p \geq \sqrt{3q} = \sqrt{3}.\ (1)\ \&\ (2)\ \Rightarrow \frac{a+b+2c}{c+2} \geq \frac{(k+1)abc-2}{kabc-2}\ (and\ analogs)$$

$$Therefore,\ \sqrt{\frac{a+b+2c}{c+2}} + \sqrt{\frac{b+c+2a}{a+2}} + \sqrt{\frac{c+a+2b}{b+2}} \geq 3\sqrt{\frac{(k+1)abc-2}{kabc-2}}.$$

$$Equality\ holds\ iff\ (a,b,c) = (1,1,0)\ and\ permutation.$$

1113. *Let $a,b,c \geq 0 : ab+bc+ca = 1$. Prove that :*

$$\sqrt{\frac{c+2}{a+b+2c}} + \sqrt{\frac{a+2}{b+c+2a}} + \sqrt{\frac{b+2}{c+a+2b}} \leq 3\sqrt{\frac{a+b+c}{(a+b)(b+c)(c+a)}}$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ will\ prove\ that\ if\ a,b,c \geq 0, ab+bc+ca = 1 :$$

$$\frac{c+2}{a+b+2c} \leq \frac{a+b+c}{(a+b)(b+c)(c+a)}\ (*)$$

$$(*)\ \Leftrightarrow\ (c+2)(a+b) \leq (a+b+c)\left(\frac{1}{b+c} + \frac{1}{c+a}\right)$$

$$\Leftrightarrow (c+2)(a+b) \leq 2 + \frac{a}{b+c} + \frac{b}{c+a} = 2(ab+bc+ca) + \frac{a}{b+c} + \frac{b}{c+a}$$

$$\Leftrightarrow 2a+2b \leq a\left[(b+c) + \frac{1}{b+c}\right] + b\left[(c+a) + \frac{1}{c+a}\right]$$

$$Which\ is\ true\ by\ AM-GM.\ \therefore\ (b+c) + \frac{1}{b+c}, (c+a) + \frac{1}{c+a} \geq 2.$$

$$Then,\ \frac{c+2}{a+b+2c} \leq \frac{a+b+c}{(a+b)(b+c)(c+a)}\ (and\ analogs)$$

$$Therefore,\ \sqrt{\frac{c+2}{a+b+2c}} + \sqrt{\frac{a+2}{b+c+2a}} + \sqrt{\frac{b+2}{c+a+2b}} \leq 3\sqrt{\frac{a+b+c}{(a+b)(b+c)(c+a)}}.$$

$$Equality\ holds\ iff\ (a,b,c) = (1,1,0)\ and\ permutation.$$

1114. *Let $a,b,c\ge 0: ab+bc+ca>0$. Prove that:*

$$\sqrt{\frac{a(b+c)}{b^2+c^2}}+\sqrt{\frac{b(c+a)}{c^2+a^2}}+\sqrt{\frac{c(a+b)}{a^2+b^2}}\ge\frac{4\sqrt{ab+bc+ca}}{a+b+c}$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution 1 by Vivek Kumar-India

$$\sqrt{\frac{a(b+c)}{b^2+c^2}}+\sqrt{\frac{b(c+a)}{c^2+a^2}}+\sqrt{\frac{c(a+b)}{a^2+b^2}}\ge\frac{4\sqrt{ab+bc+ca}}{a+b+c}\Leftrightarrow$$

$$\Leftrightarrow\sum_{cyc}\frac{2\sqrt{a(b+c)}(a+b+c)}{2\sqrt{b^2+c^2}\sqrt{ab+bc+ca}}\ge 4$$

$$\sum_{cyc}\frac{2\sqrt{a(b+c)}(a+b+c)}{2\sqrt{b^2+c^2}\sqrt{ab+bc+ca}}\overset{AM-GM}{\ge}\sum_{cyc}\frac{2\sqrt{a(b+c)}\cdot 2\sqrt{a(b+c)}}{b^2+c^2+ab+bc+ca}=$$

$$=\sum_{cyc}\frac{4a(b+c)}{(b+c)^2+a(b+c)-bc}\ge\sum_{cyc}\frac{4a(b+c)}{(b+c)^2+a(b+c)}=$$

$$=\sum_{cyc}\frac{4a}{a+b+c}=\frac{4(a+b+c)}{a+b+c}=4$$

Equality holds iff $(a=0,b=c>0)$ and permutation.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have:

$$\sqrt{\frac{a(b+c)}{b^2+c^2}}\overset{GM-HM}{\ge}\frac{2a(b+c)}{[a+(b+c)]\sqrt{b^2+c^2}}=\frac{2a(b+c)(a+b+c)}{(a+b+c)^2\sqrt{b^2+c^2}}=$$

$$=\frac{2a[(ab+bc+ca)+(b^2+c^2)+bc]}{(a+b+c)^2\sqrt{b^2+c^2}}\overset{AM-GM\ \&\ bc\ge 0}{\ge}\frac{2a.2\sqrt{(ab+bc+ca)(b^2+c^2)}}{(a+b+c)^2\sqrt{b^2+c^2}}.$$

$$Then:\ \sqrt{\frac{a(b+c)}{b^2+c^2}}\ge\frac{4a\sqrt{ab+bc+ca}}{(a+b+c)^2}.$$

$$\text{Similarly we have}: \sqrt{\frac{b(c+a)}{c^2+a^2}} \ge \frac{4b\sqrt{ab+bc+ca}}{(a+b+c)^2} \;\&\; \sqrt{\frac{c(a+b)}{a^2+b^2}} \ge \frac{4c\sqrt{ab+bc+ca}}{(a+b+c)^2}.$$

Adding these inequalities yields the desired inequality.

$$\text{Equality holds iff } (a=0, b=c>0) \text{ and permutation.}$$

1115. If $x, y, z > 0, x+y+z=3$ then:

$$\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x} \le \frac{9}{2(xy+yz+zx)}$$

Proposed by Dorina Goiceanu, Simona Dascălu – Romania

Solution by Hikmat Mammadov-Azerbaijan

$$x; y; z > 0 \text{ and } x+y+z=3$$

$$\frac{1}{x+y}+\frac{1}{y+x}+\frac{1}{z+x} \le \frac{9}{2(xy+yz+zx)}$$

The inequality is equivalent with

$$LHS = \frac{xy+yz+zx}{x+y}+\frac{xy+yz+zx}{y+z}+\frac{xy+yz+zx}{z+x} \le 6$$

$$\frac{xy+yz+zx}{x+y} = \frac{xy}{x+y}+\frac{z(x+y)}{x+y} = \frac{xy}{x+y}+z$$

But, $\frac{xy}{x+y} \le \frac{x+y}{4}$

So, $LHS \le \frac{x+y}{4}+\frac{y+z}{4}+\frac{z+x}{4}+x+y+z = 3+\frac{3}{2} = \frac{9}{2}$

So, $LHS \le \frac{9}{2}$

Equality $\Leftrightarrow x=y=z=1$

1116. In $a, b, c > 0$ such that $a+b+c=1$ and $\lambda \ge 4$ then

$$\lambda(ab+bc+ca)^2 \le ab+bc+ca+3(\lambda-3)abc$$

Proposed by Marin Chirciu-Romania

Solution 1 by Vivek Kumar-India

Let $\lambda = 6, a = \frac{1}{2}, b = \frac{1}{3}, c = \frac{1}{6}$ such that $a+b+c=1$ and $\lambda \ge 4$ then

$$LHS=\lambda(ab+bc+ca)^2=6\left(\frac{1}{2}\cdot\frac{1}{3}+\frac{1}{3}\cdot\frac{1}{6}+\frac{1}{6}\cdot\frac{1}{2}\right)^2$$

$$=6\left(\frac{1}{6}+\frac{1}{18}+\frac{1}{12}\right)^2=6\cdot\left(\frac{11}{36}\right)^2=\frac{6\times 121}{1296}=\frac{121}{216}$$

$$RHS=ab+bc+ca+3(\lambda-3)\,abc$$

$$=\frac{1}{6}+\frac{1}{18}+\frac{1}{12}+3(6-3)\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{6}=\frac{11}{36}+\frac{9}{36}=\frac{20}{36}=\frac{5}{9}$$

But $\frac{121}{216}>\frac{5}{9}\Rightarrow LHS>RHS\Rightarrow 121>120$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\lambda(ab+bc+ca)^2\le ab+bc+ca+3(\lambda-3)abc\overset{1=a+b+c}{\Leftrightarrow}$$

$$\lambda\left(\sum_{cyc}ab\right)^2\le\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a\right)^2+3(\lambda-3)abc\left(\sum_{cyc}a\right)$$

$$=\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2+2\sum_{cyc}ab\right)+3(\lambda-3)abc\left(\sum_{cyc}a\right)\Leftrightarrow$$

$$(\lambda-2)\left(\sum_{cyc}ab\right)^2\le\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2\right)+3(\lambda-3)abc\left(\sum_{cyc}a\right)$$

$$\Leftrightarrow(\lambda-3)\left(\left(\sum_{cyc}ab\right)^2-3abc\left(\sum_{cyc}a\right)\right)\le\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)$$

$$\Leftrightarrow(\lambda-4)\left(\left(\sum_{cyc}ab\right)^2-3abc\left(\sum_{cyc}a\right)\right)+\left(\sum_{cyc}ab\right)^2-$$

$$-3abc\left(\sum_{cyc}a\right)\overset{(*)}{\le}\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)$$

$$\because\left(\sum_{cyc}ab\right)^2-3abc\left(\sum_{cyc}a\right)\ge 0\ \ge\text{ and }\lambda-4\le 0$$

$$\therefore(\lambda-4)\left(\left(\sum_{cyc}ab\right)^2-3abc\left(\sum_{cyc}a\right)\right)\le 0\Rightarrow\text{LHS of }(*)\le\left(\sum_{cyc}ab\right)^2-3abc\left(\sum_{cyc}a\right)$$

$\therefore$ in order to prove $(*)$,

it suffices to prove : $\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^2 - \sum_{cyc} ab\right) \overset{(**)}{\geq} \left(\sum_{cyc} ab\right)^2 - 3abc\left(\sum_{cyc} a\right)$

Assigning $b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0, y + z - x = 2a > 0$ and $z + x - y = 2b > 0 \Rightarrow x + y > z, y + z > x, z + x > y \Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius $=$ s, R, r (say) yielding

$$2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \boxed{\sum_{cyc} a \overset{(\bullet)}{=} s} \Rightarrow a = s - x, b = s - y, c = s - z \Rightarrow \boxed{abc \overset{(\bullet\bullet)}{=} r^2 s}$$

Via aforementioned substitutions, $\sum_{cyc} ab = \sum_{cyc}(s - x)(s - y) = 4Rr + r^2$

$$\Rightarrow \boxed{\sum_{cyc} ab \overset{(\bullet\bullet\bullet)}{=} 4Rr + r^2} \text{ and } \sum_{cyc} a^2$$

$$= \left(\sum_{cyc} a\right)^2 - 2\sum_{cyc} ab \overset{\text{via}(\bullet),(\bullet\bullet\bullet)}{=} s^2 - 2(4Rr + r^2)$$

$$\Rightarrow \sum_{cyc} a^2 \overset{(\bullet\bullet\bullet\bullet)}{=} s^2 - 8Rr - 2r^2 \therefore (\bullet), (\bullet\bullet), (\bullet\bullet\bullet), (\bullet\bullet\bullet\bullet) \Rightarrow (**)$$

$$\Leftrightarrow (4Rr + r^2)\left(s^2 - 8Rr - 2r^2 - (4Rr + r^2)\right) \geq (4Rr + r^2)^2 - 3r^2s^2$$

$$\Leftrightarrow (R + r)s^2 \overset{(***)}{\geq} r(4R + r)^2$$

Now, $(R + r)s^2 \overset{\text{Gerretsen}}{\geq} (R + r)(16Rr - 5r^2) \overset{?}{\geq} r(4R + r)^2 \Leftrightarrow 3r(R - 2r) \overset{?}{\geq} 0$

$\rightarrow$ true, via Euler $\Rightarrow (***) \Rightarrow (**) \Rightarrow (*)$ is true

$\therefore \forall\, a, b, c > 0 \mid a + b + c = 1$ and $\lambda \leq 4$,

$\lambda(ab + bc + ca)^2 \leq ab + bc + ca + 3(\lambda - 3)abc$, equality iff $a = b = c = \frac{1}{3}$ (QED)

1117. Find all $x, y, z \in \left(0, \frac{\pi}{2}\right]$ such that:

$$\frac{1}{1 + \sin x + \sin y + \sin z} + \sum_{cyc} \frac{1}{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^4} = 1$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Florentin Vișescu-Romania

The equality is equivalent with

$$\frac{1}{1 + \sin x + \sin y + \sin z} + \frac{1}{(1 + \sin x)^2} + \frac{1}{(1 + \sin y)^2} + \frac{1}{(1 + \sin z)^2} = 1$$

We consider the function $f: (0; 1] \rightarrow \mathbb{R}, f(t) = \frac{1}{(1+t)^2}$

$f'(t) = -\frac{2}{(1+t)^3}; f''(t) = \frac{6}{(1+t)^4} > 0 \Rightarrow f$ convex $\Rightarrow$

$$f(t_1) + f(t_2) + f(t_3) \geq 3f\left(\frac{t_1 + t_2 + t_3}{3}\right)$$

Let be $t_1 = \sin x, t_2 = \sin y, t_3 = \sin z$. Then:

$$\frac{1}{(1+\sin x)^2} + \frac{1}{(1+\sin y)^2} + \frac{1}{(1+\sin z)^2} \geq 3\frac{1}{\left(1 + \frac{\sin x + \sin y + \sin z}{3}\right)^2}$$

or $1 - \frac{1}{1+\sin x+\sin y+\sin z} \geq 3\frac{1}{\left(1+\frac{\sin x+\sin y+\sin z}{3}\right)^2}$

we denote $\sin x + \sin y + \sin z = s$

$$1 - \frac{1}{1+s} \geq 3\frac{1}{\left(1+\frac{s}{3}\right)^2} \Leftrightarrow \frac{s}{1+s} \geq \frac{27}{(3+s)^2}$$

$$\Leftrightarrow 9s + 6s^2 + s^3 \geq 27 + 27s \Leftrightarrow s^3 + 6s^2 - 18s - 27 \geq 0$$

$$\Leftrightarrow (s-3)(s^2 + 9s + 9) \geq 0$$

But $s^2 + 9s + 9 > 0$. Then $s - 3 \geq 0 \Leftrightarrow s \geq 3 \Leftrightarrow \sin x + \sin y + \sin z \geq 3 \Rightarrow$

$$\sin x = \sin y = \sin z = 1 \Rightarrow x = y = z = \frac{\pi}{2}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Since\ x, y, z \in (0, \frac{\pi}{2}]\ \ we\ have:\ \ 0 < \sin x, \sin y, \sin z \leq 1.$$

$$Then:\ \ 1 = \frac{1}{1 + \sin x + \sin y + \sin z} + \sum_{cyc} \frac{1}{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^4} =$$

$$= \frac{1}{1 + \sin x + \sin y + \sin z} + \sum_{cyc} \frac{1}{(1+\sin x)^2} \geq \frac{1}{1+1+1+1} + \sum_{cyc} \frac{1}{(1+1)^2} = 1.$$

$$Equality\ holds\ iff\ \sin x = \sin y = \sin z = 1\ \Leftrightarrow\ x = y = z = \frac{\pi}{2}.$$

Solution 3 by Hikmat Mammadov-Azerbaijan

$$\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^4 = \left(\sin^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2} + \cos^2\frac{x}{2}\right)^2 = (1+\sin x)^2 \leq 4 \quad (1)$$

And the similar

$$1 \le 1 + \sin x + \sin y + \sin z \le 4 \quad (2) \quad \to \forall x; y; z \in \left(0, \frac{\pi}{2}\right]$$

$$\text{So, } LHS \ge \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$$

With equality iff in (1) and (2) we have equality, i.e., $x = y = z = \frac{\pi}{2}$.

1118. *If* $x, y, z > 0 : x + y + z = 3$ *and* $\lambda \ge 7$ *then* :

$$\frac{1}{x^2 + \lambda} + \frac{1}{y^2 + \lambda} + \frac{1}{z^2 + \lambda} \le \frac{3}{1 + \lambda}$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Lemma: \ If \ x \in (0, 3) \ and \ \lambda \ge 7 \ then: \ \frac{1}{x^2 + \lambda} \le \frac{-2x + \lambda + 3}{(1 + \lambda)^2} \quad (*)$$

$$We \ have: \ (*) \Leftrightarrow (-2x + \lambda + 3)(x^2 + \lambda) \ge (1 + \lambda)^2$$

$$\Leftrightarrow -2x^3 + (\lambda + 3)x^2 - 2\lambda x + \lambda - 1 \ge 0 \Leftrightarrow (x - 1)^2(\lambda - 1 - 2x) \ge 0$$

$$Which \ is \ true \ because \ \lambda - 1 - 2x > 7 - 1 - 2.3 = 0.$$

$$Then: \ \frac{1}{x^2 + \lambda} \le \frac{-2x + \lambda + 3}{(1 + \lambda)^2} \quad (and \ analogs)$$

$$Therefore, \ \frac{1}{x^2 + \lambda} + \frac{1}{y^2 + \lambda} + \frac{1}{z^2 + \lambda} \le \frac{-2.3 + 3(\lambda + 3)}{(1 + \lambda)^2} = \frac{3}{1 + \lambda}.$$

$$Equality \ holds \ iff \ x = y = z = 1.$$

1119. Prove that if $x, y, z > 0$ then:

$$\frac{x^2 + y^2 + z^2}{xy + yz + zx} + \frac{27x}{4(7x + y + z)} + \frac{27y}{4(x + 7y + z)} + \frac{27z}{4(x + y + 7z)} \ge \frac{13}{4}$$

Proposed by Titu Zvonaru, Neculai Stanciu – Romania

Solution 1 by Vivek Kumar-India

$$LHS = \frac{x^2 + y^2 + z^2}{xy + yz + zx} + \frac{27}{4}\left(\frac{x}{7x + y + z} + \frac{y}{x + 7y + z} + \frac{z}{x + y + 7z}\right)$$

$$=\frac{x^2+y^2+z^2}{xy+yz+zx}+\frac{27}{4}\left(\frac{x^2}{7x^2+xy+zx}+\frac{y^2}{xy+7y^2+yz}+\frac{z^2}{zx+yz+7z^2}\right)$$

$$\overset{CBS}{\geq}\frac{x^2+y^2+z^2}{xy+yz+zx}+\frac{27}{4}\frac{(x+y+z)^2}{7(x^2+y^2+z^2)+2(xy+yz+zx)}$$

$$=\frac{x^2+y^2+z^2}{xy+yz+zx}+\frac{27(x^2+y^2+z^2+2xy+2yz+2zx)}{4\big(7(x^2+y^2+z^2)+2(xy+yz+zx)\big)}$$

Let $x^2+y^2+z^2=a, xy+yz+zx=b$ then $a\geq b$ and it remains to prove that

$$\frac{a}{b}+\frac{27(a+2b)}{4(7a+2b)}\geq\frac{13}{4}$$

$$\Leftrightarrow 4a(7a+2b)+27b(a+2b)\geq 13b(7a+2b)$$

$$\Leftrightarrow 28a^2+8ab+27ab+54b^2\geq 91ab+26b^2\Leftrightarrow 28a^2-56ab+27b^2\geq 0$$

$$\Leftrightarrow 28(a-b)^2\geq 0 \text{ which is true}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:$$

$$\frac{27x}{4(7x+y+z)}+\frac{3x(7x+y+z)}{4(x+y+z)^2}\overset{AM-GM}{\geq}\frac{9x}{2(x+y+z)}.$$

$$Then:\ \frac{27x}{4(7x+y+z)}\geq\frac{9x}{2(x+y+z)}-\frac{3x(7x+y+z)}{4(x+y+z)^2}\quad(and\ analogs)$$

$$Thus,\ \ LHS\geq\frac{x^2+y^2+z^2}{xy+yz+zx}+\frac{9(x+y+z)}{2(x+y+z)}-\frac{21(x^2+y^2+z^2)+6(xy+yz+zx)}{4(x+y+z)^2}=$$

$$=\frac{x^2+y^2+z^2}{xy+yz+zx}+\frac{9}{2}-\left(\frac{3}{4}+\frac{9(x^2+y^2+z^2)}{2(x+y+z)^2}\right)$$

$$=\frac{13}{4}+\left(\frac{1}{2}+\frac{x^2+y^2+z^2}{xy+yz+zx}-\frac{9(x^2+y^2+z^2)}{2(x+y+z)^2}\right)=$$

$$=\frac{13}{4}+\frac{x^2+y^2+z^2}{2}.\left(\frac{1}{x^2+y^2+z^2}+\frac{2}{xy+yz+zx}-\frac{9}{(x+y+z)^2}\right)\geq$$

$$\overset{CBS}{\geq}\frac{13}{4}+\frac{x^2+y^2+z^2}{2}\left(\frac{(1+2)^2}{(x^2+y^2+z^2)+2(xy+yz+zx)}-\frac{9}{(x+y+z)^2}\right)=\frac{13}{4}.$$

$$Equality\ holds\ iff\ x=y=z.$$

Solution 3 by Sakthi Vel-India

$$x^2+y^2+z^2\ge xy+yz+zx$$

$$\therefore \frac{x^2+y^2+z^2}{xy+yz+zx}+\frac{27}{4}\sum\frac{x}{7x+y+z}\ge\frac{xy+yz+zx}{xy+yz+zx}+\frac{27}{4}\sum\frac{x}{7x+y+z}$$

Assume: $z>y>x,\therefore 7z+y+x>7y+x+z>7x+y+z$

$$\ge 1+\frac{27}{4}\sum\frac{x}{7x+y+z}\quad \text{[using Chebyshev inequality]}$$

$$\ge 1+\frac{27}{4}\left[\sum x\sum\frac{1}{7x+y+z}\right]\cdot\frac{1}{3}$$

$$\ge 1+\frac{27}{4}\cdot\frac{1}{3}(x+y+z)\left(\frac{1}{7x+y+z}+\frac{1}{x+7y+z}+\frac{1}{x+y+7z}\right)$$

$$\ge 1+\frac{9}{4}(x+y+z)\frac{(1+1+1)^2}{9x+9y+9z}\ge 1+\frac{9}{4}(x+y+z)\cdot\frac{9}{9(x+y+z)}\ge\frac{13}{4}$$

$$\text{Equality holds } iff\ x=y=z.$$

1120. Let $a,b,c,d>0$ such that $abcd=1$. Prove that:

$$\sqrt[3]{\frac{a}{b+2022}}+\sqrt[3]{\frac{b}{c+2022}}+\sqrt[3]{\frac{c}{d+2022}}+\sqrt[3]{\frac{d}{a+2022}}\ge\frac{4}{\sqrt[3]{2023}}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Vivek Kumar-India

We have

$$\left(\sum\sqrt[3]{\frac{a}{b+2022}}\right)^3\left(\sum(b+2022)\right)\overset{HOLDER}{\ge}\left(a^{\frac{1}{4}}+b^{\frac{1}{4}}+c^{\frac{1}{4}}+d^{\frac{1}{4}}\right)^4$$

$$\Rightarrow\left(\sum\sqrt[3]{\frac{a}{b+2022}}\right)^3\ge\frac{\left(a^{\frac{1}{4}}+b^{\frac{1}{4}}+c^{\frac{1}{4}}+d^{\frac{1}{4}}\right)^4}{a+b+c+d+4\times 2022}$$

$$\Rightarrow\sum\sqrt[3]{\frac{a}{b+2022}}\ge\sqrt[3]{\frac{\left(a^{\frac{1}{4}}+b^{\frac{1}{4}}+c^{\frac{1}{4}}+d^{\frac{1}{4}}\right)^4}{a+b+c+d+4\times 2022}}$$

Hence, it is enough to show that

$$\sqrt[3]{\frac{\left(a^{\frac{1}{4}}+b^{\frac{1}{4}}+c^{\frac{1}{4}}+d^{\frac{1}{4}}\right)^4}{a+b+c+d+4\times 2022}}\geq\frac{4}{\sqrt[3]{2023}}\Rightarrow\frac{\left(a^{\frac{1}{4}}+b^{\frac{1}{4}}+c^{\frac{1}{4}}+d^{\frac{1}{4}}\right)^4}{a+b+c+d+4\times 2022}\geq\frac{64}{2023}$$

Now, we have

$$a+b+c+d=\left(a^{\frac{1}{2}}+b^{\frac{1}{2}}\right)^2-2a^{\frac{1}{2}}b^{\frac{1}{2}}+\left(c^{\frac{1}{2}}+d^{\frac{1}{2}}\right)^2-2c^{\frac{1}{2}}d^{\frac{1}{2}}$$

$$=\left(a^{\frac{1}{2}}+b^{\frac{1}{2}}\right)^2+\left(c^{\frac{1}{2}}+d^{\frac{1}{2}}\right)^2-2\left(a^{\frac{1}{2}}b^{\frac{1}{2}}+c^{\frac{1}{2}}d^{\frac{1}{2}}\right)$$

$$\overset{AM-GM}{\leq}\left(a^{\frac{1}{2}}+b^{\frac{1}{2}}\right)^2+\left(c^{\frac{1}{2}}+d^{\frac{1}{2}}\right)^2-4(abcd)^{\frac{1}{4}}=\left(a^{\frac{1}{2}}+b^{\frac{1}{2}}\right)^2+\left(c^{\frac{1}{2}}+d^{\frac{1}{2}}\right)^2-4$$

$$=\left(a^{\frac{1}{2}}+b^{\frac{1}{2}}+c^{\frac{1}{2}}+d^{\frac{1}{2}}\right)^2-2\left(a^{\frac{1}{2}}+b^{\frac{1}{2}}\right)\left(c^{\frac{1}{2}}+d^{\frac{1}{2}}\right)-4$$

$$\overset{AM-GM}{\leq}\left(a^{\frac{1}{2}}+b^{\frac{1}{2}}+c^{\frac{1}{2}}+d^{\frac{1}{2}}\right)^2-8(abcd)^{\frac{1}{4}}-4=\left(a^{\frac{1}{2}}+b^{\frac{1}{2}}+c^{\frac{1}{2}}+d^{\frac{1}{2}}\right)^2-12$$

$$=\left(\left(a^{\frac{1}{4}}+b^{\frac{1}{4}}\right)^2-2^{\frac{1}{4}}b^{\frac{1}{4}}+\left(c^{\frac{1}{4}}+d^{\frac{1}{4}}\right)^2-2c^{\frac{1}{4}}d^{\frac{1}{4}}\right)-12$$

$$=\left(\left(a^{\frac{1}{4}}+b^{\frac{1}{4}}\right)^2+\left(c^{\frac{1}{4}}+d^{\frac{1}{4}}\right)^2-2\left(a^{\frac{1}{4}}b^{\frac{1}{4}}+c^{\frac{1}{4}}d^{\frac{1}{4}}\right)\right)^2-12$$

$$\overset{AM-GM}{\leq}\left(\left(a^{\frac{1}{4}}+b^{\frac{1}{4}}\right)^2+\left(c^{\frac{1}{4}}+d^{\frac{1}{4}}\right)^2-4(abcd)^{\frac{1}{8}}\right)^2-12$$

$$=\left(\left(a^{\frac{1}{4}}+b^{\frac{1}{4}}\right)^2+\left(c^{\frac{1}{4}}+d^{\frac{1}{4}}\right)^2-4\right)^2-12$$

$$=\left(\left(a^{\frac{1}{4}}+b^{\frac{1}{4}}+c^{\frac{1}{4}}+d^{\frac{1}{4}}\right)^2-2\left(a^{\frac{1}{4}}+b^{\frac{1}{4}}\right)\left(c^{\frac{1}{4}}+d^{\frac{1}{4}}\right)-4\right)^2-12$$

$$\overset{AM-GM}{\leq}\left(\left(a^{\frac{1}{4}}+b^{\frac{1}{4}}+c^{\frac{1}{4}}+d^{\frac{1}{4}}\right)^2-8(abcd)^{\frac{1}{8}}-4\right)^2-12$$

$$=\left(\left(a^{\frac{1}{4}}+b^{\frac{1}{4}}+c^{\frac{1}{4}}+d^{\frac{1}{4}}\right)^2-12\right)^2-12$$

$$=\left(a^{\frac{1}{4}}+b^{\frac{1}{4}}+c^{\frac{1}{4}}+d^{\frac{1}{4}}\right)^4-24\left(a^{\frac{1}{4}}+b^{\frac{1}{4}}+c^{\frac{1}{4}}+d^{\frac{1}{4}}\right)^2+144-12$$

$$\overset{AM-GM}{\leq}\left(a^{\frac{1}{4}}+b^{\frac{1}{4}}+c^{\frac{1}{4}}+d^{\frac{1}{4}}\right)^4-24\times 16(abcd)^{\frac{1}{8}}+132$$

$$=\left(a^{\frac{1}{4}}+b^{\frac{1}{4}}+c^{\frac{1}{4}}+d^{\frac{1}{4}}\right)^4-24\times 16+132=\left(a^{\frac{1}{4}}+b^{\frac{1}{4}}+c^{\frac{1}{4}}+d^{\frac{1}{4}}\right)^4-252$$

$$\text{So, } a+b+c+d\leq\left(a^{\frac{1}{4}}+b^{\frac{1}{4}}+c^{\frac{1}{4}}+d^{\frac{1}{4}}\right)^4-252$$

$$\frac{\left(a^{\frac{1}{4}}+b^{\frac{1}{4}}+c^{\frac{1}{4}}+d^{\frac{1}{4}}\right)^4}{a+b+c+d+4\times 2022}\geq\frac{\left(a^{\frac{1}{4}}+b^{\frac{1}{4}}+c^{\frac{1}{4}}+d^{\frac{1}{4}}\right)^4}{\left(a^{\frac{1}{4}}+b^{\frac{1}{4}}+c^{\frac{1}{4}}+d^{\frac{1}{4}}\right)^4-252+4\times 2022}$$

$$\Rightarrow\frac{\left(a^{\frac{1}{4}}+b^{\frac{1}{4}}+c^{\frac{1}{4}}+d^{\frac{1}{4}}\right)^4}{a+b+c+d+4\times 2022}\geq\frac{\left(a^{\frac{1}{4}}+b^{\frac{1}{4}}+c^{\frac{1}{4}}+d^{\frac{1}{4}}\right)^4}{\left(a^{\frac{1}{4}}+b^{\frac{1}{4}}+c^{\frac{1}{4}}+d^{\frac{1}{4}}\right)^4+4\times 1959}$$

$$=\frac{1}{1+\frac{4\times 1959}{\left(a^{\frac{1}{4}}+b^{\frac{1}{4}}+c^{\frac{1}{4}}+d^{\frac{1}{4}}\right)^4}}\geq\frac{1}{1+\frac{4\times 1959}{\left(4(abcd)^{\frac{1}{16}}\right)^4}}=\frac{1}{1+\frac{4\times 1959}{256}}$$

$$=\frac{64}{64+1959}=\frac{64}{2023}$$

1121. **If $a,b\geq 0$ and $\lambda>0$ then**

$$(a+b+\lambda)^3>\frac{27\lambda}{8}(a^2+ab+b^2)$$

Proposed by Marin Chirciu-Romania

Solution 1 by Marian Dincă-Romania

$$a^2+ab+b^2\leq(a+b)^2$$

$$(a^2+ab+b^2)\frac{27\lambda}{8}\leq(a+b)^2\cdot\frac{27\lambda}{8}=\left(\frac{a+b}{2}\right)^2\cdot\lambda\cdot\frac{27}{2}\leq$$

$$\leq\left(\frac{\frac{a+b}{2}+\frac{a+b}{2}+\lambda}{3}\right)^3\cdot\frac{27}{2}=\frac{(a+b+\lambda)^3}{2}<(a+b+\lambda)^3$$

Solution 2 by Vivek Kumar-India

$$(a+b+\lambda)^3-\frac{27\lambda}{8}(a^2+ab+b^2)$$

$$=(a+b+\lambda)^2(a+b+\lambda)-\frac{27\lambda}{8}((a+b)^2-ab)$$

$$\overset{AM-GM}{\geq}4\lambda(a+b)(a+b+\lambda)-\frac{27\lambda}{8}(a+b)^2+\frac{27\lambda}{8}ab$$

$$=4\lambda(a+b)^2+4\lambda^2(a+b)-\frac{27\lambda}{8}(a+b)^2+\frac{27\lambda}{8}ab$$

$$=\left(4\lambda-\frac{27\lambda}{8}\right)(a+b)^2+4\lambda^2(a+b)+\frac{27\lambda}{8}ab$$

$$=\frac{5\lambda}{8}(a+b)^2+4\lambda^2(a+b)+\frac{27\lambda}{8}ab>0$$

Solution 3 by Tapas Das-India

$$8(a+b+\lambda)^3=(2a+2b+2\lambda)^3=[(a+b)+(a+b)+2\lambda]^3$$

$$\geq(3)^3(a+b)^2\cdot 2\lambda\quad \text{(AM-GM)}=54\lambda(a^2+2ab+b^2)>54\lambda(a^2+ab+b^2)$$

$$\therefore (a+b+\lambda)^3>\frac{54\lambda}{8}(a^2+ab+b^2)=\frac{27\lambda}{4}(a^2+ab+b^2)>\frac{27\lambda}{8}(a^2+ab+b^2)$$

1122. If $x_1,x_2,\dots,x_n>\lambda>0$ then:

$$\frac{x_1^4}{(x_2-\lambda)^2}+\frac{x_2^4}{(x_3-\lambda)^2}+\dots+\frac{x_n^4}{(x_1-\lambda)^2}\geq 16n\lambda^2$$

Proposed by Marin Chirciu-Romania

Solution 1 by Vivek Kumar-India

Let $x_1-\lambda=y_1,x_2-\lambda=y_2,\dots,x_n-\lambda=y_n$ such that $y_1,y_2,\dots,y_n>0$ as $x_1,x_2,\dots,x_n>\lambda>0$ then,

$$LHS=\frac{(y_1+\lambda)^4}{y_2^2}+\frac{(y_2+\lambda)^4}{y_3^2}+\dots+\frac{(y_{n-1}+\lambda)^4}{y_n^2}+\frac{(y_n+\lambda)^4}{y_1^2}$$

$$\overset{AM-GM}{\geq}\frac{(4y_1\lambda)^2}{y_2^2}+\frac{(4y_2\lambda)^2}{y_3^2}+\dots+\frac{(4y_{n-1}\lambda)^2}{y_n^2}+\frac{(4y_n\lambda)^2}{y_1^2}$$

$$=16\lambda^2\left(\frac{y_1^2}{y_2^2}+\frac{y_2^2}{y_3^2}+\dots+\frac{y_{n-1}^2}{y_n^2}+\frac{y_n^2}{y_1^2}\right)$$

$$\overset{AM-GM}{\geq}16\lambda^2\cdot n\left(\frac{y_1^2}{y_2^2}\cdot\frac{y_2^2}{y_3^2}\cdot\dots\cdot\frac{y_{n-1}^2}{y_n^2}\cdot\frac{y_n^2}{y_1^2}\right)^{\frac{1}{n}}=16n\lambda^2\cdot(1)^{\frac{1}{n}}=16n\lambda^2$$

Solution 2 by Tapas Das-India

$$x_1, x_2, \dots, x_n > \lambda$$

$$\dots x_1 = (x_1 - \lambda) + \lambda \ge 2\sqrt{(x_1 - \lambda) \cdot \lambda} \quad \text{(AM-GM)}$$

$$\dots x_1^4 \ge 16(x_1 - \lambda)^2 \cdot \lambda^2 \quad \text{(analog)}$$

$$\therefore \frac{x_1^4}{(x_2 - \lambda)^2} + \frac{x_2^4}{(x_3 - \lambda)^2} + \dots + \frac{x_n^4}{(x_1 - \lambda)^2}$$

$$\ge \frac{16(x_1 - \lambda)^2 \cdot \lambda^2}{(x_2 - \lambda)^2} + \frac{16(x_2 - \lambda)^2 \cdot \lambda^2}{(x_3 - \lambda)^2} + \dots + \frac{16(x_n - \lambda)^2 \cdot \lambda^2}{(x_1 - \lambda)^2}$$

$$\ge 16\lambda^2 \cdot n \left[\frac{(x_1 - \lambda)^2}{(x_2 - \lambda)^2} \cdot \frac{(x_2 - \lambda)^2}{(x_3 - \lambda)^2} \cdots \frac{(x_n - \lambda)^2}{(x_1 - \lambda)^2}\right]^{\frac{1}{n}}$$

$$(AM \ge GM)$$

$$= 16\lambda^2 \cdot n \cdot (1) = 16\lambda^2 n$$

1123. If $a, b, c > 0$ then:

$$(3ab)^c \cdot (3bc)^a \cdot (3ca)^b \le (a^2 + b^2 + c^2)^{a+b+c}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Michael Sterghiou-Greece

$$(3ab)^c \cdot (3bc)^a \cdot (3ca)^b \le (a^2 + b^2 + c^2)^{a+b+c} \quad (1)$$

$$c \cdot (3ab) + a \cdot (3bc) + b \cdot (3ca) \overset{Generalized}{\underset{AM-GM}{\ge}} (a + b + c) \cdot \sqrt[a+b+c]{LHS\ of\ (1)}$$

So it suffices that $\left(\frac{9abc}{a+b+c}\right)^{a+b+c} \le (a^2 + b^2 + c^2)^{a+b+c}$ or

$(a^2 + b^2 + c^2)(a + b + c) \ge 9r$ true as $a^2 + b^2 + c^2 \ge 3r^{\frac{2}{3}}$; $a + b + c \ge 3r^{\frac{1}{3}}$.

Equality for $a = b = c$.

Solution 2 by Ruxandra Daniela Tonilă-Romania

We have $a^2 + b^2 + c^2 \overset{Bergstrom}{\ge} \frac{(a+b+c)^2}{3} \overset{AM-GM}{\ge} 3\sqrt[3]{a^2b^2c^2}$

So it is enough to prove that:

$$\left(3\sqrt[3]{a^2b^2c^2}\right)^{a+b+c} \ge (3ab)^c \cdot (3bc)^2 \cdot (3ca)^b \Leftrightarrow (abc)^{\frac{2}{3}(a+b+c)} \ge (ab)^c \cdot (bc)^a \cdot (ca)^b$$

$$\Leftrightarrow \frac{(abc)^{a+b+c}}{(abc)^{\frac{a+b+c}{3}}} \geq \frac{(abc)^{a+b+c}}{a^a \cdot b^b \cdot c^c} \Leftrightarrow a^a \cdot b^b \cdot c^c \geq (abc)^{\frac{a+b+c}{3}}$$

Let $f: (0, +\infty) \to \mathbb{R}, f(x) = x \log x$ with $f'(x) = \log x + 1$,

$f''(x) > \frac{1}{x} > 0, \forall x > 0 \Rightarrow f$ convex function. Then

$$f\left(\frac{a+b+c}{3}\right) \overset{Jensen}{\leq} \frac{f(a) + f(b) + f(c)}{3}$$

$$\frac{a+b+c}{3} \log \frac{a+b+c}{3} \leq \frac{a \log a + b \log b + c \log c}{3}$$

$$3 \log \left(\frac{a+b+c}{3}\right)^{\frac{a+b+c}{3}} \leq \log\left(a^a \cdot b^b \cdot c^c\right)$$

$$3 \log\left(\sqrt[3]{abc}\right)^{\frac{a+b+c}{3}} \overset{AM-GM}{\leq} 3 \log \left(\frac{a+b+c}{3}\right)^{\frac{a+b+c}{3}}$$

$$\Rightarrow \log(abc)^{\frac{a+b+c}{3}} \leq \log\left(a^a \cdot b^b \cdot c^c\right) \Leftrightarrow a^a \cdot b^b \cdot c^c \geq (abc)^{\frac{a+b+c}{3}}$$

Therefore,

$$(3ab)^c \cdot (3bc)^a \cdot (3ca)^b \leq (a^2 + b^2 + c^2)^{a+b+c}, \forall a, b, c > 0$$

Solution 3 by Tapas Das-India

$$(3ab)^c \cdot (3bc)^a \cdot (3ca)^b = 3^{a+b+c} \cdot a^{b+c} \cdot b^{c+a} \cdot c^{a+b}$$

Now we consider 3 positive numbers a, b, c with associated weight $(b+c), (c+a), (a+b)$ respectively, then

$$\left[a^{b+c} \cdot b^{c+a} \cdot c^{a+b}\right] \leq \left[\frac{(b+c)a + (c+a)b + (a+b)c}{(b+c) + (c+a) + (a+b)}\right]^{2(a+b+c)}$$

$$= \left[\frac{(ab+bc+ca)}{a+b+c}\right]^{2(a+b+c)} = \left[\frac{(ab+bc+ca)^2}{(a+b+c)^2}\right]^{(a+b+c)} \leq \left[\frac{(ab+bc+ca)^2}{3(ab+bc+ca)}\right]^{(a+b+c)}$$

$$= \left[\frac{ab+bc+ca}{3}\right]^{a+b+c} \leq \frac{(a^2+b^2+c^2)^{a+b+c}}{3^{a+b+c}}$$

$$\Rightarrow 3^{a+b+c} \cdot a^{b+c} \cdot b^{c+a} \cdot c^{a+b} \leq (a^2+b^2+c^2)^{a+b+c}$$

Note: $ab + bc + ca \leq a^2 + b^2 + c^2; (a+b+c)^2 \geq 3(ab+bc+ca)$

Equality when $a = b = c$.

Solution 4 by Sanong Huayrerai-Nakon Pathom-Thailand

$$(3ab)^c(3bc)^a(3ca)^b = 3^{a+b+c}a^{b+c}\cdot b^{c+a}\cdot c^{a+b}$$

$$\le 3^{(a+b+c)}\left(\frac{a(b+c)+b(c+a)+c(a+b)}{2(a+b+c)}\right)^{2a+b+c}$$

$$= 3^{a+b+c}\left(\frac{ab+bc+ca}{a+b+c}\right)^{2(a+b+c)} \le (a^2+b^2+c^2)^{(a+b+c)}$$

Iff $3\frac{(ab+bc+ca)^2}{(a+b+c)^2} \le a^2+b^2+c^2$

Iff $3\frac{(ab+bc+ca)}{(a+b+c)^2} \le 1$

Iff $3(ab+bc+ca) \le (a+b+c)^2$ ok

Therefore it is true.

Solution 5 by Hikmat Mammadov-Azerbaijan

$$a;b;c>0$$

$$(3ab)^c\cdot(3bc)^a\cdot(3ca)^b \le (a^2+b^2+c^2)^{a+b+c}$$

$\log(\cdot)$ is concave

$$\Rightarrow \frac{c}{a+b+c}\log(3ab)+\frac{a}{a+b+c}\log(3bc)+\frac{b}{a+b+a}\log(3ca)$$

$$\le \log\left(\frac{(c)\cdot(3ab)+(a)\cdot(3bc)+(b)\cdot(3ca)}{a+b+c}\right) = \log\frac{9abc}{a+b+c}$$

$$\Rightarrow LHS \overset{\frac{1}{a+b+c}}{\le} \frac{9abc}{a+b+c} \overset{AM-GM}{\le} \frac{9}{a+b+c}\left(\frac{a+b+c}{3}\right)^3$$

$$=\frac{1}{3}(a+b+c)^2 \overset{Cauchy-Schwarz}{\le} \frac{1}{3}(1^2+1^2+1^2)(a^2+b^2+c^2) = a^2+b^2+c^2$$

$\Rightarrow (3ab)^c(3bc)^a(3ca)^b \le (a^2+b^2+c^2)^{a+b+c}$.With equality iff $a=b=c$.

1124. If $a,b,c>0, a+b+c=3$ then:

$$\sqrt{a+\sqrt{b+\sqrt{c}}}+\sqrt{b+\sqrt{c+\sqrt{a}}}+\sqrt{c+\sqrt{a+\sqrt{b}}} \le 3\sqrt{1+\sqrt{2}}$$

Proposed by Daniel Sitaru – Romania

âSolution 1 by Tapas Das-India

$$\sqrt{a+\sqrt{b+\sqrt{c}}}+\sqrt{b+\sqrt{c+\sqrt{a}}}+\sqrt{c+\sqrt{a+\sqrt{b}}}\le$$

$$\le\sqrt{3}\left[a+b+c+\left(\sqrt{b+\sqrt{c}}+\sqrt{c+\sqrt{a}}+\sqrt{a+\sqrt{b}}\right)\right]^{\frac{1}{2}}$$

$$=\sqrt{3}\left[3+\left(\sqrt{b+\sqrt{c}}+\sqrt{c+\sqrt{a}}+\sqrt{a+\sqrt{b}}\right)\right]^{\frac{1}{2}}$$

$$(\because a+b+c=3)\quad \text{(1)}$$

Now, $\sqrt{b+\sqrt{c}}+\sqrt{c+\sqrt{a}}+\sqrt{a+\sqrt{b}}\le$

$$\le\sqrt{3}\left[(a+b+c)+\left(\sqrt{c}+\sqrt{a}+\sqrt{b}\right)\right]^{\frac{1}{2}}=\sqrt{3}\left[3+\left(\sqrt{c}+\sqrt{a}+\sqrt{b}\right)\right]^{\frac{1}{2}}$$

$$\le\sqrt{3}\left[3+\sqrt{3}(a+b+c)^{\frac{1}{2}}\right]^{\frac{1}{2}}\le\sqrt{3}\left[3+\sqrt{3}\cdot\sqrt{3}\right]^{\frac{1}{2}}\le\sqrt{3}(6)^{\frac{1}{2}}=\sqrt{3}\cdot\sqrt{3}\cdot\sqrt{2}=3\sqrt{2}$$

From (1)

$$\sqrt{a+\sqrt{b+\sqrt{c}}}+\sqrt{b+\sqrt{c+\sqrt{a}}}+\sqrt{c+\sqrt{a+\sqrt{b}}}\le\sqrt{3}\left[3+3\sqrt{2}\right]^{\frac{1}{2}}$$

$$=\sqrt{3}\cdot\sqrt{3}\left(1+\sqrt{2}\right)^{\frac{1}{2}}=3\sqrt{1+\sqrt{2}}$$

Note:

$$\frac{x_1^m+x_2^m+\dots+x_n^m}{n}\le\left(\frac{x_1+x_2+\dots+x_n}{n}\right)^m$$

when m was between 0 and 1.

Solution 2 by Vivek Kumar-India

$$\sqrt{a+\sqrt{b+\sqrt{c}}}+\sqrt{b+\sqrt{c+\sqrt{a}}}+\sqrt{c+\sqrt{a+\sqrt{b}}}\le$$

$$\overset{AM-GM}{\le}\sqrt{3\left(a+\sqrt{b+\sqrt{c}}+b+\sqrt{c+\sqrt{a}}+c+\sqrt{a+\sqrt{b}}\right)}$$

$$=\sqrt{3(a+b+c)+3\left(\sqrt{b+\sqrt{c}}+\sqrt{c+\sqrt{a}}+\sqrt{a+\sqrt{b}}\right)}$$

$$=\sqrt{9+3\left(\sqrt{b+\sqrt{c}}+\sqrt{c+\sqrt{a}}+\sqrt{a+\sqrt{b}}\right)}$$

$$\overset{AM-GM}{\leq}\sqrt{9+3\sqrt{3(b+\sqrt{c}+c+\sqrt{a}+a+\sqrt{b})}}$$

$$=\sqrt{9+3\sqrt{3(a+b+c)+3(\sqrt{a}+\sqrt{b}+\sqrt{c})}}$$

$$\overset{AM-GM}{\leq}\sqrt{9+3\sqrt{9+3\sqrt{3(a+b+c)}}}=\sqrt{9+3\sqrt{9+9}}=3\sqrt{1+\sqrt{2}}$$

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

For $x>0$ we give $f(x)=\sqrt{x}\Rightarrow f'(x)=\frac{1}{2}x^{-\frac{1}{2}}\Rightarrow f''(x)=\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)x^{-\frac{3}{2}}=-\frac{1}{4}x^{\frac{3}{2}}$

$\Rightarrow f''(x)<0$ for $x>0\Rightarrow f(x)$ is concave function

$$\Rightarrow f(x)+f(y)\leq 2f\left(\frac{x+y}{2}\right)=2\sqrt{\left(\frac{x+y}{2}\right)}$$

Hence for $a,b,c>0$ and $a+b+c=3$, we will get

$$\sqrt{a+\sqrt{b+\sqrt{c}}}+\sqrt{b+\sqrt{c+\sqrt{a}}}+\sqrt{c+\sqrt{a+\sqrt{b}}}\leq$$

$$\leq 3\sqrt{\frac{a+b+c+\sqrt{b+\sqrt{c}}+\sqrt{c+\sqrt{a}}+\sqrt{a+\sqrt{b}}}{3}}$$

$$=3\sqrt{\frac{3+\sqrt{b+\sqrt{c}}+\sqrt{c+\sqrt{a}}+\sqrt{a+\sqrt{b}}}{3}}\leq 3\sqrt{1+\frac{3\sqrt{\frac{a+b+c+\sqrt{a}+\sqrt{b}+\sqrt{c}}{3}}}{3}}$$

$$= 3\sqrt{1+\frac{3\sqrt{2+\sqrt{a}+\sqrt{b}+\sqrt{c}}}{3}} \le 3\sqrt{1+\frac{3\sqrt{1+\frac{3\sqrt{\left(\frac{a+b+c}{3}\right)}}{3}}}{3}} =$$

$$= 3\sqrt{1+\frac{3\sqrt{1+\frac{3\sqrt{\frac{3}{3}}}{3}}}{3}} = 3\sqrt{1+\frac{3\sqrt{2}}{3}} = 3\sqrt{1+\sqrt{2}}$$

Therefore it is to be true.

1125. **If $a,b,c,\lambda > 0, a+b+c=3$ then:**

$$\frac{a^2}{a+\lambda b^2}+\frac{b^2}{b+\lambda c^2}+\frac{c^2}{c+\lambda a^2} \ge 3\left(1-\frac{1}{2}\sqrt{\lambda}\right)$$

Proposed by Marin Chirciu – Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have: \frac{a^2}{a+\lambda b^2} = a - \frac{\lambda ab^2}{a+\lambda b^2} \overset{AM-GM}{\ge} a - \frac{\lambda ab^2}{2\sqrt{a.\lambda b^2}} =$$

$$= a - \frac{\sqrt{\lambda}}{2}.\sqrt{ab}.\sqrt{b} \overset{AM-GM}{\ge} a - \frac{\sqrt{\lambda}}{2}.\frac{ab+b}{2}.$$

$$Similarly\ we\ have: \frac{b^2}{b+\lambda c^2} \ge b - \frac{\sqrt{\lambda}}{2}.\frac{bc+c}{2}\ \ and\ \ \frac{c^2}{c+\lambda a^2} \ge c - \frac{\sqrt{\lambda}}{2}.\frac{ca+a}{2}.$$

Adding these inequalities we get :

$$\frac{a^2}{a+\lambda b^2}+\frac{b^2}{b+\lambda c^2}+\frac{c^2}{c+\lambda a^2} \ge 3 - \frac{\sqrt{\lambda}}{2}.\frac{(ab+bc+ca)+3}{2} \ge$$

$$\ge 3 - \frac{\sqrt{\lambda}}{2}.\frac{\frac{(a+b+c)^2}{3}+3}{2} = 3\left(1-\frac{\sqrt{\lambda}}{2}\right).$$

Equality holds iff $a=b=c=\lambda=1$.

Solution 2 by Tapas Das-India

We need to show

$$\frac{a^2}{a+\lambda b^2}+\frac{b^2}{b+\lambda c^2}+\frac{c^2}{c+\lambda a^2}\ge 3\left(1-\frac{\sqrt{\lambda}}{2}\right)$$

$$\text{or } \frac{a^2}{a+\lambda b^2}+\frac{b^2}{b+\lambda c^2}+\frac{c^2}{c+\lambda a^2}\ge 3-\frac{3\sqrt{\lambda}}{2}$$

$$\text{or } \frac{a^2}{a+\lambda b^2}+\frac{b^2}{b+\lambda c^2}+\frac{c^2}{c+\lambda a^2}\ge (a+b+c)-\frac{3\sqrt{\lambda}}{2}$$

$$[\because a+b+c=3]$$

$$\text{or } \left(\frac{a^2}{a+\lambda b^2}-a\right)+\left(\frac{b^2}{b+\lambda c^2}-b\right)+\left(\frac{c^2}{c+\lambda a^2}-c\right)\ge\frac{-3\sqrt{\lambda}}{2}$$

$$\text{or } \frac{\lambda ab^2}{a+\lambda b^2}+\frac{\lambda bc^2}{b+\lambda c^2}+\frac{\lambda a^2c}{c+\lambda a^2}\le\frac{3\sqrt{\lambda}}{2}$$

Now,

$$\frac{\lambda ab^2}{a+\lambda b^2}+\frac{\lambda bc^2}{b+\lambda c^2}+\frac{\lambda a^2c}{c+\lambda a^2}\le\frac{\lambda ab^2}{2\sqrt{a}\cdot b\sqrt{\lambda}}+\frac{\lambda bc^2}{2\sqrt{b}\sqrt{\lambda}c}+\frac{\lambda a^2c}{2\sqrt{c}\sqrt{\lambda}a}$$

(AM-GM)

$$=\frac{\sqrt{\lambda}}{2}\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)\overset{AM-GM}{\le}\frac{\sqrt{\lambda}}{2}\left[\frac{ab+b}{2}+\frac{bc+c}{2}+\frac{ca+a}{2}\right]$$

$$\le\frac{\sqrt{\lambda}}{2}\left(\frac{a+b+c}{2}+\frac{ab+bc+ca}{2}\right)$$

$$\le\frac{\sqrt{\lambda}}{2}\left[\frac{3}{2}+\frac{(\sum a)^2}{6}\right]\left(\because\left(\sum a\right)^2\ge 3\sum ab\right)=\frac{\sqrt{\lambda}}{2}\left[\frac{3}{2}+\frac{3}{2}\right]=\frac{3\sqrt{\lambda}}{2}$$

Solution 3 by Sakthi Vel-India

$$\frac{a^2}{a+\lambda b^2}=\frac{a}{1+\frac{\lambda b^2}{a}}\ge a\left(1-\frac{\sqrt{\lambda}b}{2\sqrt{a}}\right)\ge a-\frac{\sqrt{\lambda}}{2}\sqrt{a}b$$

$$\ge a-\frac{\sqrt{\lambda}}{2}\left(\frac{a+b^2}{2}\right)$$

$$\therefore\frac{b^2}{b+\lambda c^2}\ge b-\frac{\sqrt{\lambda}}{2}\left(\frac{b+c^2}{2}\right),\frac{c^2}{c+\lambda a^2}\ge c-\frac{\sqrt{\lambda}}{2}\left(\frac{c+a^2}{2}\right)$$

$$\therefore\frac{a^2}{a+\lambda b^2}+\frac{b^2}{b+\lambda c^2}+\frac{c^2}{c+\lambda a^2}\ge a+b+c-\frac{\sqrt{\lambda}}{2}\left[\frac{a^2+b^2+c^2+a+b+c}{2}\right]$$

$$\geq 3-\frac{\sqrt{\lambda}}{2}\left[\frac{\frac{(a+b+c)^2}{3}+a+b+c}{2}\right]\geq 3-\frac{\sqrt{\lambda}}{2}\left[\frac{3+3}{2}\right]\geq 3-\frac{3\sqrt{\lambda}}{2}\geq 3\left(1-\frac{\sqrt{\lambda}}{2}\right)$$

Solution 4 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a^2}{a+\lambda b^2}=\sum_{cyc}\frac{a(a+\lambda b^2-\lambda b^2)}{a+\lambda b^2}=\sum_{cyc}a-\sum_{cyc}\frac{\lambda ab^2}{a+\lambda b^2}$$

$$\overset{a+b+c=3}{=}3-\sum_{cyc}\frac{\lambda ab^2}{a+\lambda b^2}\overset{A-G}{\geq}3-\sum_{cyc}\frac{\lambda ab^2}{2b.\sqrt{\lambda a}}=3-\frac{\sqrt{\lambda}}{2}\sum_{cyc}b\sqrt{a}$$

$$=3-\frac{\sqrt{\lambda}}{2}\sum_{cyc}\sqrt{ab}.\sqrt{b}\overset{CBS}{\geq}3-\frac{\sqrt{\lambda}}{2}.\sqrt{\sum_{cyc}ab}.\sqrt{\sum_{cyc}a}\overset{a+b+c=3}{=}3-\frac{\sqrt{\lambda}}{2}.\sqrt{3\sum_{cyc}ab}$$

$$\geq 3-\frac{\sqrt{\lambda}}{2}.\sqrt{\left(\sum_{cyc}a\right)^2}\overset{a+b+c=3}{=}3-\frac{3.\sqrt{\lambda}}{2}=3\left(1-\frac{1}{2}\sqrt{\lambda}\right)\text{ (QED)}$$

1126. **If $a,b,c,d>0, a+b+c+d=1$ then:**

$$\sum_{cyc}a^2cd\cot^{-1}(b)\geq 4abcd\cot^{-1}\left(\frac{1}{4}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Tapas Das-India

$$\sum a^2cd\cot^{-1}(b)=$$

$$=a^2cd\cot^{-1}(b)+b^2da\cot^{-1}(c)+c^2ab\cot^{-1}(d)+d^2bc\cot^{-1}(a)$$

$$=abcd\left[\frac{a}{b}\cot^{-1}(b)+\frac{b}{c}\cot^{-1}(c)+\frac{c}{d}\cot^{-1}(d)+\frac{d}{a}\cot^{-1}(a)\right]\quad (1)$$

Let $f(x)=\cot^{-1}x, x>0$

$\therefore f'(x)=-\frac{1}{1+x^2}<0 \therefore \cot^{-1}x$ is a decreasing function

Again,

Let $f(t)=\cot^{-1}(t), t>0, \therefore f'(t)=-\frac{1}{1+t^2}, f''(t)=\frac{2t}{(1+t^2)^2}>0$

$\therefore f$ is convex; using Jensen's inequality

$$\frac{a}{b}f(b)+\frac{b}{c}f(c)+\frac{c}{d}f(d)+\frac{d}{a}f(a)\geq$$

$$\geq\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\right)f\left(\frac{b\cdot\frac{a}{b}+c\cdot\frac{b}{c}+d\cdot\frac{c}{d}+a\cdot\frac{d}{a}}{\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}}\right)$$

$$=\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\right)f\left(\frac{a+b+c+d}{\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}}\right)$$

$$\overset{AM-GM}{\geq}4\left(\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{d}\cdot\frac{d}{a}\right)^{\frac{1}{4}}\cot^{-1}\left(\frac{1}{\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}}\right)$$

$$[\because a+b+c+d=1]$$

$$=4\cot^{-1}\left(\frac{1}{\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}}\right)\geq 4\cot^{-1}\left(\frac{1}{4}\right)$$

Since $\cot^{-1}x$ is decreasing function and $\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\geq 4$ (AM-GM)

$$\frac{a}{b}f(b)+\frac{b}{c}f(c)+\frac{c}{d}f(d)+\frac{d}{a}f(a)\geq 4\cot^{-1}\left(\frac{1}{4}\right)$$

$$\Rightarrow\frac{a}{b}\cot^{-1}(b)+\frac{b}{c}\cot^{-1}(c)+\frac{c}{d}\cot^{-1}(d)+\frac{d}{a}\cot^{-1}(a)\geq 4\cot^{-1}\left(\frac{1}{4}\right)\quad (2)$$

Now, from (1)

$$\sum a^2cd\cot^{-1}(b)=abcd\left[\frac{a}{b}\cot^{-1}(b)+\frac{b}{c}\cot^{-1}c+\frac{c}{d}\cot^{-1}d+\frac{d}{a}\cot^{-1}(a)\right]$$

$$\geq 4abcd\cot^{-1}\frac{1}{4}\quad \text{(using (2))}$$

Solution 2 by Hikmat Mammadov-Azerbaijan

$$a,b,c,d>0 \text{ and } a+b+c+d=1$$

$$\sum_{cyc}a^2cd\cot^{-1}(b)\geq 4abcd\cot^{-1}\left(\frac{1}{4}\right)$$

$$\frac{d}{dx}\left(\frac{1}{x}\cot^{-1}x\right)=-\frac{\cot^{-1}x}{x^2}-\frac{1}{x(x^2+1)}<0$$

For $x>0\Rightarrow\frac{d^2}{dx^2}\left(\frac{1}{x}\cot^{-1}x\right)=\frac{2\cot^{-1}x}{x^3}+\frac{2(2x^2+1)}{x^2(x^2+1)^2}>0$

Then $\frac{1}{x}\cot^{-1}(x)$ over $(0,1)$ is a decreasing, convex function

Convexity implies that for nonnegative

$\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and $\lambda_1+\lambda_2+\lambda_3+\lambda_4 = 1$ we've,

$$\lambda_1\frac{1}{a}\cot^{-1}a + \lambda_2\frac{1}{b}\cot^{-1}b + \lambda_3\frac{1}{c}\cot^{-1}c + \lambda_4\frac{1}{d}\cot^{-1}d \geq$$

$$\geq \frac{1}{\lambda_1 a + \lambda_2 b + \lambda_3 c + \lambda_4 d}\cot^{-1}(\lambda_1 a + \lambda_2 b + \lambda_3 c + \lambda_4 d)$$

If $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in (0,1)$ we've equality iff $a = b = c = d$.

Set $\lambda_1 = d, \lambda_2 = a, \lambda_3 = b, \lambda_4 = c$

$$\Rightarrow \frac{d}{a}\cot^{-1}a + \frac{1}{b}\cot^{-1}b + \frac{b}{c}\cot^{-1}c + \frac{c}{d}\cot^{-1}d$$

$$\geq \frac{1}{ad+ba+cb+dc}\cot^{-1}(ad+ba+cb+dc)$$

Because $\frac{1}{x}\cot^{-1}(x)$ is decreasing

Then $RHS \geq \frac{1}{\eta}\cot^{-1}\eta$ and $\eta = \max$

$ad+ba+cb+dc$ subject to $a,b,c,d > 0$ and $a+b+c+d = 1$

$$ad+ba+cb+dc = (a+c)(b+d) \overset{AM-GM}{\leq} \left(\frac{a+b+c+d}{2}\right)^2 = \frac{1}{4}$$

Thus $\frac{d}{a}\cot^{-1}a + \frac{a}{b}\cot^{-1}b + \frac{b}{c}\cot^{-1}c + \frac{c}{d}\cot^{-1}d$

$$\geq \frac{1}{\frac{1}{4}}\cot^{-1}\left(\frac{1}{4}\right) = 4\cot^{-1}\left(\frac{1}{4}\right)$$

So $\Downarrow$

$$4abcd\cot^{-1}\left(\frac{1}{4}\right) \leq bcd^2\cot^{-1}a + a^2cd\cot^{-1}b + ab^2c\cot^{-1}c + abc^2\cot^{-1}d$$

Equality iff $a = b = c = \frac{1}{4}$

1127. If $x, y, z > 0, x^2+y^2+z^2 \geq 3$, then :

$$x^3+y^3+z^3 \geq xy+yz+zx$$

Proposed by Marin Chirciu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$x^2+y^2+z^2\ge 3\Rightarrow\left(\sum_{cyc}x\right)^2-2\sum_{cyc}xy\ge 3\Rightarrow\left(\sum_{cyc}x\right)^2\ge 3+2\sum_{cyc}xy$$
$$\Rightarrow\sum_{cyc}x\overset{(\bullet)}{\ge}\sqrt{3+2\sum_{cyc}xy}$$

$\boxed{\text{Case 1}}$ $xy+yz+zx\le 3$

$$\text{Now, } x^3+y^3+z^3\overset{\text{Chebyshev}}{\ge}\frac{1}{3}\left(\sum_{cyc}x\right)\left(\sum_{cyc}x^2\right)\overset{x^2+y^2+z^2\ge 3}{\ge}\frac{1}{3}\left(\sum_{cyc}x\right).3$$
$$\overset{\text{via }(\bullet)}{\ge}\sqrt{3+2\sum_{cyc}xy}\overset{?}{\ge}\sum_{cyc}xy\Leftrightarrow t^2-2t-3\overset{?}{\le}0\quad\left(t=\sum_{cyc}xy\right)$$

$$\Leftrightarrow(t-3)(t+1)\overset{?}{\le}0\Leftrightarrow t\overset{?}{\le}3\Leftrightarrow xy+yz+zx\le 3\to\text{true}$$
$$\therefore x^3+y^3+z^3\ge xy+yz+zx$$

$\boxed{\text{Case 2}}$ $xy+yz+zx>3$

$$\text{Again, } x^3+y^3+z^3\overset{\text{Chebyshev}}{\ge}\frac{1}{3}\left(\sum_{cyc}x\right)\left(\sum_{cyc}x^2\right)\ge\frac{1}{3}\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)$$
$$\overset{\text{via }(\bullet)}{\ge}\frac{1}{3}.\sqrt{3+2\sum_{cyc}xy}.\left(\sum_{cyc}xy\right)\overset{xy+yz+zx>3}{\ge}\frac{1}{3}.\sqrt{3+6}.\left(\sum_{cyc}xy\right)=\sum_{cyc}xy$$
$$\therefore x^3+y^3+z^3\ge xy+yz+zx\ \therefore\text{ combining cases 1 and 2,}$$
$$x^3+y^3+z^3\ge xy+yz+zx\ \forall\, x,y,z>0\ \middle|\ x^2+y^2+z^2\ge 3\ (\text{QED})$$

Solution 2 by Michael Sterghiou-Greece

$$\sum_{cyc}x^3\ge\sum_{cyc}xy;\quad(1)$$

$$\text{Let: }(p,q,r)=\left(\sum_{cyc}x,\sum_{cyc}xy,\prod_{cyc}x\right)\text{, then }\sum_{cyc}x^2=p^2-2q\ge 3;\quad(2)$$

From Jensen's inequality for the function $f(t)=t^3$, we have:

$$\sum_{cyc} x^3 \ge \frac{p^3}{9} = 3\cdot\left(\frac{\sum x}{3}\right)^3 \overset{(?)}{\ge} q$$ or from (2) the stronger:

$$\left(\sqrt{2q+3}\right)^3 \ge 9q \Leftrightarrow (2q+3)^3 - 81q^2 \ge 0 \text{ or}$$

$(q-3)^2(8q+3) \ge 0$, which is true. Equality holds for $x = y = z = 1$.

1128. *Let $a,b,c \ge 0 : ab+bc+ca = 1$. Prove that :*

$$\sqrt{1-\frac{abc}{ab+1}} + \sqrt{1-\frac{abc}{bc+1}} + \sqrt{1-\frac{abc}{ca+1}} \le 3\sqrt{\frac{a+b+c}{2}}$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By CBS inequality we have :

$$LHS \le \sqrt{3\left(3 - abc\sum_{cyc}\frac{1}{ab+1}\right)} \le \sqrt{3\left(3 - abc.\frac{3^2}{ab+bc+ca+3}\right)} = 3\sqrt{1-\frac{3abc}{4}}.$$

$$\text{So it suffices to prove : } 1 - \frac{3abc}{4} \le \frac{a+b+c}{2} \text{ or } 2p + 3r \ge 4,$$

$$\text{where} : p = a+b+c,\ r = abc \text{ and let } q = ab+bc+ca = 1.$$

If $p \ge 2$ the inequality is obvious. Assume now that $p \le 2$.

$$\text{Then} : 2p + 3r \overset{Schur}{\ge} 2p + \frac{4p - p^3}{3} = 4 + \frac{(2-p)(p^2+2p-6)}{3} \ge 4.$$

Which is true because $2 \ge p \ge \sqrt{3q} = \sqrt{3}$. So the proof is completed.

Equality holds iff $(a,b,c) = (1,1,0)$ and permutation.

1129. *If $a,b,c > 0$ then :*

$$\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b} \ge \sqrt[4]{\frac{a^4+b^4}{2}} + \sqrt[4]{\frac{b^4+c^4}{2}} + \sqrt[4]{\frac{c^4+a^4}{2}}$$

Proposed by Tran Quoc Thinh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Lemma: If\ a,b>0\ then: \sqrt[4]{\frac{a^4+b^4}{2}} \le \sqrt{a^2-ab+b^2}\ (1)$$

$$Proof:\ We\ have\ (1) \Leftrightarrow 2(a^2-ab+b^2)^2-(a^4+b^4)\ge 0 \Leftrightarrow (a-b)^4\ge 0,$$

which is true and the proof of the lemma is complete.

From the lemma, it suffices to prove:

$$\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\ge \sqrt{a^2-ab+b^2}+\sqrt{b^2-bc+c^2}+\sqrt{c^2-ca+a^2}.$$

$$Let\ x=\frac{1}{a},\ y=\frac{1}{b},\ z=\frac{1}{c},\ the\ inequality\ becomes:$$

$$x^2+y^2+z^2\ge z\sqrt{x^2-xy+y^2}+x\sqrt{y^2-yz+z^2}+y\sqrt{z^2-zx+x^2}$$

By CBS inequality we have:

$$\sum_{cyc} x\sqrt{y^2-yz+z^2}\le \sqrt{\left(\sum x\right)\left(\sum x(y^2-yz+z^2)\right)} \overset{?}{\le} x^2+y^2+z^2$$

$$\Leftrightarrow\ xy(x^2+y^2)+yz(y^2+z^2)+zx(z^2+x^2)\le x^4+y^4+z^4+xyz(x+y+z),$$

which is the fourth degree Schur's inequality.

So the proof is completed. Equality holds iff $a=b=c$.

1130. Let $a,b,c\ge 0: ab+bc+ca=1$. Set $p=a+b+c$. Prove that:

$$\sqrt{(2a+bc)(2b+ca)(2c+ab)}\le p\sqrt{\frac{2p+1}{p^2+1}}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM inequality we have:

$$2b+ca\le\left(\frac{b}{c+a}+b(c+a)\right)+ca=\frac{b}{c+a}+1=\frac{p}{c+a}.$$

$$Similarly\ we\ have:\ \ 2c+ab\le\frac{p}{a+b}.$$

$$Then:\ \sqrt{(2b+ca)(2c+ab)}\le\frac{p}{\sqrt{(c+a)(a+b)}}=\frac{p}{\sqrt{a^2+1}}.$$

$$So\ it\ suffices\ to\ prove:\ \frac{2a+bc}{a^2+1}\le\frac{2p+1}{p^2+1}\ \ or\ \ \frac{2a+1-a(p-a)}{a^2+1}\le\frac{2p+1}{p^2+1}$$

$$or\ \ (2p+1)(a^2+1)-(p^2+1)(a^2+2a+1-ap)\ge 0$$

$$or\ \ (p-2)(p-a)(ap-1)\ge 0\ \ or\ \ (p-2)(b+c)(a^2-bc)\ge 0\ (1)$$

$$WLOG\ we\ can\ assume\ that\ a=min\{a,b,c\}\ or\ a=max\{a,b,c\},so:$$

$$If\ p\ge 2\ we\ can\ choose\ a=max\{a,b,c\}\ then\ (1)\ is\ true,$$

$$similarly\ if\ p\le 2\ we\ can\ choose\ a=min\{a,b,c\}.\ So\ the\ proof\ is\ completed.$$

$$Equality\ holds\ iff\ (a,b,c)=(1,1,0)\ and\ permutation.$$

1131. Let $a,b,c\ge 0: ab+bc+ca=1.$ Prove that:

$$\sqrt{\frac{a}{b-bc+c}}+\sqrt{\frac{b}{c-ca+a}}+\sqrt{\frac{c}{a-ab+b}}\ge\frac{2}{\sqrt{a+b+c-1}}$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{We have}:$$

$$\sqrt{\frac{a}{b-bc+c}}=\sqrt{\frac{a}{b+c-1+a(b+c)}}=\sqrt{\frac{a}{(b+c)(1+a+b+c)-[(b+c)^2+1]}}\ge$$

$$\overset{AM-GM}{\ge}\sqrt{\frac{a}{(b+c)(1+a+b+c)-2(b+c)}}=$$

$$=\frac{a}{\sqrt{a(b+c)(a+b+c-1)}}\overset{AM-GM}{\geq}\frac{2a}{(a+b+c)\sqrt{a+b+c-1}}.$$

Similarly we have :

$$\sqrt{\frac{b}{c-ca+a}}\geq\frac{2b}{(a+b+c)\sqrt{a+b+c-1}}\ \ and\ \ \sqrt{\frac{c}{a-ab+b}}$$
$$\geq\frac{2c}{(a+b+c)\sqrt{a+b+c-1}}.$$

Adding these inequalities yields the desired result.

Equality holds iff $(a,b,c)=(1,1,0)$ and their permutation.

Solution 2 by Michael Sterghiou-Greece

$$\sqrt{\frac{a}{b-bc+c}}+\sqrt{\frac{b}{c-ca+a}}+\sqrt{\frac{c}{a-ab+b}}\geq\frac{2}{\sqrt{a+b+c-1}};\ \ (1)$$

Let $(p,q,r)=(\sum a,\sum ab,\prod a),q=1\Rightarrow p^2\geq 3q\Rightarrow p\geq\sqrt{3},r\geq 0$

$$(1)\Rightarrow\sum_{cyc}\frac{a}{\sqrt{ab+ac-r}}\geq\frac{2}{\sqrt{p-1}}$$

which by Jensesn's inequality for funtion $t\to\frac{1}{\sqrt{t}};t>0$ with weights a,b,c becomes the stronger :

$$p\cdot\frac{1}{\sqrt{\frac{\sum ab+ac-r}{p}}}\overset{?}{\geq}\frac{q}{\sqrt{p-1}};\ \ (2)$$

But $\sum a(b+c)=pq-3r=p-3r$ and by squarin (2): $\frac{p^3}{p-3r-pr}\geq\frac{4}{p-1}$ or

$$f(r)=p^3(p-1)-4(p-3r-pr)\geq 0;\ \ (3)$$

$f(r)\searrow$. Assume $p\in\left[\sqrt{3},2\right]$ then by 3rd Schur's inquality

$$r\geq\frac{4pq-p^3}{9}=\frac{4p-p^3}{9}=r_0$$

and $f(r) \geq f(r_0) = \cdots = \frac{1}{9}\underbrace{(p-2)}_{\geq 0}(5p^2 - 11p - 6) \geq 0$ as

$$5p^2 - 11p - 6 = (5p-1)(p-2) - 8 < 0$$

Now, if $p \geq 2$ then $f(r) \geq f(0) = (p-2)(p^2 + p + 2) \geq 0$

Equality holds for $p = 2$ and $r = 0$ or $(a,b,c) = (1,1,0)$.

1132. *Prove that the following inequality :*

$$\frac{x^2 + 2x - 4}{x} \geq \sum_{cyc} \frac{a^2}{a + \sqrt{bc}} \geq \frac{y^2 + 2y - 7}{y - 3}$$

holds for all $a, b, c \geq 0,\ \ ab + bc + ca = 1,$

where $x = a + b + c$ and $y = \sqrt{ab} + \sqrt{bc} + \sqrt{ca}.$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Firstly, we will prove that $x + y \geq 3$:

$$We\ have:\ 2 \overset{AM-GM}{\leq} \frac{ab + bc + ca}{a + \sqrt{bc}} + \left(a + \sqrt{bc}\right) = \frac{a(a+b+c)}{a + \sqrt{bc}} + 2\sqrt{bc}.$$

$$Then:\ 1 \leq \frac{x}{2} \cdot \frac{a}{a + \sqrt{bc}} + \sqrt{bc}\ \ (and\ analogs)\ \ (1)$$

$$Thus,\ 3 \leq \frac{x}{2}\sum_{cyc} \frac{a}{a + \sqrt{bc}} + y = \frac{x}{2}\left(3 - \sum_{cyc} \frac{\sqrt{bc}}{a + \sqrt{bc}}\right) + y \overset{CBS}{\leq}$$

$$\leq \frac{x}{2}\left(3 - \frac{y^2}{y^2 - \sqrt{abc}\sum_{cyc}\sqrt{a}}\right) + y \leq \frac{x}{2}(3 - 1) + y = x + y.$$

$$Hence,\ \ x + y \geq 3\ \ (2)$$

Now we have :

$$\sum_{cyc} \frac{a^2}{a + \sqrt{bc}} \overset{(1)}{\geq} \sum_{cyc} a.\frac{2\left(1 - \sqrt{bc}\right)}{x} = 2 - \frac{2\sqrt{abc}\sum_{cyc}\sqrt{a}}{x} = 2 - \frac{y^2 - 1}{x} \geq$$

$$\overset{(2)\ and\ y\geq 1}{\leq} 2-\frac{y^2-1}{3-y}=\frac{y^2+2y-7}{y-3}.$$

Also we have :

$$\sum_{cyc}\frac{a^2}{a+\sqrt{bc}}=\sum_{cyc}\left(a-\frac{a\sqrt{bc}}{a+\sqrt{bc}}\right)\overset{(1)}{\leq} x-\sum_{cyc}\sqrt{bc}.\frac{2\left(1-\sqrt{bc}\right)}{x}=x-\frac{2y-2}{x}\leq$$

$$\overset{(2)}{\leq} x-\frac{2(3-x)-2}{x}=\frac{x^2+2x-4}{x}.$$

So the proof is done. Equality holds iff $(a,b,c)=(1,1,0)$ *and their permutation.*

1133. ***If*** $a,b,c>0, ab+bc+ca=1$ ***and*** $\lambda\geq\frac{1}{3}$ ***then :***

$$\sum_{cyc}\frac{1}{\lambda+a^2+b^2}\leq\frac{9}{3\lambda+2}$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let\ x:=a^2+b^2+c^2\geq ab+bc+ca=1.$$

$$We\ have:\ \sum_{cyc}\frac{1}{\lambda+a^2+b^2}=\sum_{cyc}\frac{1}{\lambda}\left(1-\frac{a^2+b^2}{\lambda+a^2+b^2}\right)=\frac{3}{\lambda}-\frac{1}{\lambda}\sum_{cyc}\frac{\sqrt{a^2+b^2}^2}{\lambda+a^2+b^2}\leq$$

$$\overset{CBS}{\leq}\frac{3}{\lambda}-\frac{1}{\lambda}.\frac{\left(\sum_{cyc}\sqrt{a^2+b^2}\right)^2}{3\lambda+2x}=\frac{3}{\lambda}-\frac{2x+2\sum_{cyc}\sqrt{(a^2+b^2)(a^2+c^2)}}{\lambda(3\lambda+2x)}\leq$$

$$\overset{CBS}{\leq}\frac{3}{\lambda}-\frac{2x+2\sum_{cyc}(a^2+bc)}{\lambda(3\lambda+2x)}=\frac{3}{\lambda}-\frac{4x+2}{\lambda(3\lambda+2x)}=\frac{1}{\lambda}+\frac{2(3\lambda-1)}{\lambda(3\lambda+2x)}\leq$$

$$\overset{x\geq 1\ \&\ 3\lambda\geq 1}{\leq}\frac{1}{\lambda}+\frac{2(3\lambda-1)}{\lambda(3\lambda+2)}=\frac{9}{3\lambda+2}.$$

$$Equality\ holds\ a=b=c=\frac{\sqrt{3}}{3}.$$

1134. If $a, b, c > 0$ and $n \in \mathbb{N}^*$ then:

$$\sum_{cyc} \frac{a^{2n}}{(b+c)^3} \geq \frac{3}{8}\left(\frac{1}{3} \cdot \sum_{cyc} a\right)^{2n-3}$$

Proposed by Marin Chirciu-Romania

Solution by Tran Quoc Anh-Vietnam

WLOG, we can suppose: $a \geq b \geq c > 0 \Rightarrow a^2 \geq b^2 \geq c^2$

$$\frac{a^{2n}}{(b+c)^3} \geq \frac{b^{2n}}{(c+a)^3} \geq \frac{c^{2n}}{(a+b)^3}$$

For $n = 1$:

$$\sum_{cyc} \frac{a^2}{(b+c)^3} \overset{Chebyshev}{\geq} \frac{a^2+b^2+c^2}{3} \cdot \left(\sum_{cyc} \frac{1}{(b+c)^3}\right) =$$

$$= \frac{a^2+b^2+c^2}{3} \cdot \left(\sum_{cyc} \frac{1^4}{(b+c)^3}\right) \overset{CBS;\,Radon}{\geq} \left(\frac{a+b+c}{3}\right)^2 \cdot \frac{81}{8(a+b+c)^3} =$$

$$= \frac{9}{8(a+b+c)} = \frac{3}{8}\left(\frac{1}{3}\sum_{cyc} a\right)^{-1} = \frac{3}{8}\left(\frac{1}{3} \cdot \sum_{cyc} a\right)^{2\cdot 1-3}$$

Thus inequality true with $n = 1$, suppose the inequality true to n, it mean:

$$\sum_{cyc} \frac{a^{2n}}{(b+c)^3} \geq \frac{3}{8}\left(\frac{1}{3} \cdot \sum_{cyc} a\right)^{2n-3}$$

We will prove that is true with $n+1$. In fact:

$$\sum_{cyc} \frac{a^{2(n+1)}}{(b+c)^3} = \sum_{cyc} a^2 \cdot \frac{a^{2n}}{(b+c)^3} \overset{Chebyshev}{\geq} \frac{a^2+b^2+c^2}{3} \cdot \left(\sum_{cyc} \frac{a^{2n}}{(b+c)^3}\right) \geq$$

$$\overset{CBS, math.ind.}{\geq} \left(\frac{a+b+c}{3}\right)^2 \cdot \left[\frac{3}{8}\left(\frac{1}{3} \cdot \sum_{cyc} a\right)^{2n-3}\right] = \frac{3}{8}\left(\frac{1}{3} \cdot \sum_{cyc} a\right)^{2n-1}$$

Therefore the inequality is true with $n+1$, thus is true with al positive integer.QED.

1135. ***Let*** $a,b,c\geq 0: ab+bc+ca=1.$ ***Prove that :***

$$\frac{a}{4a^2+a^2bc+bc}+\frac{b}{4b^2+b^2ca+ca}+\frac{c}{4c^2+c^2ab+ab}\geq\frac{1}{a+b+c}$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sqrt{4a^2+a^2bc+bc}=\sqrt{[4a^2+bc(a^2+1)][a(b+c)+bc]}\overset{CBS}{\geq}$$

$$\geq 2a\sqrt{a(b+c)}+bc\sqrt{a^2+1}\geq$$

$$\overset{GM-HM}{\geq}2a.\frac{2a.(b+c)}{a+(b+c)}+\frac{bc(a^2+1)}{\sqrt{a^2+1}}\overset{a(b+c)\leq 1}{\geq}\frac{4a^2(b+c)}{a+b+c}+\frac{bc(a^2+1)}{\sqrt{\frac{a}{b+c}+1}}\geq$$

$$\geq\frac{4a^2(b+c)}{a+b+c}+\frac{bc(a^2+1)}{\frac{a}{b+c}+1}=\frac{(4a^2+a^2bc+bc)(b+c)}{a+b+c}.$$

$$Then:\frac{1}{\sqrt{4a^2+a^2bc+bc}}\geq\frac{b+c}{a+b+c}\quad(and\ analogs)$$

$$Therefore,\ \sum_{cyc}\frac{a}{4a^2+a^2bc+bc}\geq\sum_{cyc}\frac{a(b+c)^2}{(a+b+c)^2}$$

$$=\frac{(a+b+c)(ab+bc+ca)+3abc}{(a+b+c)^2}\geq\overset{abc\geq 0}{\geq}\frac{ab+bc+ca}{a+b+c}$$

$$=\frac{1}{a+b+c},\quad as\ desired.$$

$$Equality\ holds\ iff\ (a,b,c)=(1,1,0)\ and\ permutation.$$

1136. **Let** $x_i>0$ **and** $\prod_{i=1}^{n}x_i=1, n\in\mathbb{N}, n\geq 1.$ **Prove that:**

$$\prod_{i=1}^{n}\left(x_i^2-x_i+1\right)+\sum_{i=1}^{n}x_i^{2022}\geq n+1$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Tapas Das-India

$$x_i^2 - x_i + 1 = x_i^2 + 1 - x_i \overset{AM-GM}{\geq} 2\sqrt{x_i^2 \cdot 1} - x_i = 2x_i - x_i = x_i$$

$$\prod_{i=1}^{n}(x_i^2 - x_i + 1) \geq x_1 \cdot x_2 \cdot \ldots \cdot x_n = 1$$

$$\sum_{i=1}^{n} x_i^{2022} \overset{AM-GM}{\geq} n\sqrt[n]{x_1^{2022} \cdot x_2^{2022} \cdot \ldots \cdot x_n^{2022}} = n\sqrt[n]{(x_1 \cdot x_2 \cdot \ldots \cdot x_n)^{2022}} = n$$

$$Therefore, \quad \prod_{i=1}^{n}(x_i^2 - x_i + 1) + \sum_{i=1}^{n} x_i^{2022} \geq n + 1$$

1137. *Prove that if $x, y, z > 0$ and $xyz = 1$ then:*

$$\frac{yz}{\sqrt{1 + y^2z^2}} + \frac{zx}{\sqrt{1 + z^2x^2}} + \frac{xy}{\sqrt{1 + x^2y^2}} \leq \frac{3}{\sqrt{2}}$$

Proposed by Nikos Ntorvas-Greece

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Since\ y^2z^2.z^2x^2.x^2y^2 = 1\ then\ \exists a, b, c > 0\ such\ that:$$

$$y^2z^2 = \frac{a}{b}, z^2x^2 = \frac{b}{c}, x^2y^2 = \frac{c}{a}.$$

$$The\ given\ inequality\ becomes: \sqrt{\frac{a}{a+b}} + \sqrt{\frac{b}{b+c}} + \sqrt{\frac{c}{c+a}} \leq \frac{3}{\sqrt{2}} \quad (1)$$

$$By\ CBS\ inequality\ we\ have:$$

$$LHS_{(1)} \leq \sqrt{\left(\sum(a+c)\right)\left(\sum\frac{a}{(a+b)(a+c)}\right)} = \sqrt{\frac{4(a+b+c)(ab+bc+ca)}{(a+b)(b+c)(c+a)}} =$$

$$= \sqrt{4 + \frac{4abc}{(a+b)(b+c)(c+a)}} \underbrace{\leq}_{Cesaro} \sqrt{4 + \frac{1}{2}} = \frac{3}{\sqrt{2}}.$$

$$So\ the\ proof\ is\ done.\ Equality\ holds\ iff\ x = y = z = 1.$$

1138. If $a, b, c > 0, abc = 1$ and $n \in \mathbb{N}^*$, then :

$$\sum_{cyc} \frac{(3a - 2b)^{2n}}{2a + 3b} \geq \frac{3}{5}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{(3a-2b)^{2n}}{2a+3b} = \sum_{cyc} \frac{(|3a-2b|^n)^2}{2a+3b} \overset{\text{Bergstrom}}{\geq} \frac{\left(\sum_{cyc}|3a-2b|^n\right)^2}{\sum_{cyc}(2a+3b)}$$

$$\overset{\text{Chebyshev }(n-1)\text{ times}}{\geq} \frac{\left(\frac{1}{3^{n-1}}\left(\sum_{cyc}|3a-2b|\right)^n\right)^2}{2\sum_{cyc} a + 3\sum_{cyc} a} \quad (\because n \in \mathbb{N}^*)$$

$$\geq \frac{\left(\frac{1}{3^{n-1}}\left|\sum_{cyc}(3a-2b)\right|^n\right)^2}{5\sum_{cyc} a}$$

$$\left(\because |x| + |y| + |z| \overset{\text{Triangle inequality}}{\geq} |x + y| + |z| \overset{\text{Triangle inequality}}{\geq} |x + y + z|\right)$$

$$= \frac{\left(\frac{1}{3^{n-1}}\left(\sum_{cyc} a\right)^n\right)^2}{5\sum_{cyc} a} = \frac{\left(\sum_{cyc} a\right).\frac{1}{3^{2n-2}}.\left(\sum_{cyc} a\right)^{2n-1}}{5\sum_{cyc} a} \overset{A-G}{\geq} \frac{\left(\sum_{cyc} a\right).\frac{1}{3^{2n-2}}\left(3.\sqrt[3]{abc}\right)^{2n-1}}{5\sum_{cyc} a}$$

$$= \frac{3}{5} \overset{abc=1}{=} \frac{\frac{3^{2n-1}}{3^{2n-2}}}{5} \therefore \sum_{cyc} \frac{(3a-2b)^{2n}}{2a+3b} \geq \frac{3}{5}\ \forall\, a, b, c > 0 \,\middle|\, abc = 1 \text{ and } n \in \mathbb{N}^*,$$

with equality iff $a = b = c = 1$ (QED)

1139. If $a, b, c > 0, a + b + c = 3$ and $\lambda \geq 0$ then:

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \lambda(a^2 + b^2 + c^2) \geq 3(\lambda + 1)$$

Proposed by Marin Chirciu-Romania

Solution 1 by Tapas Das-India

$$a + b + c = 3;\ \sqrt[3]{abc} \overset{AM-GM}{\leq} \frac{a + b + c}{3} = 1$$

$$a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca) \overset{(1)}{\geq}$$

$$\geq (a + b + c)^2 - \frac{2(a + b + c)^2}{3} = 3^2 - \frac{2}{3} \cdot 3^2 = 3;$$

$$(1) \Leftrightarrow ab+bc+ca \le \frac{1}{3}(a+b+c)^2$$

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \ge 3\cdot\frac{1}{\sqrt[3]{(abc)^2}} \overset{(AM-GM)}{\ge} 3; (\because abc \le 1)$$

Therefore,

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\lambda(a^2+b^2+c^2) \ge 3(\lambda+1)$$

Solution 2 by Ravi Prakash-New Delhi-India

$$\frac{a^2+b^2+c^2}{3} \ge \left(\frac{a+b+c}{3}\right)^2 \Rightarrow a^2+b^2+c^3 \ge 3; \quad (1)$$

$$\text{Also,} \quad \frac{1}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \ge \frac{3}{a+b+c} = 1$$

$$\Rightarrow \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \ge 3 \Rightarrow \frac{1}{3}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) \ge \left[\frac{1}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\right]^2 \ge 1; \quad (2)$$

From (1) and (2), we get:

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\lambda(a^2+b^2+c^2) \ge 3(\lambda+1)$$

1140. **If $a,b,c > \frac{1}{2\lambda}, a+b+c = 3$ and $\lambda > 0$, then :**

$$\sum_{cyc}\frac{1}{\lambda a^2+\lambda+1} \ge \frac{3}{2\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{\lambda a^2+\lambda+1} \ge \frac{1}{2\lambda+1}+\frac{2\lambda(1-a)}{(2\lambda+1)^2} \Leftrightarrow \frac{1}{\lambda a^2+\lambda+1}-\frac{1}{2\lambda+1}-\frac{2\lambda(1-a)}{(2\lambda+1)^2} \ge 0$$

$$\Leftrightarrow \frac{2\lambda+1-\lambda a^2-\lambda-1}{(2\lambda+1)(\lambda a^2+\lambda+1)}-\frac{2\lambda(1-a)}{(2\lambda+1)^2} \ge 0$$

$$\Leftrightarrow \frac{\lambda(1-a)(1+a)}{(2\lambda+1)(\lambda a^2+\lambda+1)}-\frac{2\lambda(1-a)}{(2\lambda+1)^2} \ge 0$$

$$\Leftrightarrow \frac{\lambda(1-a)}{2\lambda+1}\cdot\left(\frac{1+a}{\lambda a^2+\lambda+1}-\frac{2}{2\lambda+1}\right) \ge 0 \Leftrightarrow \frac{\lambda(1-a)}{2\lambda+1}\cdot\frac{2\lambda a(1-a)-(1-a)}{(2\lambda+1)(\lambda a^2+\lambda+1)} \ge 0$$

$$\Leftrightarrow \frac{\lambda(1-a)^2(2\lambda a-1)}{(\lambda a^2+\lambda+1)(2\lambda+1)^2} \ge 0 \to \text{true} \because a > \frac{1}{2\lambda} \Rightarrow 2\lambda a-1 > 0 \text{ and } \lambda > 0$$

$$\therefore \frac{1}{\lambda a^2+\lambda+1} \geq \frac{1}{2\lambda+1}+\frac{2\lambda(1-a)}{(2\lambda+1)^2} \text{ and analogs}$$

$$\overset{\text{via cyclic summation}}{\Rightarrow} \sum_{\text{cyc}} \frac{1}{\lambda a^2+\lambda+1} \geq \frac{3}{2\lambda+1}+\frac{2\lambda}{(2\lambda+1)^2}\cdot\sum_{\text{cyc}}(1-a)$$

$$=\frac{3}{2\lambda+1}+\frac{2\lambda}{(2\lambda+1)^2}\cdot\left(3-\sum_{\text{cyc}} a\right) \overset{a+b+c=3}{=} \frac{3}{2\lambda+1}$$

$$\therefore \sum_{\text{cyc}} \frac{1}{\lambda a^2+\lambda+1} \geq \frac{3}{2\lambda+1} \ \forall\, a,b,c>\frac{1}{2\lambda}, a+b+c=3 \text{ and } \lambda>0,$$

equality iff $a=b=c=1$ (QED)

1141. ***Prove that :***

$$\frac{1+a\sqrt{bc}}{a+\sqrt{bc}}+\frac{1+b\sqrt{ca}}{b+\sqrt{ca}}+\frac{1+c\sqrt{ab}}{c+\sqrt{ab}} \geq 1+\frac{4}{a+b+c},$$

holds for all $a,b,c\geq 0, ab+bc+ca=1$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By CBS inequality we have :

$$a+b+c=\sqrt{[(-a+b+c)^2+4a(b+c)][bc+a(b+c)]}$$
$$\geq (-a+b+c)\sqrt{bc}+2a(b+c).$$

$$\textit{Then}:\ 1 \geq \frac{(-a+b+c)\sqrt{bc}+2a(b+c)}{a+b+c}=\frac{(-a+b+c)\left(a+\sqrt{bc}\right)}{a+b+c}+a.$$

$$\textit{Hence},\ \frac{1+a\sqrt{bc}}{a+\sqrt{bc}} \geq \frac{\left(\frac{(-a+b+c)\left(a+\sqrt{bc}\right)}{a+b+c}+a\right)+a\sqrt{bc}}{a+\sqrt{bc}}$$
$$=\frac{-a+b+c}{a+b+c}+a.\frac{1+\sqrt{bc}}{a+\sqrt{bc}} \geq$$

$$\geq \frac{-a+b+c}{a+b+c}+a.\left(\frac{-a+b+c}{a+b+c}+1\right)=\frac{-a+b+c}{a+b+c}+\frac{2a(b+c)}{a+b+c}.$$

$$\textit{Therefore},\ \sum_{cyc}\frac{1+a\sqrt{bc}}{a+\sqrt{bc}} \geq \sum_{cyc}\left(\frac{-a+b+c}{a+b+c}+\frac{2a(b+c)}{a+b+c}\right)=1+\frac{4}{a+b+c}$$

$$Equality\ holds\ iff\ (a,b,c)=(1,1,0)\ and\ permutation.$$

1142. $Let\ a,b,c>0: ab+bc+ca+2abc=1.\ Prove\ that:$

$$\sqrt{\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}}+\sqrt{\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}\geq 2\sum_{cyc}\frac{\sqrt{a+1}}{\sqrt{abc}+\sqrt{b+c+abc}}$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$By\ the\ AM-GM\ inequality, we\ have:$$

$$2\sqrt{b+c+abc}\leq\frac{(b+c+abc)a}{(a+1)\sqrt{abc}}+\frac{(a+1)\sqrt{abc}}{a}=\frac{ab+bc+ca}{(a+1)\sqrt{abc}}+2\sqrt{abc}.$$

$$Then:$$

$$\frac{2\sqrt{a+1}}{\sqrt{abc}+\sqrt{b+c+abc}}=\frac{\left(2\sqrt{b+c+abc}-2\sqrt{abc}\right)\sqrt{a+1}}{b+c}\leq\frac{ab+bc+ca}{(b+c)\sqrt{(a+1)abc}}.$$

$$Thus,\ 2\sum_{cyc}\frac{\sqrt{a+1}}{\sqrt{abc}+\sqrt{b+c+abc}}\leq\sum_{cyc}\frac{ab+bc+ca}{(b+c)\sqrt{(a+1)abc}}$$

$$=\sum_{cyc}\left(\sqrt{\frac{a}{(a+1)bc}}+\frac{\sqrt{bc}}{(b+c)\sqrt{(a+1)a}}\right)\leq$$

$$\overset{AM-GM}{\leq}\sum_{cyc}\left(\sqrt{\frac{a}{(a+1)bc}}+\frac{1}{2\sqrt{(a+1)a}}\right)\overset{CBS}{\leq}\sqrt{\sum\frac{a}{a+1}\cdot\sum\frac{1}{bc}}+\frac{1}{2}\sqrt{\sum\frac{1}{a+1}\cdot\sum\frac{1}{a}}.$$

$$Since\ ab+bc+ca+2abc=1\ then\ we\ have:$$

$$\sum\frac{1}{a+1}=2\ \ and\ \ \sum\frac{a}{a+1}=1,$$

$$Therefore,\ 2\sum_{cyc}\frac{\sqrt{a+1}}{\sqrt{abc}+\sqrt{b+c+abc}}\leq\sqrt{\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}}+\sqrt{\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}.$$

$$Equality\ holds\ iff\ a=b=c=\frac{1}{2}.$$

1143. $If\ a,b,c>0, a+b+c=3\ and\ \lambda\geq 2\ then:$

$$\frac{(2\lambda+1)a^3-b^3}{ab+(2\lambda-1)a^2}+\frac{(2\lambda+1)b^3-c^3}{bc+(2\lambda-1)b^2}+\frac{(2\lambda+1)c^3-a^3}{ca+(2\lambda-1)c^2}\leq 3$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Lemma:\ If\ a,b>0\ then\ \frac{(2\lambda+1)a^3-b^3}{ab+(2\lambda-1)a^2}\leq a+\frac{2(a-b)}{\lambda}\ (1)$$

$$Proof:\ (1)\Leftrightarrow \lambda\left[(2\lambda+1)a^3-b^3\right]\leq\left[ab+(2\lambda-1)a^2\right][(\lambda+2)a-2b]$$

$$\Leftrightarrow 2(\lambda-1)a^3-(3\lambda-4)a^2b-2ab^2+\lambda b^3\geq 0\Leftrightarrow (a-b)^2[2(\lambda-1)a+\lambda b]\geq 0$$

$$Which\ is\ true\ and\ the\ proof\ of\ the\ lemma\ is\ done. Using\ the\ lemma, we\ have:$$

$$\sum_{cyc}\frac{(2\lambda+1)a^3-b^3}{ab+(2\lambda-1)a^2}\leq\sum_{cyc}\left(a+\frac{2(a-b)}{\lambda}\right)=a+b+c=3.$$

$$Equality\ holds\ iff\ a=b=c=1.$$

1144. For $all\ a,\mathrm{b},\mathrm{c}>0, such\ that\ \mathrm{abc}=1,$

$$15+12(a^4+\mathrm{b}^4+\mathrm{c}^4)(a^2+\mathrm{b}^2+\mathrm{c}^2)\geq 11(a^6+\mathrm{b}^6+\mathrm{c}^6)+30(a^3+\mathrm{b}^3+\mathrm{c}^3)$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$The\ given\ inequality\ is\ successively\ equivalent\ to:$$

$$15(abc)^2+12(a^4+b^4+c^4)(a^2+b^2+c^2)\geq 11(a^6+b^6+c^6)+30abc(a^3+b^3+c^3)$$

$$\Leftrightarrow\sum_{cyc}a^6+12\sum_{cyc}(a^4b^2+a^2b^4)+15(abc)^2\geq 30abc(a^3+b^3+c^3)$$

$$\Leftrightarrow\frac{1}{2}\sum_{cyc}(a^6+b^6-a^4b^2-a^2b^4)+15\sum_{cyc}(c^4a^2+c^4b^2-2abc.c^3)$$

$$-\frac{5}{2}\sum_{cyc}(a^4c^2+b^4c^2-2(abc)^2)\geq 0$$

$$\Leftrightarrow \frac{1}{2}\sum_{cyc}(a^2+b^2)(a^2-b^2)^2+15\sum_{cyc}c^4(a-b)^2-\frac{5}{2}\sum_{cyc}c^2(a^2-b^2)^2\geq 0$$

$$\Leftrightarrow \sum_{cyc}\left(\frac{(a^2+b^2)(a+b)^2}{2}+15c^4-\frac{5}{2}c^2(a+b)^2\right)(a-b)^2\geq 0,\ \textit{which is true because}:$$

$$\frac{(a^2+b^2)(a+b)^2}{2}+15c^4\overset{CBS}{\geq}\frac{(a+b)^4}{4}+15c^4\overset{AM-GM}{\geq}\sqrt{15}c^2(a+b)^2$$
$$\geq\frac{5}{2}c^2(a+b)^2\ (\textit{and analogs})$$

So the proof is done. Equality holds iff $a=b=c=1$.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\text{Assigning } b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0,$$
$$y+z-x=2a>0 \text{ and } z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y$$
$$\Rightarrow x,y,z \text{ form sides of a triangle with semiperimeter, circumradius and inradius}$$
$$=s,R,r\ (say) \text{ yielding } 2\sum_{cyc}a=\sum_{cyc}x=2s\Rightarrow\sum_{cyc}a=s$$
$$\Rightarrow a=s-x, b=s-y, c=s-z$$

$$\text{Via aforementioned substitutions, }\sum_{cyc}a^2=\left(\sum_{cyc}a\right)^2-2\sum_{cyc}ab$$
$$=s^2-2\sum_{cyc}(s-x)(s-y)=s^2-2(4Rr+r^2)\Rightarrow\sum_{cyc}a^2\overset{(i)}{=}s^2-8Rr-2r^2 \text{ and}$$
$$\sum_{cyc}a^3=\left(\sum_{cyc}a\right)^3-3(ab+bc)(bc+ca)(ca+ab)=s^3-3r^2s.4Rrs$$
$$\Rightarrow\sum_{cyc}a^3\overset{(ii)}{=}s(s^2-12Rr) \text{ and }\sum_{cyc}a^2b^2=\left(\sum_{cyc}ab\right)^2-2abc\left(\sum_{cyc}a\right)$$
$$=(4Rr+r^2)^2-2r^2s.s\Rightarrow\sum_{cyc}a^2b^2\overset{(iii)}{=}(4Rr+r^2)^2-2r^2s^2$$
$$\because abc=1\therefore 15+12(a^4+b^4+c^4)(a^2+b^2+c^2)$$
$$\geq 11(a^6+b^6+c^6)+30(a^3+b^3+c^3)$$
$$\Leftrightarrow 15a^2b^2c^2+12\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a^4\right)$$
$$\geq 11\left(3a^2b^2c^2+\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a^4-\sum_{cyc}a^2b^2\right)\right)+30abc\left(\sum_{cyc}a^3\right)$$

$$\Leftrightarrow\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a^4\right)+11\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a^2b^2\right)\geq 18a^2b^2c^2+30abc\left(\sum_{cyc}a^3\right)$$

$$\Leftrightarrow\left(\sum_{cyc}a^2\right)\left(\left(\sum_{cyc}a^4+2\sum_{cyc}a^2b^2\right)+9\sum_{cyc}a^2b^2\right)\geq 18a^2b^2c^2+30abc\left(\sum_{cyc}a^3\right)$$

$$\Leftrightarrow\left(\sum_{cyc}a^2\right)^3+9\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a^2b^2\right)\geq 18a^2b^2c^2+30abc\left(\sum_{cyc}a^3\right)$$

$$\overset{\text{via (i),(ii),(iii)}}{\Leftrightarrow}(s^2-8Rr-2r^2)^3+9(s^2-8Rr-2r^2)\left((4Rr+r^2)^2-2r^2s^2\right)$$

$$\geq 18r^4s^2+30r^2s^2(s^2-12Rr)$$

$$\Leftrightarrow s^6-(24Rr+54r^2)s^4+r^2s^2(336R^2+672Rr+39r^2)$$

$$-r^3(1664R^3+1248R^2r+312Rr^2+26r^3)\overset{(*)}{\geq}0\ \text{and}$$

$$\because (s^2-16Rr+5r^2)^3\overset{\text{Gerretsen}}{\geq}0\ \therefore\ \text{in order toprove }(*),$$

$$\text{it suffices to prove : LHS of }(*)\geq(s^2-16Rr+5r^2)^3$$

$$\Leftrightarrow(24Rr-69r^2)s^4-r^2s^2(432R^2-1152Rr+36r^2)$$

$$+r^3(2432R^3-5088R^2r+888Rr^2-151r^3)\overset{(**)}{\geq}0$$

$\boxed{\text{Case 1}}$ $24R-69r>0$ *and* $\because(24Rr-69r^2)(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\geq}0$

$$\therefore\ \text{in order toprove }(*),\text{it suffices to prove :}$$

$$\text{LHS of }(**)\geq(24Rr-69r^2)(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow(336R-624r)(R-2r)s^2-594r^2s^2$$

$$-r(3712R^3-16416R^2r+10752Rr^2-1574r^3)\overset{(***)}{\geq}0$$

$$\text{Now, LHS of }(***)\overset{\text{Gerretsen}}{\geq}(336R-624r)(R-2r)(16Rr-5r^2)$$

$$-594r^2(4R^2+4Rr+3r^2)-r(3712R^3-16416R^2r+10752Rr^2-1574r^3)\overset{?}{\geq}0$$

$$\Leftrightarrow 1664t^3-8376t^2+13320t-6448\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)\left(\left(\frac{208t}{3}-11\right)(24t-69)+2465\right)\overset{?}{\geq}0\rightarrow\text{true}\ \because t>\frac{69}{24}\Rightarrow(***)$$

$$\Rightarrow(**)\text{ is true (strict inequality)}$$

$\boxed{\text{Case 2}}$ $24R-69r\leq 0$ *and* then, $(**)$

$$\Leftrightarrow(69r^2-24Rr)s^4+r^2s^2(432R^2-1152Rr+36r^2)$$

$$-r^3(2432R^3-5088R^2r+888Rr^2-151r^3)\overset{(\bullet)}{\leq}0$$

$$\text{Now, Rouche}\Rightarrow s^2-(m-n)\geq 0\text{ and }s^2-(m+n)\leq 0,$$

$$\text{where }m=2R^2+10Rr-r^2\text{ and }n=2(R-2r).\sqrt{R^2-2Rr}$$

$$\therefore\left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\leq 0$$

$$\Rightarrow s^4-s^2(2m)+m^2-n^2\leq 0\Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\leq 0$$

$$\Rightarrow(69r^2-24Rr)s^4-s^2(4R^2+20Rr-2r^2)(69r^2-24Rr)$$

$$+r(69r^2-24Rr)(4R+r)^3 \le 0 \therefore \text{ in order toprove } (\bullet), \text{it suffices to prove :}$$
$$\text{LHS of } (\bullet) \le (69r^2-24Rr)s^4 - s^2(4R^2+20Rr-2r^2)(69r^2-24Rr)$$
$$+r(69r^2-24Rr)(4R+r)^3$$
$$\Leftrightarrow (96R^3-228R^2r-276Rr^2+102r^3)s^2$$
$$+r^2(2432R^3-5088R^2r+888Rr^2-151r^3)-r(24R-69r)(4R+r)^3 \overset{(\bullet\bullet)}{\ge} 0$$
$$\text{Let } f(t)=96t^3-228t^2-276t+102\ \forall\, t\in\left[2,\frac{69}{24}\right]\left(t=\frac{R}{r}\right);\text{then :}$$
$$f(t)=(24t-69)\left(\frac{16t^2+8t-23}{4}\right)-\frac{1179}{4}\le-\frac{1179}{4}<0$$
$$\left(\because 24t-69\le 0 \text{ and } 16t^2+8t-23 \overset{\text{Euler}}{\ge} 16(4)+8(2)-23>0\right.$$
$$\left.\Rightarrow (24t-69)\left(\frac{16t^2+8t-23}{4}\right)\le 0\right)$$
$$\therefore 96R^3-228R^2r-276Rr^2+102r^3<0 \text{ for } 24R-69r\le 0$$
$$\Rightarrow \text{LHS of } (\bullet\bullet) = -\left(-(96R^3-228R^2r-276Rr^2+102r^3)\right)s^2$$
$$+r^2(2432R^3-5088R^2r+888Rr^2-151r^3)-r(24R-69r)(4R+r)^3$$
$$\overset{\text{Rouche}}{\ge} -\left(-(96R^3-228R^2r-276Rr^2+102r^3)\right)\left(2R^2+10Rr-r^2+2(R-2r).\sqrt{R^2-2Rr}\right)$$
$$+r^2(2432R^3-5088R^2r+888Rr^2-151r^3)-r(24R-69r)(4R+r)^3 \overset{?}{\ge} 0$$
$$\Leftrightarrow (R-2r)(48R^4-162R^3r+368R^2r^2-362Rr^3+23r^4)$$
$$\ge (R-2r).\sqrt{R^2-2Rr}.\left(-(48R^3-114R^2r-138Rr^2+51r^3)\right)$$
$$\Leftrightarrow (R-2r)\left((R-2r)(15R^3+33R^2(R-2r)+236Rr^2+110r^3)+243r^4\right)$$
$$\ge (R-2r).\sqrt{R^2-2Rr}.\left(-(48R^3-114R^2r-138Rr^2+51r^3)\right)$$
$$\overset{\because R-2r\ge 0}{\Leftrightarrow} \left((R-2r)(15R^3+33R^2(R-2r)+236Rr^2+110r^3)+243r^4\right)^2$$
$$>(R^2-2Rr)(48R^3-114R^2r-138Rr^2+51r^3)^2 \Leftrightarrow$$
$$39936t^6-190848t^5+320224t^4-244976t^3+117219t^2-11450t+529>0$$
$$\Leftrightarrow (t-2)\left((t-2)(39936t^4-31104t^3+36064t^2+23696t+67747)+164754\right)$$
$$+59049>0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2 \Rightarrow (\bullet\bullet) \Rightarrow (\bullet) \Rightarrow (**) \text{ is true}$$
$$\therefore \text{combining cases 1 and 2}, (**) \text{ is true for all triangles}$$
$$\Rightarrow (*) \text{ is true for all triangles, equality iff concerned triangle is equilateral}$$
$$\Rightarrow 15+12(a^4+b^4+c^4)(a^2+b^2+c^2)\ge 11(a^6+b^6+c^6)+30(a^3+b^3+c^3)$$
$$\forall\, a,b,c>0 \mid abc=1, \text{equality iff } a=b=c=1 \text{ (QED)}$$

1145. If $a,b>0$ then:

$$16\left(\sqrt{ab}-\frac{a+b}{2}+\sqrt{\frac{a^2+b^2}{2}}\right)^4+(a+b)^4\le 16a^2b^2+4(a^2+b^2)^2$$

Proposed by Daniel Sitaru-Romania

Solution by Tapas Das-India

$$\sqrt{ab}+\sqrt{\frac{a^2+b^2}{2}}\overset{CBS}{\leq}\sqrt{2\left(ab+\frac{a^2+b^2}{2}\right)}=\sqrt{\frac{2(a^2+b^2+2ab)}{2}}=$$

$$=\sqrt{(a+b)^2}=a+b;\quad (1)$$

$$16\left(\sqrt{ab}-\frac{a+b}{2}+\sqrt{\frac{a^2+b^2}{2}}\right)^4+(a+b)^4\leq 16\left(a+b-\frac{a+b}{2}\right)^4+(a+b)^4\overset{(1)}{=}$$

$$=16\cdot\frac{(a+b)^4}{16}+(a+b)^4=2(a+b)^4.$$

We need to prove:

$$2(a+b)^4\leq 16a^2b^2+4(a^2+b^2)^2\Leftrightarrow (a+b)^4+8a^2b^2+2(a^2+b^2)^2\Leftrightarrow$$

$$a^4+4a^3b+6a^2b^2+4ab^3+b^4\leq 8a^2b^2+2(a^4+b^4)+4a^2b^2\Leftrightarrow$$

$$a^4+b^4-4a^3b+6a^2b^2-4ab^3\geq 0\Leftrightarrow (a-b)^4\geq 0 \text{ true}.$$

1146. *Let $a,b,c\geq 0:(a+b)(b+c)(c+a)=8.$ Prove that :*

$$2\sqrt{2}\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)\geq ab\sqrt{a+b}+bc\sqrt{b+c}+ca\sqrt{c+a}+3\sqrt{2}abc$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:$$

$$2\sqrt{2bc}-bc\sqrt{b+c}=\sqrt{bc(b+c)}\left(\sqrt{(a+b)(c+a)}-\sqrt{bc}\right)=$$

$$=\frac{\sqrt{bc(b+c)}.a(a+b+c)}{\sqrt{(a+b)(c+a)}+\sqrt{bc}}\overset{CBS}{\geq}\frac{a(a+b+c)\sqrt{bc(b+c)}}{\sqrt{[(a+b)+c][(c+a)+b]}}=a\sqrt{bc(b+c)}.$$

$$Similarly\ we\ have:$$

$$2\sqrt{2ab}-ab\sqrt{a+b}\geq c\sqrt{ab(a+b)}\ \ and\ \ 2\sqrt{2ca}-ca\sqrt{c+a}\geq b\sqrt{ca(c+a)}.$$

$$Adding\ these\ inequalities, we\ have:$$

$$2\sqrt{2}\sum_{cyc}\sqrt{ab}-\sum_{cyc}ab\sqrt{a+b}\geq\sum_{cyc}a\sqrt{bc(b+c)}\overset{AM-GM}{\geq}3\sqrt[6]{(abc)^4.(a+b)(b+c)(c+a)}=$$

$$= 3\sqrt{2}.\sqrt[3]{(abc)^2} \overset{?}{\geq} 3\sqrt{2}abc \Leftrightarrow abc \leq 1 \text{ which is true because}:$$

$$abc \overset{Cesaro}{\leq} \frac{(a+b)(b+c)(c+a)}{8} = 1.$$

So the proof is done. Equality holds iff $a = b = c = 1$.

1147. **If $x, y > 0, x + y = 2, \lambda \geq 0, k \geq n \geq 0$, then :**

$$\lambda(x^2 + y^2) - n\left(\frac{x}{y} + \frac{y}{x}\right) + k\left(\frac{1}{x^2} + \frac{1}{y^2}\right) \geq 2(\lambda - n + k)$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$2 = x + y \overset{A-G}{\geq} 2.\sqrt{xy} \Rightarrow \sqrt{xy} \leq 1 \Rightarrow (\sqrt{xy} - 1)(\sqrt{xy} + 1) \leq 0$$

$$\Rightarrow xy \leq 1 \Rightarrow x^2y^2 \overset{(*)}{\leq} xy$$

$$\frac{1}{x^2} + \frac{1}{y^2} = \frac{x^2 + y^2}{x^2y^2} \overset{\text{via }(*)}{\geq} \frac{x^2 + y^2}{xy} \Rightarrow \frac{1}{x^2} + \frac{1}{y^2} - \left(\frac{x}{y} + \frac{y}{x}\right) \geq 0$$

$$\overset{\because k \geq 0}{\Rightarrow} k\left(\frac{1}{x^2} + \frac{1}{y^2} - \left(\frac{x}{y} + \frac{y}{x}\right)\right) \geq 0 \Rightarrow k\left(\frac{1}{x^2} + \frac{1}{y^2}\right) \geq k\left(\frac{x}{y} + \frac{y}{x}\right)$$

$$\Rightarrow k\left(\frac{1}{x^2} + \frac{1}{y^2}\right) - n\left(\frac{x}{y} + \frac{y}{x}\right) \geq (k - n)\left(\frac{x}{y} + \frac{y}{x}\right)$$

$$\overset{A-G}{\geq} (k - n)\left(2.\sqrt{\frac{x}{y}.\frac{y}{x}}\right) (\because k - n \geq 0) \therefore k\left(\frac{1}{x^2} + \frac{1}{y^2}\right) - n\left(\frac{x}{y} + \frac{y}{x}\right) \overset{(**)}{\geq} 2(k - n)$$

$$\text{Again}, \lambda(x^2 + y^2) \geq \frac{\lambda}{2}(x + y)^2 \ (\because \lambda \geq 0) \overset{x+y=2}{=} 2\lambda \therefore \lambda(x^2 + y^2) \overset{(***)}{\geq} 2\lambda$$

$$\therefore (**) + (***) \Rightarrow k\left(\frac{1}{x^2} + \frac{1}{y^2}\right) - n\left(\frac{x}{y} + \frac{y}{x}\right) + \lambda(x^2 + y^2) \geq 2(k - n) + 2\lambda$$

$$\Rightarrow \lambda(x^2 + y^2) - n\left(\frac{x}{y} + \frac{y}{x}\right) + k\left(\frac{1}{x^2} + \frac{1}{y^2}\right) \geq 2(\lambda - n + k)$$

$$\forall\, x, y > 0 \,\middle|\, x + y = 2, \lambda \geq 0, k \geq n \geq 0 \text{ (QED)}$$

1148. If $a,b,c>0$ and $\lambda\ge 0$ then:

$$\sum_{cyc}\frac{b+c}{\sqrt{a^2+\lambda}}\ge\sum_{cyc}\frac{b+c}{\sqrt{bc+\lambda}}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have :

$$\sum_{cyc}\frac{b+c}{\sqrt{a^2+\lambda}}=\sum_{cyc}\left(\frac{b}{\sqrt{c^2+\lambda}}+\frac{c}{\sqrt{b^2+\lambda}}\right)=\sum_{cyc}\left(\frac{b^{\frac{3}{2}}}{(bc^2+\lambda b)^{\frac{1}{2}}}+\frac{c^{\frac{3}{2}}}{(b^2c+\lambda c)^{\frac{1}{2}}}\right)\ge$$

$$\overset{Hölder}{\ge}\sum_{cyc}\frac{(b+c)^{\frac{3}{2}}}{[(bc^2+\lambda b)+(b^2c+\lambda c)]^{\frac{1}{2}}}=\sum_{cyc}\frac{(b+c)^{\frac{3}{2}}}{[(b+c)(bc+\lambda)]^{\frac{1}{2}}}=\sum_{cyc}\frac{b+c}{\sqrt{bc+\lambda}}.$$

Equality holds iff $a=b=c$.

Solution 2 by Hikmat Mammadov-Azerbaijan

For a convex function $g,\mu_{1,2},\mu_1+\mu_2=1$:

$$\mu_1g(x)+\mu_2g(y)\ge g(\mu_1x+\mu_2y)$$

$$g(x)=\frac{1}{\sqrt{x}}-\textit{is convex function, then:}$$

$$\frac{b}{b+c}\cdot\frac{1}{\sqrt{c^2+\lambda}}+\frac{c}{b+c}\cdot\frac{1}{\sqrt{b^2+\lambda}}\ge\frac{1}{\sqrt{\frac{b}{b+c}(c^2+\lambda)+\frac{c}{b+c}(b^2+\lambda)}}=$$

$$=\frac{\sqrt{b+c}}{\sqrt{bc(b+c)+\lambda(b+c)}}=\frac{1}{\sqrt{bc+\lambda}}$$

Thus,

$$\frac{b}{\sqrt{c^2+\lambda}}+\frac{c}{\sqrt{b^2+\lambda}}\ge\frac{b+c}{\sqrt{bc+\lambda}}$$

$$\frac{c}{\sqrt{a^2+\lambda}}+\frac{a}{\sqrt{c^2+\lambda}}\ge\frac{a+c}{\sqrt{ac+\lambda}}$$

$$\frac{a}{\sqrt{b^2+\lambda}}+\frac{b}{\sqrt{a^2+\lambda}}\ge\frac{a+b}{\sqrt{ab+\lambda}}$$

$$By\ adding: \sum_{cyc}\frac{b+c}{\sqrt{a^2+\lambda}} \ge \sum_{cyc}\frac{b+c}{\sqrt{bc+\lambda}}$$

1149. If $a,b,c,d>0, a^2+b^2+c^2+d^2=4$ then:

$$\sum_{cyc}\sqrt{a^2+b+c+2022} \le 180$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Tapas Das-India

$$\frac{a^2+b^2+c^2+d^2}{4} \overset{QM-AM}{\ge} \left(\frac{a+b+c+d}{4}\right)^2 \Leftrightarrow$$

$$1 \ge \left(\frac{a+b+c+d}{4}\right)^2 \Leftrightarrow 16 \ge (a+b+c+d)^2 \Leftrightarrow 4 \ge a+b+c+d)$$

$$\sum_{cyc}\sqrt{a^2+b+c+2022} \overset{CBS}{\le} \sqrt{4\left[\sum_{cyc}a^2+2\sum_{cyc}a+4\cdot 2022\right]} \le$$

$$\le \sqrt{4(4+2\cdot 4+8088)} = 2\sqrt{8100} = 2\cdot 90 = 180$$

Solution 2 by Tran Quoc Anh-Vietnam

$$\sum_{cyc}\sqrt{a^2+b+c+2022} = \sum_{cyc}\frac{(a^2+b+c+2022)^{\frac{1}{2}}}{1^{-\frac{1}{2}}} \overset{Radon}{\le}$$

$$\le \frac{(\sum a^2+2\sum a+8088)^{\frac{1}{2}}}{4^{-\frac{1}{2}}} = 2\left(4+2\sum_{cyc}a+8088\right)^{\frac{1}{2}}$$

Alternatively:

$$4 = \sqrt{4(a^2+b^2+c^2+d^2)} \overset{CBS}{\ge} \sqrt{(a+b+c+d)^2} = a+b+c+d$$

Thus:

$$2\left(4+2\sum_{cyc}a+8088\right)^{\frac{1}{2}} \le 2(4+2\cdot 4+8088)^{\frac{1}{2}} = 180$$

Therefore,

$$\sum_{cyc}\sqrt{a^2+b+c+2022}\le 180$$

Equality holds for $a=b=c=d=1$.

1150. If $a,b,c,\lambda>0$ such that $a^2+b^2+c^2=\lambda$, then:

$$\sum_{cyc}\left(\frac{\lambda-bc}{b+c}\right)^2\ge\lambda$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$We\ know\ that:\ ab+bc+ca\le a^2+b^2+c^2,then$$

$$ab+bc+ca\le\lambda$$

$$(a+b)^2+(b+c)^2+(c+a)^2=2(a^2+b^2+c^2)+2(ab+bc+ca)\le$$

$$\le 2(a^2+b^2+c^2)+2(a^2+b^2+c^2)=4(a^2+b^2+c^2)=4\lambda;\qquad (1)$$

$$\sum_{cyc}\frac{(\lambda-bc)^2}{(b+c)^2}\overset{AM-GM}{\ge}\sum_{cyc}\frac{\left(\lambda-\frac{b^2+c^2}{2}\right)^2}{(b+c)^2}\ge\frac{\left(3\lambda-\frac{a^2+b^2+b^2+c^2+c^2+a^2}{2}\right)^2}{(b+c)^2+(c+a)^2+(a+b)^2}\ge$$

$$\ge\frac{\left(3\lambda-(a^2+b^2+c^2)\right)^2}{a^2+b^2+b^2+c^2+c^2+a^2+2(ab+bc+ca)}\ge\frac{(3\lambda-\lambda)^2}{4\lambda}=\lambda$$

1151. *Let* $a,b,c>0: a+b+c=ab+bc+ca$. *Prove that* :

$$\frac{a+b}{4b+c+a}+\frac{b+c}{4c+a+b}+\frac{c+a}{4a+b+c}\le\frac{a+b+c+1}{4}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\ 4b+(c+a)\overset{AM-HM}{\ge}4b+\frac{4ca}{c+a}=\frac{4(ab+bc+ca)}{c+a}.$$

$$Then:\ \frac{a+b}{4b+c+a}\le\frac{(a+b)(a+c)}{4(ab+bc+ca)}\ (and\ analogs)$$

$$Therefore,$$

$$\sum_{cyc}\frac{a+b}{4b+c+a}\le\sum_{cyc}\frac{(a+b)(a+c)}{4(ab+bc+ca)}=\frac{(a+b+c)^2+(ab+bc+ca)}{4(ab+bc+ca)}=\frac{a+b+c+1}{4}.$$

So the proof is done. Equality holds iff $a=b=c=1$.

1152. *Let* $a,b,c\ge 0, ab+bc+ca=1$. *Prove that:*

$$\frac{bc+1}{a^2+3a\sqrt{bc}+2bc}+\frac{ca+1}{b^2+3b\sqrt{ca}+2ca}+\frac{ab+1}{c^2+3c\sqrt{ab}+2ab}\ge 2$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have:

$$3\sqrt{bc}=\sqrt{bc}+2\sqrt{bc}\overset{AM-GM}{\le}\frac{b+c}{2}+\left(\frac{b+c}{2}+\frac{2bc}{b+c}\right)=b+c+\frac{2bc}{b+c}.$$

$$\text{Then: }\frac{bc+1}{a^2+3a\sqrt{bc}+2bc}\ge\frac{bc+1}{a^2+a\left(b+c+\frac{2bc}{b+c}\right)+2bc}=\frac{bc+1}{(a+b+c)\left(a+\frac{2bc}{b+c}\right)}=$$

$$=\frac{(bc+1)(b+c)}{(a+b+c)(ab+2bc+ca)}=\frac{b+c}{a+b+c}.$$

$$\text{Therefore, }\sum_{cyc}\frac{bc+1}{a^2+3a\sqrt{bc}+2bc}\ge\sum_{cyc}\frac{b+c}{a+b+c}=2.$$

Equality holds iff $a=b=c=\frac{\sqrt{3}}{3}$.

1153. *Let* $a,b,c\ge 0: ab+bc+ca=1$. *Prove that:*

$$\frac{1}{\sqrt{4a^2+a^2bc+bc}}+\frac{1}{\sqrt{4b^2+ab^2c+ca}}+\frac{1}{\sqrt{4c^2+abc^2+ab}}\ge 2\sqrt{\frac{a+b+c}{(a+b)(b+c)(c+a)}}$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have:

$$\frac{1}{4a^2+a^2bc+bc}=\frac{b+c}{(b+c)(4a^2+bc)+(ab+ca).abc}\overset{ab+ca\le 1}{\ge}\frac{b+c}{(b+c)(4a^2+bc)+abc}=$$

$$=\frac{b+c}{4a(b+c).a+(a+b+c)bc}\overset{AM-GM}{\geq}\frac{b+c}{[a+(b+c)]^2.a+(a+b+c)bc}$$
$$=\frac{b+c}{(a+b+c)(a+b)(a+c)}.$$

$$Then:\ \frac{1}{\sqrt{4a^2+a^2bc+bc}}\geq\frac{b+c}{\sqrt{(a+b+c)(a+b)(b+c)(c+a)}}.\ (and\ analogs)$$

$$Therefore,$$

$$\sum_{cyc}\frac{1}{\sqrt{4a^2+a^2bc+bc}}\geq\sum_{cyc}\frac{b+c}{\sqrt{(a+b+c)(a+b)(b+c)(c+a)}}=2\sqrt{\frac{a+b+c}{(a+b)(b+c)(c+a)}}.$$

$$Equality\ holds\ iff\ (a,b,c)=(1,1,0)\ and\ permutation.$$

1154. If $a,b,c>0, a^2+b^2+c^2=3$ then:

$$\sqrt[5]{a^2+2bc+1021}+\sqrt[5]{b^2+2ca+1021}+\sqrt[5]{c^2+2ab+1021}\leq 12$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Ravi Prakash-New Delhi-India

$$2bc\leq b^2+c^2\Rightarrow a^2+2bc+1021\leq a^2+b^2+c^2+1021=1024=2^{10}$$
$$\Rightarrow\sqrt[5]{a^2+2bc+1021}\leq\sqrt[5]{2^{10}}=4$$
$$Similarly\ for\ the\ other\ two\ terms:$$
$$\sqrt[5]{a^2+2bc+1021}+\sqrt[5]{b^2+2ca+1021}+\sqrt[5]{c^2+2ab+1021}\leq 4+4+4=12$$
$$Equality\ holds\ for\ a=b=c.$$

Solution 2 by Tapas Das-India

$$a^2+b^2+c^2=3, ab+bc+ca\leq a^2+b^2+c^2$$
$$\Rightarrow ab+bc+ca\leq 3$$
$$\sqrt[5]{a^2+2bc+1021}+\sqrt[5]{b^2+2ca+1021}+\sqrt[5]{c^2+2ab+1021}\leq$$
$$\leq 3\cdot\sqrt[5]{\frac{a^2+b^2+c^2+2(ab+bc+ca)+3\cdot 1021}{3}}\leq$$
$$\leq 3\cdot\sqrt[5]{\frac{3+2\cdot 3+3063}{3}}=3\cdot\sqrt[5]{1024}=3\cdot 2^2=12$$

$$Equality\ holds\ for\ a=b=c.$$

Solution 3 by Max Wong-Hong Kong

$$\sqrt[5]{a^2+2bc+1021}+\sqrt[5]{b^2+2ca+1021}+\sqrt[5]{c^2+2ab+1021}=$$

$$=\sqrt[5]{1024-(b-c)^2}+\sqrt[5]{1024-(c-a)^2}+\sqrt[5]{1024-(a-b)^2}\le 3\cdot\sqrt[5]{1024}=12$$

$$Equalty\ holds\ for\ a=b=c.$$

Solution 4 by Tran Quoc Anh-Vietnam

$$\sum_{cyc}\sqrt[5]{a^2+2bc+1021}=\sum_{cyc}\frac{(a^2+2bc+1021)^{\frac{1}{5}}}{1^{-\frac{4}{5}}}\overset{Radon}{\le}$$

$$\le\frac{(\sum a^2+2\sum ab+3063)^{\frac{1}{5}}}{3^{-\frac{4}{5}}}=81^{\frac{1}{5}}\left(3+2\sum_{cyc}ab+3063\right)^{\frac{1}{5}}$$

$$Alternatively:$$

$$2(ab+bc+ca)\le 2(a^2+b^2+c^2)=6$$

$$\Rightarrow 81^{\frac{1}{5}}\left(3+2\sum_{cyc}ab+3063\right)^{\frac{1}{5}}\le 81^{\frac{1}{5}}(3+6+3063)^{\frac{1}{5}}=12$$

$$Therefore,$$

$$\sum_{cyc}\sqrt[5]{a^2+2bc+1021}\le 12$$

$$Equality\ holds\ for\ a=b=c.$$

1155. *Let $a,b,c\ge 0: ab+bc+ca=1.$ Prove that:*

$$\frac{a+b+c}{2}\ge\sqrt{\frac{(2a+bc)(2b+ca)(2c+ab)}{(2a+\sqrt{bc})(2b+\sqrt{ca})(2c+\sqrt{ab})}}$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$(*):\ \frac{a+b+c}{2}\ge\sqrt{\frac{(2a+bc)(2b+ca)(2c+ab)}{(2a+\sqrt{bc})(2b+\sqrt{ca})(2c+\sqrt{ab})}}$$

$$Let\ p:=a+b+c.\ \ If\ p\ge 2, we\ have:$$

$$RHS_{(*)} \overset{bc\le 1\ (and\ analogs)}{\le} \sqrt{\frac{(2a+bc)(2b+ca)(2c+ab)}{(2a+bc)(2b+ca)(2c+ab)}} = 1 \le LHS_{(*)}.$$

$$Assume\ now\ that\ p \le 2.\ We\ have:$$

$$2a+bc \overset{AM-GM}{\le} \left(\frac{a}{b+c} + a(b+c)\right) + bc = \frac{a}{b+c} + 1 = \frac{p}{b+c}$$

$$and\ \ 2a+\sqrt{bc} \overset{GM-HM}{\ge} 2a + \frac{2bc}{b+c} = \frac{2(ab+bc+ca)}{b+c} = \frac{2}{b+c}.$$

$$Then: \frac{2a+bc}{2a+\sqrt{bc}} \le \frac{p}{2}\ (and\ analogs) \Rightarrow RHS_{(*)} \le \sqrt{\left(\frac{p}{2}\right)^3} \overset{p\le 2}{\le} \frac{p}{2} = LHS_{(*)}.$$

$$So\ the\ proof\ is\ done.\ Equality\ holds\ iff\ (a,b,c) = (1,1,0)\ and\ permutation.$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let\ p := a+b+c.$$

$$We\ have:\ \ 2a+bc \overset{AM-GM}{\le} \left(\frac{a}{b+c} + a(b+c)\right) + bc = \frac{a}{b+c} + 1 = \frac{p}{b+c}.$$

$$Similarly\ we\ have:\ \ 2b+ca \le \frac{p}{c+a}.$$

$$Then: \frac{1}{(2a+bc)(2b+ca)} \ge \frac{(b+c)(c+a)}{p^2}\ (ana\ analogs)$$

$$\Rightarrow \sum_{cyc} \frac{1}{(2a+bc)(2b+ca)} \ge \sum_{cyc} \frac{(b+c)(c+a)}{p^2}$$

$$\Leftrightarrow \frac{2p+(ab+bc+ca)}{(2a+bc)(2b+ca)(2c+ab)} \ge \frac{p^2+(ab+bc+ca)}{p^2}$$

$$Then:\ (2a+bc)(2b+ca)(2c+ab) \le \frac{p^2(2p+1)}{p^2+1}\ \ (1)$$

$$Now\ we\ have: 2a+\sqrt{bc} \overset{GM-HM}{\ge} 2a + \frac{2bc}{b+c} = \frac{2(ab+bc+ca)}{b+c} = \frac{2}{b+c}.$$

$$Similarly\ we\ have:\ \ 2b+\sqrt{ca} \ge \frac{2}{c+a}.$$

$$Then: \frac{1}{(2a+\sqrt{bc})(2b+\sqrt{ca})} \le \frac{(b+c)(c+a)}{4}\ (and\ analogs)$$

$$\Rightarrow \sum_{cyc} \frac{1}{(2a+\sqrt{bc})(2b+\sqrt{ca})} \le \sum_{cyc} \frac{(b+c)(c+a)}{4}$$

$$\Leftrightarrow \frac{2p+(\sqrt{ab}+\sqrt{bc}+\sqrt{ca})}{(2a+\sqrt{bc})(2b+\sqrt{ca})(2c+\sqrt{ab})} \le \frac{p^2+(ab+bc+ca)}{4}$$

$$Then:$$

$$\left(2a+\sqrt{bc}\right)\left(2b+\sqrt{ca}\right)\left(2c+\sqrt{ab}\right)\geq\frac{4\left[2p+\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)\right]}{p^2+1}\geq\frac{4(2p+1)}{p^2+1}\ (2)$$

From (1) *and* (2) *we have* :

$$\sqrt{\frac{(2a+bc)(2b+ca)(2c+ab)}{\left(2a+\sqrt{bc}\right)\left(2b+\sqrt{ca}\right)\left(2c+\sqrt{ab}\right)}}\leq\frac{p}{2}=\frac{a+b+c}{2}.$$

Equality holds iff $(a,b,c)=(1,1,0)$ *and permutation.*

1156. *If* $a,b,c>0$ *and* $\frac{1}{4}\leq\lambda\leq\frac{5}{4}$ *then* :

$$\sum_{cyc}\frac{a(b+c)}{a^2+\lambda(b+c)^2}\leq\frac{6}{4\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{Lemma}:\ \textit{If}\ x,y>0\ \textit{then}:\ \frac{xy}{x^2+\lambda y^2}\leq\frac{4(8\lambda-1)x+(5-4\lambda)y}{(4\lambda+1)^2(x+y)}\ (1)$$

$$\textit{Proof}:\ (1)\Leftrightarrow\left(x^2+\lambda y^2\right)[4(8\lambda-1)x+(5-4\lambda)y]-(4\lambda+1)^2(x+y)xy\geq 0$$

$$\Leftrightarrow 4(8\lambda-1)x^3-4\left(4\lambda^2+3\lambda-1\right)x^2y+\left(16\lambda^2-12\lambda-1\right)xy^2+\lambda(5-4\lambda)y^3\geq 0$$

$$\Leftrightarrow (2x-y)^2[(8\lambda-1)x+\lambda(5-4\lambda)y]\geq 0,\ \textit{which is true.}\left(\because\frac{1}{4}\leq\lambda\leq\frac{5}{4}\right)$$

Let $x=a$ *and* $y=b+c$, *we have* :

$$\frac{a(b+c)}{a^2+\lambda(b+c)^2}\leq\frac{4(8\lambda-1)a+(5-4\lambda)(b+c)}{(4\lambda+1)^2(a+b+c)}\ (\textit{and analogs})$$

$$\textit{Therefore},\ \sum_{cyc}\frac{a(b+c)}{a^2+\lambda(b+c)^2}\leq\sum_{cyc}\frac{4(8\lambda-1)a+(5-4\lambda)(b+c)}{(4\lambda+1)^2(a+b+c)}=$$

$$=\frac{4(8\lambda-1)(a+b+c)+(5-4\lambda).2(a+b+c)}{(4\lambda+1)^2(a+b+c)}=\frac{6}{4\lambda+1}.$$

So the proof is completed. Equality holds iff $a=b=c$.

1157. $Let\ a,b,c>0: abc=1.\ Prove\ that:$

$$\frac{4}{(a+1)(b+1)(c+1)}+\frac{1}{4}\ge\frac{a}{(a+1)^2}+\frac{b}{(b+1)^2}+\frac{c}{(c+1)^2}$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let\ x:=\frac{1}{a+1},\ \ y:=\frac{1}{b+1},\ \ z:=\frac{1}{c+1}.$$

$$The\ problem\ becomes\ to\ prove:$$

$$4xyz+\frac{1}{4}\ge(x-x^2)+(y-y^2)+(z-z^2)\ \ or\ \ x^2+y^2+z^2+4xyz+\frac{1}{4}\ge x+y+z\ \ (1)$$

$$We\ have: a=\frac{1}{x}-1=\frac{1-x}{x}\ \ (and\ analogs)\ then\ from\ the\ given\ condition,$$

$$we\ have:\ \frac{(1-x)(1-y)(1-z)}{xyz}=1\ \ or\ \ 2xyz=1-(x+y+z)+(xy+yz+zx)$$

$$Then\ (1)\ \Leftrightarrow\ x^2+y^2+z^2+2(xy+yz+zx)+\frac{9}{4}\ge 3(x+y+z)$$

$$\Leftrightarrow\ (x+y+z)^2-3(x+y+z)+\frac{9}{4}\ge 0\ \Leftrightarrow\ \left(x+y+z-\frac{3}{2}\right)^2\ge 0,$$

$$which\ is\ true\ and\ the\ proof\ is\ done.$$

1158. $If\ a,b,c,\lambda>0\ then:$

$$(\lambda+1)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\ge 3(\lambda-1)+\frac{(a+b)^2}{a^2+bc}+\frac{(b+c)^2}{b^2+ca}+\frac{(c+a)^2}{c^2+ab}$$

Proposed by Marin Chirciu, Octavian Stroe-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$By\ CBS\ inequality, we\ have:$$

$$\frac{(a+b)^2}{a^2+bc}\le\frac{a^2}{a^2}+\frac{b^2}{bc}=1+\frac{b}{c}\ \ (and\ analogs)$$

$$Then: \ RHS \le 3(\lambda-1)+3+\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)=\lambda.3+\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\le$$

$$\overset{AM-GM}{\le} \lambda.\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)=(\lambda+1)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)=LHS.$$

So the proof is done. Equality holds iff $a=b=c$.

1159. *If $a,b,c>0$ and $\lambda\ge 0$ then:*

$$\sum_{cyc}\frac{1}{a^3+\lambda abc}\ge\frac{3(a+b+c)}{(\lambda+1)(a^2b^2+b^2c^2+c^2a^2)}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $x:=\frac{1}{a},\ y:=\frac{1}{b},\ z:=\frac{1}{c}$. *The problem becomes to prove:*

$$\sum_{cyc}\frac{x^2}{yz+\lambda x^2}\ge\frac{3(xy+yz+zx)}{(\lambda+1)(x^2+y^2+z^2)}.$$

By Bergström's inequality, we have:

$$\sum_{cyc}\frac{x^2}{yz+\lambda x^2}\ge\frac{(x+y+z)^2}{(xy+yz+zx)+\lambda(x^2+y^2+z^2)}\ge\frac{3(xy+yz+zx)}{(\lambda+1)(x^2+y^2+z^2)},$$

because: $(x+y+z)^2\ge 3(xy+yz+zx)$ *and* $xy+yz+zx\le x^2+y^2+z^2$.

So the proof is done. Equality holds iff $x=y=z \Leftrightarrow a=b=c$.

Solution 2 by Tapas Das-India

$$\sum_{cyc}\frac{1}{a^3+\lambda abc}=\sum_{cyc}\frac{\frac{1}{a^2}}{a+\lambda\cdot\frac{bc}{a}}\ge\frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}{(a+b+c)+\lambda\left(\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c}\right)}=$$

$$=\frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}{(a+b+c)+\lambda\cdot\frac{b^2c^2+c^2a^2+a^2b^2}{abc}}=\frac{abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}{abc(a+b+c)+\lambda(b^2c^2+c^2a^2+a^2b^2)}=$$

$$=\frac{\frac{(ab+bc+ca)^2}{abc}}{ab\cdot ac+ab\cdot bc+bc\cdot ca+\lambda(a^2b^2+b^2c^2+c^2a^2)}\geq$$

$$\geq\frac{\frac{(ab+bc+ca)^2}{abc}}{(a^2b^2+b^2c^2+c^2a^2)+\lambda(a^2b^2+b^2c^2+c^2a^2)}=$$

$$=\frac{\frac{(ab+bc+ca)^2}{abc}}{(\lambda+1)(a^2b^2+b^2c^2+c^2a^2)}\geq\frac{3(ab\cdot bc+bc\cdot ca+ca\cdot ab)}{(\lambda+1)(a^2b^2+b^2c^2+c^2a^2)abc}=$$

$$=\frac{3(a+b+c)}{(\lambda+1)(a^2b^2+b^2c^2+c^2a^2)}$$

Solution 3 by Soumitra Mandal-Chandar Nagore-India

$$\sum_{cyc}\frac{1}{a^3+\lambda abc}=\sum_{cyc}\frac{\frac{1}{a}}{a^2+\lambda bc}=\sum_{cyc}\frac{\frac{1}{a^2}}{a+\lambda\cdot\frac{bc}{a}}\overset{CBS}{\geq}$$

$$\geq\frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}{\sum a+\lambda\sum\frac{bc}{a}}=\frac{abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}{abc\sum a+\lambda\sum a^2b^2}\geq\frac{3abc\sum\frac{1}{ab}}{(\lambda+1)\sum a^2b^2}=$$

$$=\frac{3(a+b+c)}{(\lambda+1)\sum a^2b^2}$$

$$Equality\ holds\ for\ a=b=c.$$

1160. *Let $a,b,c>0$. Prove that :*

$$\frac{b}{(2a+b)(2a+c)}+\frac{c}{(2b+c)(2b+a)}+\frac{a}{(2c+a)(2c+b)}\geq\frac{1}{a+b+c}$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution 1 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\sum_{cyc}\frac{a}{(a+2c)(b+2c)}\geq\frac{1}{a+b+c}\Leftrightarrow\sum_{cyc}\frac{x}{(x+2z)(y+2z)}\geq 1;(x+y+z=1)$$

$$\frac{(x+y+z)^2}{\sum x(x+2z)(y+2z)}\geq 1$$

$$\sum_{cyc}x(xy+2xz+2yz+4z^2)\geq 1$$

$$x^3+y^3+z^3+x^2z+z^2y+y^2x \geq 2(x^2y+y^2z+z^2x) \text{ which is true, because:}$$

$$x^3+xy^2 \geq 2x^2y,\ y^3+z^2y \geq 2y^2z,\ z^3+zx^2 \geq 2z^2x$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{By Hölder's inequality, we have :}$$

$$\sum_{cyc}\frac{b}{(2a+b)(2a+c)} \geq \frac{\left(\sum_{cyc} b\right)^3}{\left[\sum_{cyc} b(2a+b)\right]\left[\sum_{cyc} b(2a+c)\right]} = \frac{\left(\sum_{cyc} a\right)^3}{\left(\sum_{cyc} a\right)^2 . 3\sum_{cyc} bc} =$$

$$= \frac{a+b+c}{3(ab+bc+ca)} \overset{3\sum bc \leq (\sum a)^2}{\geq} \frac{1}{a+b+c}, \text{ as desired.}$$

$$\text{Equality holds iff } a=b=c.$$

Solution 3 by Tran Quoc Anh-Vietnam

$$\sum_{cyc}\frac{b}{(2a+b)(2a+c)} \overset{CBS}{\geq} \sum_{cyc}\frac{4b}{(4a+b+c)^2} = 4\sum_{cyc}\frac{b^3}{4ab+bc+b^2} \overset{Radon}{\geq}$$

$$\geq \frac{4(a+b+c)^3}{(\sum a+5\sum ab)^2}; \quad (1)$$

$$\textbf{Alternatively:}$$

$$\text{Let } a+b+c=p>0 \text{ and } ab+bc+ca=q>0, \text{ then } p^2 \geq 3q$$

$$4p^4-(p^2+3q)^2 = 3(p^2-3q)(p^2+q) \geq 0$$

$$4p^4 \geq (p^2+3q)^2 \Rightarrow \frac{4p^3}{(p^2+3q)^2} \geq \frac{1}{p}$$

$$\frac{4(a+b+c)^3}{(\sum a^2+5\sum ab)^2} \geq \frac{1}{a+b+c}; \quad (2)$$

$$\text{From (1) and (2) we have:}$$

$$\sum_{cyc}\frac{b}{(2a+b)(2a+c)} \geq \frac{1}{a+b+c}$$

1161. If $x,y,z>0, xyz=1$ **then:**

$$(y^2+z^2)\left(\frac{yz}{x}\right)^{\log\left(\frac{y}{z}\right)} + (z^2+x^2)\left(\frac{zx}{y}\right)^{\log\left(\frac{z}{x}\right)} + (x^2+y^2)\left(\frac{xy}{z}\right)^{\log\left(\frac{x}{y}\right)} \geq 6$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$\text{Let } A=(y^2+z^2)\left(\frac{yz}{x}\right)^{\log\left(\frac{y}{z}\right)}+(z^2+x^2)\left(\frac{zx}{y}\right)^{\log\left(\frac{z}{x}\right)}+(x^2+y^2)\left(\frac{xy}{z}\right)^{\log\left(\frac{x}{y}\right)}\geq$$

$$\geq 2yz\left(\frac{yz}{x}\right)^{\log\left(\frac{y}{z}\right)}+2zx\left(\frac{zx}{y}\right)^{\log\left(\frac{z}{x}\right)}+2xy\left(\frac{xy}{z}\right)^{\log\left(\frac{x}{y}\right)}\geq$$

$$\geq 2\cdot 3\cdot (x^2y^2z^2\cdot E)^{\frac{1}{3}}=6E^{\frac{1}{3}},\qquad \text{where}$$

$$E=\left(\frac{yz}{x}\right)^{\log\left(\frac{y}{z}\right)}\left(\frac{zx}{y}\right)^{\log\left(\frac{z}{x}\right)}\left(\frac{xy}{z}\right)^{\log\left(\frac{x}{y}\right)}=x^{-2\log\left(\frac{y}{z}\right)}y^{-2\log\left(\frac{z}{x}\right)}z^{-2\log\left(\frac{x}{y}\right)};\ xyz=1$$

$$\text{Hence,}$$

$$\log E=-2[(\log y-\log z)\log x+(\log z-\log x)\log y+(\log x-\log y)\log z]$$

$$=-2\cdot 0=0,\ \text{then } E=1 \text{ and } A\geq 6.$$

Solution 2 by Hikmat Mammadov-Azerbaijan

$$\Psi=(y^2+z^2)\left(\frac{yz}{x}\right)^{\log\left(\frac{y}{z}\right)}+(z^2+x^2)\left(\frac{zx}{y}\right)^{\log\left(\frac{z}{x}\right)}+(x^2+y^2)\left(\frac{xy}{z}\right)^{\log\left(\frac{x}{y}\right)}\geq$$

$$\geq 2yz\left(\frac{yz}{x}\right)^{\log\left(\frac{y}{z}\right)}+2zx\left(\frac{zx}{y}\right)^{\log\left(\frac{z}{x}\right)}+2xy\left(\frac{xy}{z}\right)^{\log\left(\frac{x}{y}\right)}\overset{AM-GM}{\geq}$$

$$\geq 6\cdot\sqrt[3]{\frac{1}{x^{1+2\log\left(\frac{y}{z}\right)}y^{1+2\log\left(\frac{z}{x}\right)}z^{1+2\log\left(\frac{x}{y}\right)}}}=$$

$$=6\cdot\sqrt[3]{\exp\left(-\left(\sum_{cyc}(\log x+2\log x\log y-2\log x\log z)\right)\right)}=$$

$$=6\cdot\sqrt[3]{\exp(-\log(xyz))}\geq 6$$

$$\text{Therefore,}$$

$$(y^2+z^2)\left(\frac{yz}{x}\right)^{\log\left(\frac{y}{z}\right)}+(z^2+x^2)\left(\frac{zx}{y}\right)^{\log\left(\frac{z}{x}\right)}+(x^2+y^2)\left(\frac{xy}{z}\right)^{\log\left(\frac{x}{y}\right)}\geq 6$$

1162. *If $x,y,z>0, x+y+z=3$ and $n\in N$, then :*

$$\sum_{cyc}\frac{x^n(x+y)}{\left(y+\sqrt{zx}\right)^2}\ge\frac{3}{2}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Sanong Huayrerai-Nakon Pathom-Thailand

For $x,y,z>0, x+y+z=3$, we give: $a^2=x, b^2=y, c^2=z$, then:

$$\sum_{cyc}\frac{x^n(x+y)}{\left(y+\sqrt{zx}\right)^2}\ge\frac{x^n+y^n+z^n}{3}\sum_{cyc}\frac{x+y}{\left(y+\sqrt{zx}\right)^2}=$$

$$=\sum_{cyc}\frac{\left(\frac{x+y}{y+\sqrt{yz}}\right)^2}{x+y}\ge\frac{\left(\sum\frac{x+y}{y+\sqrt{zx}}\right)^2}{2(\sum x)}\ge\frac{3}{2}$$

$$Iff\left(\sum\frac{x+y}{y+\sqrt{zx}}\right)^2\ge 9\Leftrightarrow\sum\frac{x+y}{y+\sqrt{zx}}\ge 3$$

$$3\sqrt[3]{\frac{(a^2+b^2)(b^2+c^2)(c^2+a^2)}{(a^2+bc)(b^2+ca)(c^2+ab)}}\ge 3$$

$$(a^2+b^2)(b^2+c^2)(c^2+a^2)\ge(a^2+bc)(b^2+ca)(c^2+ab)$$

$$a^4b^2+b^4c^2+c^4a^2+c^4a^2+c^4b^2+b^4a^2+2a^2b^2c^2\ge$$

$$a^3b^3+b^3c^3+c^3a^3+a^4bc+b^4ca+c^4ab+2a^2b^2c^2$$

$$a^4b^2+b^4c^2+c^4a^2+c^4a^2+c^4b^2+b^4a^2\ge 2(a^4bc+b^4ca+c^4ab)\ true.$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have :

$$\sum_{cyc}\frac{x^n(x+y)}{\left(y+\sqrt{zx}\right)^2}\overset{CBS}{\ge}\sum_{cyc}\frac{x^n(x+y)}{(y+z)(y+x)}=\sum_{cyc}\frac{x^n}{y+z}\ge$$

$$\overset{\text{Hölder if } n\ge 2 \text{ or Nesbitt if } n=1}{\ge}\frac{(x+y+z)^n}{3^{n-2}.2(x+y+z)}=\frac{3}{2}.$$

So the proof is done. Equality holds iff $x=y=z=1$.

1163. *Let $a,b,c>0: abc=1$. Prove that :*

$$\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}}\ge\frac{3}{\sqrt{a^3+b^3+c^3-1}}$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:$$

$$LHS^2\overset{Hölder}{\ge}\frac{(1+1+1)^3}{\sum_{cyc}bc(b+c)}\overset{AM-GM}{\ge}\frac{27}{\sum_{cyc}(b^3+c^3)+3abc-3}\ge$$

$$\overset{AM-GM}{\ge}\frac{27}{2\sum_{cyc}a^3+\sum_{cyc}a^3-3}=\frac{9}{a^3+b^3+c^3-1}=RHS^2.$$

$$So\ the\ proof\ is\ done.\ Equality\ holds\ iff\ a=b=c=1.$$

1164. *Let $a,b,c\ge 0, ab+bc+ca>0$ then :*

$$\left(1+\frac{a}{b+c}\right)^2+\left(1+\frac{b}{c+a}\right)^2+\left(1+\frac{c}{a+b}\right)^2\ge 1+\frac{4(a^2+b^2+c^2)}{ab+bc+ca}.$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Due\ to\ homogenity, we\ may\ assume\ that:\ ab+bc+ca=1.$$

$$We\ have:\frac{a}{b+c}=a^2+\frac{abc}{b+c}\ge a^2,\ then:$$

$$\left(1+\frac{a}{b+c}\right)^2=1+\frac{2a}{b+c}+\frac{a^2}{(b+c)^2}\ge 1+2a^2+\frac{a^3}{b+c}=$$

$$=2a^2+bc+\left(a(b+c)+\frac{a^3}{b+c}\right)\overset{AM-GM}{\ge}2a^2+bc+2a^2=bc+4a^2.$$

$$Therefore,$$

$$\sum_{cyc}\left(1+\frac{a}{b+c}\right)^2 \ge \sum_{cyc}(bc+4a^2) = 1+4(a^2+b^2+c^2) = 1+\frac{4(a^2+b^2+c^2)}{ab+bc+ca}.$$

$$Equality\ holds\ iff\ (a=b\ and\ c=0)\ and\ permutation.$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

$$For\ a,b,c\ge 0, ab+bc+ca>0, we\ get:$$

$$\sum_{cyc}\left(1+\frac{a}{b+c}\right)^2 \ge 1+4\cdot\frac{a^2+b^2+c^2}{ab+bc+ca}$$

$$3+\sum_{cyc}\left(\frac{a}{b+c}\right)^2+2\sum_{cyc}\frac{a}{b+c} \ge 1+4\cdot\frac{a^2+b^2+c^2}{ab+bc+ca}$$

$$8\sum_{cyc}ab+2\left[\sum_{cyc}a^2+abc\sum_{cyc}\frac{1}{a+b}\right]+\sum_{cyc}\frac{a}{b+c}\cdot\left(a^2+\frac{abc}{b+c}\right)\ge 4\sum_{cyc}a^2$$

$$2\sum_{cyc}ab+2abc\sum_{cyc}\frac{1}{a+b}+\sum_{cyc}\frac{a^3}{b+c}+\sum_{cyc}\frac{a^2bc}{(b+c)^2}\ge 2\sum_{cyc}a^2$$

$$2\sum_{cyc}a^2 \ge 2\sum_{cyc}a^2\ \ true.$$

1165. If $m,n,x,y,z>0$ then:

$$\sum_{cyc}\left(1+\frac{x}{my+nz}\right)^{t+1} \ge 3\left(\frac{m+n+1}{m+n}\right)^{t+1};\ t\ge 0$$

Proposed by D.M. Bătineţu-Giurgiu, Florică Anastase-Romania

Solution by Tapas Das-India

$$\sum_{cyc}\left(1+\frac{x}{my+nz}\right)^{t+1} \overset{Radon}{\ge} \frac{1}{3^t}\left(3+\sum_{cyc}\frac{x}{my+nz}\right)^{t+1} =$$

$$=\frac{1}{3^t}\left(3+\sum_{cyc}\frac{x^2}{mxy+nxz}\right)^{t+1} \ge \frac{1}{3^t}\left(3+\frac{(x+y+z)^2}{(m+n)(xy+yz+zx)}\right)^{t+1} \ge$$

$$\geq \frac{1}{3^t}\left(3+\frac{3(xy+yz+zx)}{(m+n)(xy+yz+zx)}\right)^{t+1}=\frac{1}{3^t}\left(3+\frac{3}{m+n}\right)^{t+1}=$$

$$=3^{t+1}\left(\frac{m+n+1}{m+n}\right)^{t+1}\cdot\frac{1}{3^t}=3\left(\frac{m+n+1}{m+n}\right)^{t+1}$$

1166. If $a, b, c > 0$ such that : $a+b+c-abc=5$ then:

$$(a^2+1)(b^2+1)(c^2+1)\geq 25$$

Proposed by Marin Chirciu-Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$a+b+c-abc=5$$
$$(a^2+1)(b^2+1)(c^2+1)=|a+i|^2\cdot|b+i|^2\cdot|c+i|^2=$$
$$=|(a+i)(b+i)(c+i)|^2=$$
$$=|i^3+i^2(a+b+c)+i(ab+bc+ca)+abc|=$$
$$=\left|\big(abc-(a+b+c)\big)+i(ab+bc+ca-1)\right|=$$
$$=(a+b+c-abc)^2+(ab+bc+ca-1)^2\geq 5^2=25$$
$$Equality\ holds\ for\ ab+bc+ca=1.$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$(a^2+1)(b^2+1)(c^2+1)\geq 25=(a+b+c-abc)^2$$

$$\Leftrightarrow a^2b^2c^2+\sum_{cyc}a^2+\sum_{cyc}a^2b^2+1\geq\left(\sum_{cyc}a\right)^2+a^2b^2c^2-2abc\sum_{cyc}a$$

$$\Leftrightarrow \sum_{cyc}a^2+\left(\sum_{cyc}ab\right)^2-2abc\sum_{cyc}a+1\geq\sum_{cyc}a^2+2\sum_{cyc}ab-2abc\sum_{cyc}a$$

$$\Leftrightarrow\left(\sum_{cyc}ab\right)^2-2\sum_{cyc}ab+1\geq 0\Leftrightarrow\left(\sum_{cyc}ab-1\right)^2\geq 0\to \text{true}$$

$$\therefore (a^2+1)(b^2+1)(c^2+1)\geq 25\ \forall\, a,b,c\in\mathbb{R}\,\big|\, a+b+c-abc=5\ (\text{QED})$$

1167. *Let $a,b,c\in[0,1]$: $ab+bc+ca=1$. Prove that :*

$$\frac{1}{\sqrt{a^2+1}}+\frac{1}{\sqrt{b^2+1}}+\frac{1}{\sqrt{c^2+1}}\geq 1+\sqrt{2}$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Since\ ab+bc+ca=1\ then\ \exists x,y,z\in\left[0,\frac{\pi}{2}\right),x+y+z=\frac{\pi}{2}\ such\ that:$$

$$a=\tan x,\quad b=\tan y,\quad c=\tan z.$$

$$The\ problem\ becomes\ to\ prove:\ \cos x+\cos y+\cos z\ge 1+\sqrt{2}.$$

$$Since\ a,b,c\in[0,1]\ then\ we\ have:x,y,z\in\left[0,\frac{\pi}{4}\right].\ Let\ f(t)=\cos t,t\in\left[0,\frac{\pi}{4}\right].$$

$$f\ is\ concave\ on\ \left[0,\frac{\pi}{4}\right]\ and\ the\ sequence\ \left(\frac{\pi}{4},\frac{\pi}{4},0\right)\ majorizes\ the\ sequence\ (x,y,z),$$

$$then\ by\ Karamata\ inequality, we\ have:$$

$$\cos x+\cos y+\cos z\ge\cos\frac{\pi}{4}+\cos\frac{\pi}{4}+\cos 0=1+\sqrt{2}.$$

$$So\ the\ proof\ is\ done.\ Equality\ holds\ iff\ (a,b,c)=(1,1,0)\ and\ permutations.$$

1168. For *all* $a,b,c\ge 0$ such that: $ab+bc+ca>0$,

$$\frac{a(b+c)}{a^2+bc}+\frac{b(a+c)}{b^2+ac}+\frac{c(b+a)}{c^2+ba}\ge 1+\frac{16abc}{(a+b)(b+c)(c+a)}$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$\boxed{Case\ 1}$ Exactly one among $a,b,c=0$ and WLOG we may assume

$$a=0\ \text{and then}:\ \text{LHS}=\frac{bc}{b^2}+\frac{bc}{c^2}=\frac{b}{c}+\frac{c}{b}\overset{A-G}{\ge}2>1=RHS\Rightarrow LHS>RHS$$

$$\boxed{Case\ 2}\ a,b,c>0\ \text{and then}:\frac{a(b+c)}{a^2+bc}+\frac{b(a+c)}{b^2+ac}+\frac{c(b+a)}{c^2+ba}$$

$$=\frac{a(b+c)}{a^2+bc}+1+\frac{b(a+c)}{b^2+ac}+1+\frac{c(b+a)}{c^2+ba}+1-3$$

$$=\frac{ab+ac+a^2+bc}{a^2+bc}+\frac{ab+bc+b^2+ac}{b^2+ac}+\frac{bc+ca+c^2+ab}{c^2+ba}-3$$

$$=\frac{(a+b)(a+c)}{a^2+bc}+\frac{(b+a)(b+c)}{b^2+ac}+\frac{(c+b)(c+a)}{c^2+ba}-3$$

$$=(a+b)(b+c)(c+a)\sum_{cyc}\frac{1}{(a^2+bc)(b+c)}-3$$

$$\overset{Bergstrom}{\ge}\frac{9(a+b)(b+c)(c+a)}{\sum_{cyc}a^2b+\sum_{cyc}a^2c+\sum_{cyc}b^2c+\sum_{cyc}bc^2}-3=\frac{9(x+2y)}{2x}-3$$

$$\left(x=\sum_{cyc}a^2b+\sum_{cyc}ab^2 \text{ and } y=abc\right)\overset{?}{\geq}1+\frac{16abc}{(a+b)(b+c)(c+a)}$$

$$\Leftrightarrow\frac{9(x+2y)}{2x}\overset{?}{\geq}4+\frac{16abc}{(a+b)(b+c)(c+a)}=\frac{4(x+2y)+16y}{x+2y}$$

$$\Leftrightarrow 9(x+2y)^2\overset{?}{\geq}2x(4x+24y)\Leftrightarrow x^2-12xy+36y^2\overset{?}{\geq}0\Leftrightarrow(x-6y)^2\overset{?}{\geq}0$$

$$\to \text{true} \therefore \text{LHS}\geq\text{RHS} \therefore \text{combining both cases,}$$

$$\frac{a(b+c)}{a^2+bc}+\frac{b(a+c)}{b^2+ac}+\frac{c(b+a)}{c^2+ba}\geq 1+\frac{16abc}{(a+b)(b+c)(c+a)}$$

$$\forall\, a,b,c\geq 0\ \big|\ ab+bc+ca>0, \text{with equality iff } a=b=c \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

The given inequality can be rewritten as follows :

$$\sum_{cyc}\frac{(a+b)(a+c)}{a^2+bc}\geq 4+\frac{16abc}{(a+b)(b+c)(c+a)}=4\sum_{cyc}\frac{a(b+c)}{(a+b)(a+c)}.$$

By AM – HM inequality, we have : $\dfrac{(a^2+bc)+a(b+c)}{2}\geq\dfrac{2(a^2+bc).a(b+c)}{(a^2+bc)+a(b+c)}.$

Then : $\dfrac{(a+b)(a+c)}{a^2+bc}\geq\dfrac{4a(b+c)}{(a+b)(a+c)}$ *(and analogs)*

Therefore, $\displaystyle\sum_{cyc}\frac{(a+b)(a+c)}{a^2+bc}\geq 4\sum_{cyc}\frac{a(b+c)}{(a+b)(a+c)}$, *as desired.*

Equality holds iff $a=b=c$.

1169. Let $a,b,c\geq 0, ab+bc+ca>0$. Prove that:

$$\sum_{cyc}\frac{1}{2a^2+bc}\geq\frac{2}{a^2+b^2+c^2}+\frac{1}{ab+bc+ca}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Nguyen Thuong-Vietnam

$$\sum_{cyc}\frac{1}{2a^2+bc}=\sum_{cyc}\frac{(b+c)^2}{(b+c)^2(2a^2+bc)}\geq\frac{4(\sum a)^2}{(\sum a^2)(\sum a)+6(\sum ab)^2-9abc\sum a}$$

We have:

$$(a^2+b^2+c^2)(ab+bc+ca)+3abc(a+b+c)\geq 2(ab+bc+ca)^2 \Leftrightarrow$$

$$ab(a-b)^2+bc(b-c)^2+ca(c-a)^2\geq 0$$

$$(a^2+b^2+c^2)(ab+bc+ca)+6(ab+bc+ca)^2-9abc(a+b+c)\leq$$

$$\leq 4(a^2+b^2+c^2)(ab+bc+ca)$$

$$\sum_{cyc}\frac{1}{2a^2+bc}\geq\frac{(a+b+c)^2}{(a^2+b^2+c^2)(ab+bc+ca)}=\frac{2}{a^2+b^2+c^2}+\frac{1}{ab+bc+ca}$$

$$\textit{Equality holds for } a=b=c \textit{ or } (a,b,c)=(0,t,t), t>0.$$

1170. If $a,b,c>0$ then:

$$\sum_{cyc}\frac{a}{b(6(a^3+b^3)+10ab(a+b)+ac(19a+30b))}\geq\frac{1}{(a+b+c)^3}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by proposer

$$\sum_{cyc}\frac{a}{b(6(a^3+b^3)+10ab(a+b)+ac(19a+30b))}=$$

$$=\sum_{cyc}\frac{a^2}{ab(6(a^3+b^3)+10ab(a+b)+ac(19a+30b))}\overset{Bergstrom}{\geq}$$

$$\geq\frac{(a+b+c)^2}{\sum ab(6(a^3+b^3)+10ab(a+b)+ac(19a+30b))}\geq\frac{1}{(a+b+c)^3}\Leftrightarrow$$

$$(a+b+c)^5\geq\sum_{cyc}ab\left(6(a^3+b^3)+10ab(a+b)+ac(19a+30b)\right)\Leftrightarrow$$

$$\sum_{cyc}a^5+5\sum_{cyc}(a^4b+ab^4)+10\sum_{cyc}(a^3b^2+a^2b^3)+20\sum_{cyc}a^2bc+30\sum_{cyc}a^2b^2c\geq$$

$$\geq 6\sum_{cyc}(a^4b+ab^4)+10\sum_{cyc}a^3b^2+19\sum_{cyc}a^3bc+30\sum_{cyc}a^2b^2c$$

$$a^5+b^5+c^5+abc(a^2+b^2+c^2)\geq\sum_{cyc}(a^4b+ab^4)$$

$$\sum_{cyc}a^3(a-b)(a-c)\geq 0 \textit{ which is true by Schur's inequality for } n=3.$$

$$\textit{Equality holds for } a=b=c.$$

1171. If $a, b, c > 0$ then:

$$\sum_{cyc} \frac{a^3}{b+c} \geq \frac{(\sum_{cyc} a)(\sum_{cyc} a^3)}{2\sum_{cyc} bc}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$$We\ know\ if\ x, y, z \geq 0\ and\ p = x+y+z, q = xy+yz+zx\ and\ r = xyz$$
$$then\ p^4 - 5p^2q + 4q^2 + 6rp > 0;\ \ (1)$$

$$\sum_{cyc} \frac{a^3}{b+c} \geq \frac{(\sum a)(\sum a^3)}{2\sum ab} \Leftrightarrow 2\left(\sum_{cyc} \frac{a^3}{b+c}\right)\left(\sum_{cyc} ab\right) \geq \left(\sum_{cyc} a\right)\left(\sum_{cyc} a^3\right)$$

$$2\sum_{cyc} a^4 + 2\sum_{cyc} \frac{a^3bc}{b+c} \geq \left(\sum_{cyc} a\right)\left(\sum_{cyc} a^3\right)$$

$$2\left[\left(\sum_{cyc} a^2\right)^2 - 2\left(\left(\sum_{cyc} ab\right)^2 - 2abc\sum_{cyc} a\right)\right] + 2\sum_{cyc} \frac{a^3bc}{b+c} \geq$$

$$\geq \left(\sum_{cyc} a\right)\left[\left(\sum_{cyc} a\right)^3 - 3\sum_{cyc} a \cdot \sum_{cyc} ab + 3abc\right] \Leftrightarrow$$

$$2[(A^2-2B)^2 - 2(B^2-2AC)] + AC \geq A(A^3 - 3AB + 3C)$$

$$\because \sum_{cyc} \frac{a^3bc}{b+c} \geq abc(a+b+c), where\ A = a+b+c, B = ab+bc+ca\ and\ C = abc$$

$$\Leftrightarrow A^4 - 5A^2B + 4B^2 + 6AC \geq 0\ which\ is\ true\ from\ (1).$$

$$Therefore,\qquad \sum_{cyc} \frac{a^3}{b+c} \geq \frac{(\sum a)(\sum a^3)}{2\sum ab}$$

$$Equality\ holds\ for\ a = b = c.$$

Solution 2 by Soumava Chakraborty-Kolkata-India

Assigning $b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0$,
$y + z - x = 2a > 0$ and $z + x - y = 2b > 0 \Rightarrow x + y > z, y + z > x, z + x > y$
$\Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius
$= s, R, r$ (say) yielding $2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a \overset{(\bullet)}{=} s$

$$\Rightarrow a = s - x, b = s - y, c = s - z$$

Via such substitutions, $\sum_{cyc} ab = \sum_{cyc}(s-x)(s-y) = 4Rr + r^2$

$$\Rightarrow \sum_{cyc} ab \overset{(\bullet\bullet)}{=} 4Rr + r^2$$

$$\text{Also}, \sum_{cyc} a^3 = \left(\sum_{cyc} a\right)^3 - 3(a+b)(b+c)(c+a) = s^3 - 3xyz = s^3 - 12Rrs$$

$$\Rightarrow \sum_{cyc} a^3 \overset{(\bullet\bullet\bullet)}{=} s^3 - 12Rrs$$

$$\sum_{cyc} \frac{a^3}{b+c} = \sum_{cyc} \frac{(s-x)^3}{x} = \sum_{cyc} \frac{s^3 - x^3 - 3sx(s-x)}{x}$$

$$= \frac{s^3(s^2+4Rr+r^2)}{4Rrs} - 2(s^2-4Rr-r^2) - 3s\sum_{cyc}(s-x)$$

$$= \frac{s^2(s^2+4Rr+r^2)}{4Rr} - 2(s^2-4Rr-r^2) - 3s(3s-2s)$$

$$= \frac{s^2(s^2+4Rr+r^2) - 8Rr(s^2-4Rr-r^2) - 12Rrs^2}{4Rr}$$

$$\Rightarrow \sum_{cyc} \frac{a^3}{b+c} = \frac{s^4 - s^2(16Rr - r^2) + 8Rr^2(4R+r)}{4Rr}$$

$$\geq \frac{(\sum_{cyc} a)(\sum_{cyc} a^3)}{2\sum_{cyc} bc} \overset{\text{via } (\bullet),(\bullet\bullet),(\bullet\bullet\bullet)}{=} \frac{s^2(s^2-12Rr)}{2(4Rr+r^2)}$$

$$\Leftrightarrow \frac{s^4 - s^2(16Rr - r^2) + 8Rr^2(4R+r)}{4Rr} \geq \frac{s^2(s^2-12Rr)}{2(4Rr+r^2)}$$

$$\Leftrightarrow (2R+r)s^4 - rs^2(40R^2 + 12Rr - r^2) + 8Rr^2(4R+r)^2 \overset{(*)}{\geq} 0$$

$$\text{and} \because (2R+r)(s^2 - 16Rr + 5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0,$$

$$\therefore \text{in order to prove } (*), \text{it suffices to prove :}$$

$$(2R+r)s^4 - rs^2(40R^2 + 12Rr - r^2) + 8Rr^2(4R+r)^2$$
$$\geq (2R+r)(s^2 - 16Rr + 5r^2)^2$$

$$\Leftrightarrow (24R^2 - 9r^2)s^2 \overset{(**)}{\geq} r(384R^3 - 128R^2r - 118Rr^2 + 25r^3)$$

$$\text{Now, LHS of } (**) \overset{\text{Gerretsen}}{\geq} (24R^2 - 9r^2)(16Rr - 5r^2)$$

$$\overset{?}{\geq} r(384R^3 - 128R^2r - 118Rr^2 + 25r^3) \Leftrightarrow 2r(4R^2 - 13Rr + 10r^2) \overset{?}{\geq} 0$$

$$\Leftrightarrow 2r(4(R-2r)+3r)(R-2r) \overset{?}{\geq} 0 \rightarrow \text{true} \because R \overset{\text{Euler}}{\geq} 2r \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{cyc} \frac{a^3}{b+c} \geq \frac{(\sum_{cyc} a)(\sum_{cyc} a^3)}{2\sum_{cyc} bc} \ \forall\, a,b,c > 0, \text{with equality iff } a = b = c \text{ (QED)}$$

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\textit{For } a,b,c > 0, \textit{we get}: \sum_{cyc} \frac{a^3}{b+c} \geq \frac{(\sum a)(\sum a^3)}{2\sum ab} \Leftrightarrow$$

$$\frac{(\sum a^3)^2}{\sum(a^3b+ab^3)} \ge \frac{(\sum a)(\sum a^3)}{2\sum ab} \Leftrightarrow 2\left(\sum_{cyc} a^3\right)\left(\sum_{cyc} ab\right) \ge \left(\sum_{cyc} a\right)\left(\sum_{cyc} a^3b+ab^3\right)$$

$$\sum_{cyc}(a^4b+ab^4) \ge \sum_{cyc}(a^3b^2+a^2b^3), \text{which is true because:}$$

$$a^4b+ab^4 = ab(a^3+b^3) \ge ab(a^2b+ab^2) = a^3b^2+a^2b^3 \text{ (and analogs)}$$

Solution 4 by Daoudi Abdessattar-Tunisia

$$(p,q,r) = (\sum a, \sum ab, \prod a)$$

$$\sum_{cyc}\frac{a^3}{b+c} \ge \frac{(\sum a)(\sum a^3)}{2\sum ab} \Leftrightarrow 2q(\sum a^5 + q\sum a^3) \ge (pq-r)p\sum a^3 \Leftrightarrow$$

$$2q\sum a^5 \ge (p^2q-2q^2-rp)\sum a^3 \Leftrightarrow 2q\sum a^3 \ge (p^2q-2q^2-rp)p$$

$$p\sum a^5 \overset{CBS}{\ge} (\sum a^3)^2; (*)$$

$$\Leftrightarrow 6pr+2qp^3-6pq^2 \ge p^3q-2q^2p-rp^2$$

$$\Leftrightarrow qp^3-4pq^2+rp^2+6qr \ge 0; (p^2 \ge 3q)$$

$$\Leftrightarrow q(p^3-4pq+9r) \ge 0 \text{ true by Schur's inequality } p^3+9r \ge 4pq$$

1172. If $x,y,z,\lambda > 0$ then:

$$\sum_{cyc}\sqrt{\frac{x}{\lambda y+\lambda z-x}} \ge \frac{2}{\lambda}\sqrt{\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Rin Huynh-Vietnam

$$\sum_{cyc}\sqrt{\frac{x}{\lambda y+\lambda z-x}} = \sum_{cyc}\sqrt{\frac{x^2(\lambda+1)}{(\lambda+1)x(\lambda y+\lambda z-x)}} \overset{AM-GM}{\ge}$$

$$\ge \sum_{cyc}\frac{x\sqrt{\lambda+1}}{\frac{\lambda(x+y+z)}{2}} = \sum_{cyc}\frac{2x\sqrt{\lambda+1}}{\lambda(x+y+z)} = \frac{2}{\lambda}\sqrt{\lambda+1}$$

Equality holds for $x=y=z$.

Solution 2 by Marian Dincă-Romania

$$\sum_{cyc}\sqrt{\frac{x}{\lambda y+\lambda z-x}} = \sum_{cyc}\sqrt{\frac{x^2}{x(\lambda y+\lambda z-x)}} = \sum_{cyc}\frac{x}{\sqrt{\frac{(\lambda-(\lambda+1)x)(\lambda+1)x}{\lambda+1}}} =$$

$$=\sqrt{\lambda+1}\sum_{cyc}\frac{x}{\sqrt{(\lambda-(\lambda+1)x)(\lambda+1)x}}\geq\sqrt{\lambda+1}\sum_{cyc}\frac{x}{\frac{(\lambda-(\lambda+1)x)+(\lambda+1)x}{2}}=$$

$$=\sqrt{\lambda+1}\cdot\frac{2}{\lambda}\sum_{cyc}x=\frac{2\sqrt{\lambda+1}}{\lambda}$$

1173. *Let $a,b,c\geq 0, ab+bc+ca\neq 0$. Prove that:*

$$\frac{a}{bc+a^2}+\frac{b}{ca+b^2}+\frac{c}{ab+c^2}\leq\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$(*):\frac{a}{bc+a^2}+\frac{b}{ca+b^2}+\frac{c}{ab+c^2}\leq\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}$$

$$We\ have:\ (*)\Leftrightarrow\sum_{cyc}\frac{a(a+b)(b+c)(c+a)}{bc+a^2}\leq\sum_{cyc}(b+c)(c+a)$$

$$\Leftrightarrow\sum_{cyc}a(b+c)\left(1+\frac{a(b+c)}{bc+a^2}\right)\leq\sum_{cyc}(b+c)(c+a)\ \Leftrightarrow\sum_{cyc}\frac{a^2(b+c)^2}{bc+a^2}\leq\sum_{cyc}(bc+a^2)$$

$$We\ have:$$

$$\sum_{cyc}\frac{a^2(b+c)^2}{bc+a^2}=\sum_{cyc}\frac{a^2(b+c)^3}{b(c^2+a^2)+c(a^2+b^2)}\overset{CBS}{\leq}\sum_{cyc}a^2(b+c)\left(\frac{b}{c^2+a^2}+\frac{c}{a^2+b^2}\right)=$$

$$=\sum_{cyc}\left(\frac{b^2c(c+a)}{a^2+b^2}+\frac{a^2c(b+c)}{a^2+b^2}\right)=\sum_{cyc}\left(c^2+\frac{abc(a+b)}{a^2+b^2}\right)\overset{AM-GM}{\leq}\sum_{cyc}\left(c^2+\frac{c(a+b)}{2}\right)$$

$$=\sum_{cyc}(bc+a^2).$$

So the proof is done. Equality holds iff $a=b=c$.

Solution 2 by Nguyen Truong-Vietnam

$$\frac{a}{a^2+bc}=\frac{a}{\left(1+\frac{c}{b}\right)(a^2+bc)}+\frac{a}{\left(1+\frac{b}{c}\right)(a^2+bc)}\leq\frac{a}{(a+c)^2}+\frac{a}{(a+b)^2}$$

$$Similarly,\quad\frac{b}{b^2+ca}\leq\frac{b}{(b+c)^2}+\frac{b}{(a+b)^2}\ and\ \frac{c}{c^2+ab}\leq\frac{c}{(a+c)^2}+\frac{c}{(c+b)^2}$$

By adding above inequalities, we get the desired result.

Equality holds for $a = b = c > 0$.

1174. $\boldsymbol{If\ a, b, c > 0, a^2 + b^2 + c^2 = 3\ and\ 2 < \lambda \leq 4\ then:}$

$$\frac{1}{\lambda - \sqrt{ab}} + \frac{1}{\lambda - \sqrt{bc}} + \frac{1}{\lambda - \sqrt{ca}} \leq \frac{3}{\lambda - 1}$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Lemma:$$

$$If\ x \in \left(0, \sqrt{\frac{3}{2}}\right)\ then: \frac{1}{\lambda - x} \leq \frac{x^4 + 4\lambda - 5}{4(\lambda - 1)^2}\quad (1)$$

$$Proof:\ (1) \Leftrightarrow (x^4 + 4\lambda - 5)(\lambda - x) - 4(\lambda - 1)^2 \geq 0$$

$$\Leftrightarrow -x^5 + \lambda x^4 - (4\lambda - 5)x + 3\lambda - 4 \geq 0$$

$$\Leftrightarrow (x - 1)^2[3\lambda - 4 - x^3 + (\lambda - 2)x^2 + (2\lambda - 3)x] \geq 0,$$

$$which\ is\ true\ because:\ \lambda > 2\ and\ 3\lambda - 4 > 3.2 - 4 = 2 > \sqrt{\frac{3}{2}}^{\,3} > x^3.$$

$$Now, since\ \sqrt{ab} \overset{AM-GM}{\leq} \sqrt{\frac{a^2 + b^2}{2}} < \sqrt{\frac{3}{2}}\ then:$$

$$\frac{1}{\lambda - \sqrt{ab}} \overset{Lemma}{\leq} \frac{(ab)^2 + 4\lambda - 5}{4(\lambda - 1)^2}\quad (and\ analogs)$$

$$Therefore, \sum_{cyc} \frac{1}{\lambda - \sqrt{ab}} \leq \sum_{cyc} \frac{(ab)^2 + 4\lambda - 5}{4(\lambda - 1)^2} = \frac{\sum_{cyc}(ab)^2 + 3(4\lambda - 5)}{4(\lambda - 1)^2} \leq$$

$$\leq \frac{\frac{1}{3}(a^2 + b^2 + c^2)^2 + 12\lambda - 15}{4(\lambda - 1)^2} = \frac{3 + 12\lambda - 15}{4(\lambda - 1)^2} = \frac{3}{\lambda - 1},\quad as\ desired.$$

Equality holds iff $a = b = c = 1$.

1175. For *all* positive real numbers a, b, c such that :

$$a+b+c=ab+bc+ca \text{ prove:}$$

$$\frac{1}{(a+b)b}+\frac{1}{(b+c)c}+\frac{1}{(c+a)a}\geq\frac{3}{2\sqrt[3]{abc}}$$

Proposed by Nguyen Thuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{(a+b)b}+\frac{1}{(b+c)c}+\frac{1}{(c+a)a}\geq\frac{3}{2.\sqrt[3]{abc}}\overset{a+b+c=ab+bc+ca}{\Leftrightarrow}$$

$$(ab+bc+ca)\left(\frac{1}{(a+b)b}+\frac{1}{(b+c)c}+\frac{1}{(c+a)a}\right)\overset{(*)}{\geq}\frac{3}{2.\sqrt[3]{abc}}(a+b+c)$$

$$\text{Now, }2(ab+bc+ca)\left(\frac{1}{(a+b)b}+\frac{1}{(b+c)c}+\frac{1}{(c+a)a}\right)$$

$$=\left(\sum_{cyc}c(a+b)\right)\left(\sum_{cyc}\frac{1}{(a+b)b}\right)\overset{CBS}{\geq}\left(\sum_{cyc}\sqrt{\frac{c}{b}}\right)^2=\sum_{cyc}\frac{b}{a}+2\sum_{cyc}\sqrt{\frac{a}{b}}$$

$$=\left(\frac{b}{a}+\sqrt{\frac{b}{c}}+\sqrt{\frac{b}{c}}\right)+\left(\frac{c}{b}+\sqrt{\frac{c}{a}}+\sqrt{\frac{c}{a}}\right)+\left(\frac{a}{c}+\sqrt{\frac{a}{b}}+\sqrt{\frac{a}{b}}\right)$$

$$\overset{A-G}{\geq}3\left(\sqrt[3]{\frac{b}{a}\cdot\frac{b}{c}}+\sqrt[3]{\frac{c}{b}\cdot\frac{c}{a}}+\sqrt[3]{\frac{a}{c}\cdot\frac{a}{b}}\right)$$

$$=3\left(\sqrt[3]{\frac{b^3}{abc}}+\sqrt[3]{\frac{c^3}{abc}}+\sqrt[3]{\frac{a^3}{abc}}\right)=\frac{3}{\sqrt[3]{abc}}(a+b+c)$$

$$\Rightarrow(ab+bc+ca)\left(\frac{1}{(a+b)b}+\frac{1}{(b+c)c}+\frac{1}{(c+a)a}\right)\geq\frac{3}{2.\sqrt[3]{abc}}(a+b+c)$$

$$\Rightarrow(*)\text{ is true ; QED, equality iff }a=b=c=1$$

1176. *Let* $a,b,c>0: a^2+b^2+c^2=3abc$*. Prove that :*

$$\frac{a}{a^2+b+c}+\frac{b}{a+b^2+c}+\frac{c}{a+b+c^2}\leq 1$$

Proposed by Nguyen Thuong-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{We have}:\ a^2+b^2+c^2=3abc\overset{AM-GM}{\leq}3\sqrt{\left(\frac{a^2+b^2+c^2}{3}\right)^3}\Rightarrow a^2+b^2+c^2\geq 3\ (1)$$

$$We\ have:\sum_{cyc}\frac{a}{a^2+b+c}\overset{CBS}{\leq}\sum_{cyc}\frac{a(1+b+c)}{(a+b+c)^2}=\frac{\sum_{cyc}a+2\sum_{cyc}bc}{(a+b+c)^2}\overset{?}{\leq}1$$

$$\Leftrightarrow\ a+b+c\leq a^2+b^2+c^2,\ \ which\ is\ true\ because:$$

$$\sum_{cyc}a\overset{AM-GM}{\leq}\sum_{cyc}\frac{a^2+1}{2}=\frac{1}{2}\sum_{cyc}a^2+\frac{3}{2}\overset{(1)}{\leq}\sum_{cyc}a^2.$$

$$Equality\ holds\ iff\ (a,b,c)=(1,1,1).$$

1177. For $all\ a,\mathrm{b},\mathrm{c}\geq 0$ such that :

$$ab+\mathrm{bc}+\mathrm{c}a=1,\frac{4-3a\mathrm{bc}}{a+\mathrm{b}+\mathrm{c}}\leq 2$$

Proposed by Tran Quoc Thinh-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

We observe that either exactly one among $a,\mathrm{b},\mathrm{c}=0$ or $a,\mathrm{b},\mathrm{c}>0$

For the first scenario, WLOG we may assume $a=0\ \therefore\ \mathrm{bc}=1\ (\mathrm{b},\mathrm{c}>0)$

$$and\ \frac{4-3a\mathrm{bc}}{a+\mathrm{b}+\mathrm{c}}=\frac{4}{\mathrm{b}+\mathrm{c}}\overset{A-G}{\leq}\frac{4}{2\sqrt{\mathrm{bc}}}\overset{\mathrm{bc}=1}{=}2\ \therefore\ \boxed{\text{in the first case},\frac{4-3a\mathrm{bc}}{a+\mathrm{b}+\mathrm{c}}\leq 2},$$

with equality for $(a,\mathrm{b},\mathrm{c})=(0,1,1)$ or $(1,0,1)$ or $(1,1,0)$

We now shift our focus to : $a,\mathrm{b},\mathrm{c}>0$

Assigning $\mathrm{b}+\mathrm{c}=x,\mathrm{c}+a=\mathrm{y},a+\mathrm{b}=\mathrm{z}\Rightarrow x+\mathrm{y}-\mathrm{z}=2\mathrm{c}>0,$

$\mathrm{y}+\mathrm{z}-x=2a>0$ and $\mathrm{z}+x-\mathrm{y}=2\mathrm{b}>0\Rightarrow x+\mathrm{y}>z,\mathrm{y}+\mathrm{z}>x,\mathrm{z}+x>y$

$\Rightarrow x,\mathrm{y},\mathrm{z}$ form sidesof a triangle with semiperimeter, circumradius and inradius

$$=\mathrm{s},\mathrm{R},\mathrm{r}\ (say)\ \text{yielding}\ 2\sum_{\mathrm{cyc}}a=\sum_{\mathrm{cyc}}x=2\mathrm{s}\Rightarrow\sum_{\mathrm{cyc}}a=\mathrm{s}\rightarrow(1)$$

$$\Rightarrow a=\mathrm{s}-x,\mathrm{b}=\mathrm{s}-\mathrm{y},\mathrm{c}=\mathrm{s}-\mathrm{z}$$

$$\text{Via such substitutions,}\sum_{\mathrm{cyc}}ab=\sum_{\mathrm{cyc}}(\mathrm{s}-x)(\mathrm{s}-\mathrm{y})=4\mathrm{Rr}+\mathrm{r}^2\rightarrow(2)$$

$$and\ abc=(\mathrm{s}-x)(\mathrm{s}-\mathrm{y})(\mathrm{s}-\mathrm{y})=\mathrm{r}^2\mathrm{s}\rightarrow(3)$$

$$\text{Now,}\frac{4-3a\mathrm{bc}}{a+\mathrm{b}+\mathrm{c}}-2\leq 0\Leftrightarrow\frac{4-3a\mathrm{bc}-2\sum_{\mathrm{cyc}}a}{a+\mathrm{b}+\mathrm{c}}\leq 0$$

$$\overset{\because\ 1=ab+bc+ca}{\Leftrightarrow}\ 2\sum_{\mathrm{cyc}}a+\frac{3abc}{\sum_{\mathrm{cyc}}ab}\geq 4.\sqrt{\sum_{\mathrm{cyc}}ab}$$

$$\Leftrightarrow 2\left(\sum_{\mathrm{cyc}}a\right)\left(\sum_{\mathrm{cyc}}ab\right)+3abc\geq 4\left(\sum_{\mathrm{cyc}}ab\right).\sqrt{\sum_{\mathrm{cyc}}ab}$$

$$\overset{\text{via }(1),(2),(3)}{\Leftrightarrow}\ 2\mathrm{s}(4\mathrm{Rr}+\mathrm{r}^2)+3\mathrm{r}^2\mathrm{s}\geq 4(4\mathrm{Rr}+\mathrm{r}^2).\sqrt{4\mathrm{Rr}+\mathrm{r}^2}$$

$$\Leftrightarrow s(8R+5r) \geq 4(4R+r).\sqrt{4Rr+r^2}$$

$$\Leftrightarrow s^2(8R+5r) \geq 16(4Rr+r^2)(4R+r)^2 \overset{(*)}{\geq} 0$$

$$\text{Now, LHS of } (*) \overset{\text{Gerretsen}}{\geq} (8R+5r)(16Rr-5r^2) \overset{?}{\geq} 16(4Rr+r^2)(4R+r)^2$$

$$\Leftrightarrow 192R^2-192Rr-141r^2 \overset{?}{\geq} 0 \Leftrightarrow (R-2r)(192R+192r)+243r^2 \overset{?}{\geq} 0$$

$$\to \text{true} \because R \overset{\text{Euler}}{\geq} 2r \Rightarrow (*) \text{ is true (strict inequality)} \therefore \boxed{\frac{4-3abc}{a+b+c} < 2\ \forall\, a,b,c > 0}$$

$$\therefore \text{combining both cases, } \frac{4-3abc}{a+b+c} \leq 2\ \forall\, a,b,c \geq 0 \mid ab+bc+ca=1,$$
$$\text{with equality for } (a,b,c) = (0,1,1) \text{ or } (1,0,1) \text{ or } (1,1,0)\ \text{(QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{Let } q := ab+bc+ca = 1,\ \ p := a+b+c \geq \sqrt{3q} = \sqrt{3}\ \ \textit{and}\ \ r := abc.$$

$$\textit{The problem becomes to prove}: \frac{4-3r}{p} \leq 2\ \ \textit{or}\ \ 2p+3r \geq 4.$$

$$\textit{If } p \geq 2 \textit{ the inequality is obvious true.}$$

$$\textit{We assume now that } \sqrt{3} \leq p \leq 2.$$

$$\textit{By Schur's inequality, we have}: 9r \geq 4pq - p^3 = 4p - p^3.$$

$$\textit{Then}:\ 2p+3r \geq 2p + \frac{4p-p^3}{3} = 4 + \frac{(2-p)(p^2+2p-6)}{3} \geq 4.$$

$$\textit{So the proof is done. Equality holds iff } (a,b,c) = (1,1,0) \textit{ and permutation.}$$

1178. *Let $a,b,c \geq 0:\ a^2+b^2+c^2=3$. Prove that:*

$$a^2b+b^2c+c^2a+2abc+3 \geq \frac{8}{3}(ab+bc+ca)$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{The given inequality can be rewritten as follows}:$$

$$(a+b+c)(ab+bc+ca)+3 \geq \frac{8}{3}(ab+bc+ca)+(ab^2+bc^2+ca^2+abc).$$

$$Lemma: \ If\ a,b,c \geq 0\ then: \ ab^2+bc^2+ca^2+abc \leq \frac{4}{27}(a+b+c)^3.$$

$$Proof: \ WLOG\ we\ assume\ that\ b = mid\{a,b,c\}.$$

$$We\ have: \ ab^2+bc^2+ca^2+abc = -a(a-b)(b-c)+\frac{1}{2}.2b(a+c)^2 \leq$$

$$\overset{AM-GM}{\leq} 0+\frac{1}{2}.\left(\frac{2b+(a+c)+(a+c)}{3}\right)^3 = \frac{4}{27}(a+b+c)^3, \ as\ desired.$$

$$So\ it\ suffices\ to\ prove\ that:$$

$$(a+b+c)(ab+bc+ca)+3 \geq \frac{8}{3}(ab+bc+ca)+\frac{4}{27}(a+b+c)^3.$$

$$Let\ p := a+b+c\ and\ q := ab+bc+ca.\ We\ have: 3 = a^2+b^2+c^2 = p^2-2q.$$

$$So\ we\ need\ to\ prove: \ pq+3 \geq \frac{8}{3}q+\frac{4}{27}p^3\ \ or\ \ p.\frac{p^2-3}{2}+3 \geq \frac{8}{3}.\frac{p^2-3}{2}+\frac{4}{27}p^3$$

$$or\ \ \frac{19}{54}p^3-\frac{4}{3}p^2-\frac{3}{2}p+7 \geq 0\ \ or\ \ (p-3)^2\left(\frac{19}{54}p+\frac{7}{9}\right) \geq 0, \ which\ is\ true.$$

$$So\ the\ proof\ is\ done.\ Equality\ holds\ iff\ a=b=c=1.$$

1179. *Let $a,b,c,d > 0$ such that: $a^2+b^2+c^2+d^2 = abcd$. Prove that:*

$$\frac{a}{a^3+2bcd}+\frac{b}{b^3+2cda}+\frac{c}{c^3+2dab}+\frac{d}{d^3+abc} \leq \frac{1}{3}$$

Proposed by Tran Quoc Thinh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:$$

$$\frac{a}{a^3+2bcd} \overset{AM-GM}{\leq} \frac{a}{3\sqrt[3]{a^3(bcd)^2}} = \frac{a\sqrt[3]{bcd}}{3abcd} \overset{AM-GM}{\leq} \frac{ab+ac+ad}{3.3abcd} \leq$$

$$\overset{AM-GM}{\leq} \frac{(a^2+b^2)+(a^2+c^2)+(a^2+d^2)}{2.9abcd} = \frac{3a^2+b^2+c^2+d^2}{18abcd} \quad (and\ analogs)$$

$$\text{Therefore, } \sum_{cyc} \frac{a}{a^3+2bcd} \le \sum_{cyc} \frac{3a^2+b^2+c^2+d^2}{18abcd} = \frac{6(a^2+b^2+c^2+d^2)}{18abcd} = \frac{1}{3}.$$

$$\text{Equality holds iff } a=b=c=d=2.$$

1180. *If $a,b,c>0: a+b+c+1=4abc$ then:*

$$(b+c)\sqrt{a}+(c+a)\sqrt{b}+(a+b)\sqrt{c} \ge \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}$$

Proposed by Nguyen Thuong-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{We have:}$$

$$a+b+c+1 = a.4bc \overset{AM-GM}{\le} a(b+c)^2 \Leftrightarrow (b+c+1)\big(a(b+c)-(1+a)\big) \ge 0$$

$$\text{Then: } b+c \ge \frac{1+a}{a} \Rightarrow 4abc \ge a+1+\frac{1+a}{a} = \frac{(a+1)^2}{a} \overset{AM-GM}{\ge} \frac{2(a+1)}{\sqrt{a}}.$$

$$\text{Then: } \sqrt{a} \ge \frac{a+1}{2abc} \ (\text{and analogs})$$

$$\text{Therefore, } \sum_{cyc}(b+c)\sqrt{a} \ge \sum_{cyc} \frac{(a+1)(b+c)}{2abc} = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}.$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$4abc = a+b+c+1 \overset{A-G}{\ge} a+1+2\sqrt{bc} \Rightarrow 4a\left(\sqrt{bc}\right)^2 - 2\sqrt{bc}-(a+1)$$
$$\ge 0 \therefore \text{either } \sqrt{bc} \le \frac{2-\sqrt{4+16a(a+1)}}{8a} \text{ or } \sqrt{bc} \ge \frac{2+\sqrt{4+16a(a+1)}}{8a}$$
$$\Rightarrow \text{either } \sqrt{bc} \le \frac{1-(2a+1)}{4a} \text{ or } \sqrt{bc} \ge \frac{1+(2a+1)}{4a} \Rightarrow \text{either } \sqrt{bc} \le \frac{-1}{2}$$
$$\text{or } \sqrt{bc} \ge \frac{a+1}{2a} \text{ and } \because \sqrt{bc} > 0 \therefore \sqrt{bc} \ge \frac{a+1}{2a} \Rightarrow \sqrt{a}.\sqrt{a}.\sqrt{bc} \ge \frac{a+1}{2}$$
$$\Rightarrow \sqrt{a} \ge \frac{a+1}{2\sqrt{abc}} \Rightarrow (b+c)\sqrt{a} \ge \frac{(b+c)(a+1)}{2\sqrt{abc}} \text{ and analogs}$$
$$\Rightarrow \sum_{cyc}\left((b+c)\sqrt{a}\right) \ge \frac{1}{2\sqrt{abc}}\sum_{cyc}(b+c)(a+1) = \frac{1}{2\sqrt{abc}}\sum_{cyc}(ab+ca+b+c)$$

$$=\frac{2}{2\sqrt{abc}}\left(\sum_{cyc}a+\sum_{cyc}ab\right)\overset{?}{\geq}\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}=\frac{1}{abc}\left(\sum_{cyc}a+\sum_{cyc}ab\right)$$

$$\Leftrightarrow abc\overset{?}{\geq}\sqrt{abc}\Leftrightarrow abc\overset{?}{\geq}1\to \text{true} \because 4abc=a+b+c+1\overset{A-G}{\geq}4.\sqrt[4]{abc}$$

$$\Rightarrow abc\geq 1\therefore (b+c)\sqrt{a}+(c+a)\sqrt{b}+(a+b)\sqrt{c}\geq\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}$$

$\forall\, a,b,c>0\,\middle|\, a+b+c+1=4abc$, with equality iff $a=b=c$ occurring iff $3a+1=4a^3\Rightarrow$ iff $(a-1)(2a+1)^2=0\Rightarrow$ equality iff $a=b=c=1$ (QED)

1181. $If\ x,y,z>0,x+y+z=9\ then:$

$$\frac{x}{3x+yz}+\frac{y}{3y+zx}+\frac{z}{3z+xy}\geq\frac{1}{2}$$

Proposed by Tuan Kiet-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let\ a=\sqrt{\frac{yz}{x}},b=\sqrt{\frac{zx}{y}},c=\sqrt{\frac{xy}{z}}.\ We\ have:\ ab+bc+ca=9\ and,$$

$$the\ problem\ becomes\ to\ prove:\ \frac{1}{3+a^2}+\frac{1}{3+b^2}+\frac{1}{3+c^2}\geq\frac{1}{2}.$$

$$We\ have:\ \frac{1}{3+a^2}=\frac{1}{3}\left(1-\frac{a^2}{3+a^2}\right)=\frac{1}{3}-\frac{a^2}{ab+bc+ca+3a^2}=$$

$$=\frac{1}{3}-\frac{a^2}{a(a+b+c)+(2a^2+bc)}\overset{CBS}{\geq}\frac{1}{3}-\frac{1}{4}\left(\frac{a^2}{a(a+b+c)}+\frac{a^2}{2a^2+bc}\right)=$$

$$=\frac{1}{3}-\frac{a}{4(a+b+c)}-\frac{1}{8}\left(1-\frac{bc}{2a^2+bc}\right)=\frac{5}{24}-\frac{a}{4(a+b+c)}+\frac{bc}{8(2a^2+bc)}\ (and\ analogs)$$

$$Then:\ \sum_{cyc}\frac{1}{3+a^2}\geq\sum_{cyc}\left(\frac{5}{24}-\frac{a}{4(a+b+c)}+\frac{bc}{8(2a^2+bc)}\right)=\frac{5}{8}-\frac{1}{4}+\frac{1}{8}\sum_{cyc}\frac{(bc)^2}{2a^2bc+(bc)^2}\geq$$

$$\overset{CBS}{\geq}\frac{3}{8}+\frac{1}{8}\cdot\frac{\left(\sum_{cyc}bc\right)^2}{\sum_{cyc}(2a^2bc+(bc)^2)}=\frac{3}{8}+\frac{1}{8}=\frac{1}{2}.$$

$$So\ the\ proof\ is\ done.\ Equality\ holds\ iff\ x=y=z=3.$$

1182. For *all* positive real numbers x, y, z such that :

$$x + y + z = 1, \textit{prove that}$$

$$\frac{1}{(x+yz)^2} + \frac{1}{(y+zx)^2} + \frac{1}{(z+xy)^2} \le \frac{9}{16xyz}$$

Proposed by Nguyen Thuong-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{1}{(x+yz)^2} + \frac{1}{(y+zx)^2} + \frac{1}{(z+xy)^2} \overset{x+y+z=1}{=} \sum_{cyc} \frac{1}{(1-y-z+yz)^2}$$

$$= \sum_{cyc} \frac{(1-x)^2}{(1-x)^2(1-y)^2(1-z)^2} \overset{x+y+z=1}{=} \frac{\sum_{cyc}(y+z)^2}{(x+y)^2(y+z)^2(z+x)^2}$$

$$\overset{(x+y)(y+z)(z+x) \ge \frac{8}{9}(\sum_{cyc} x)(\sum_{cyc} xy)}{\le} \frac{2\sum_{cyc} x^2 + 2\sum_{cyc} xy}{\frac{8}{9}(\sum_{cyc} x)(\sum_{cyc} xy)(x+y)(y+z)(z+x)}$$

$$\overset{x+y+z=1}{=} \frac{2\sum_{cyc} x^2 + 2\sum_{cyc} xy}{\frac{8}{9}(\sum_{cyc} xy)(x+y)(y+z)(z+x)} \overset{?}{\le} \frac{9}{16xyz}$$

$$\Leftrightarrow \left(\sum_{cyc} xy\right)(x+y)(y+z)(z+x) \overset{?}{\ge} 2xyz\left(2\sum_{cyc} x^2 + 2\sum_{cyc} xy\right)$$

$$\Leftrightarrow \left(\sum_{cyc} xy\right)\left(2xyz + \sum_{cyc} xy(x+y)\right) \overset{?}{\ge} 2xyz\left(2\sum_{cyc} x^2 + 2\sum_{cyc} xy\right)$$

$$\Leftrightarrow \left(\sum_{cyc} xy\right)\left(\sum_{cyc} xy(x+y)\right) \underset{(*)}{\overset{?}{\ge}} 2xyz\left(2\sum_{cyc} x^2 + \sum_{cyc} xy\right)$$

$$\text{Now,} \left(\sum_{cyc} xy\right)\left(\sum_{cyc} xy(x+y)\right)$$

$$= (xy + yz + zx)\big(xy(x+y) + yz(y+z) + zx(z+x)\big)$$

$$= \sum_{cyc} x^2y^2(x+y) + \sum_{cyc}\big(xy.yz.(y+z)\big) + \sum_{cyc}\big(xy.zx.(z+x)\big)$$

$$= \sum_{cyc} x^3(y^2+z^2) + xyz\sum_{cyc}(y^2+yz) + xyz\sum_{cyc}(x^2+zx)$$

$$\overset{A-G}{\ge} \sum_{cyc} x^3.2yz + xyz\left(2\sum_{cyc} x^2 + 2\sum_{cyc} xy\right)$$

$$= 2xyz\sum_{cyc} x^2 + xyz\left(2\sum_{cyc} x^2 + 2\sum_{cyc} xy\right) = 2xyz\left(2\sum_{cyc} x^2 + \sum_{cyc} xy\right)$$

$$\Rightarrow (*) \text{ is true } \therefore \frac{1}{(x+yz)^2} + \frac{1}{(y+zx)^2} + \frac{1}{(z+xy)^2} \le \frac{9}{16xyz}$$

$$\forall\, x, y, z > 0 \,\middle|\, x + y + z = 1, \text{equality iff } x = y = z = \frac{1}{3} \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

The given inequality is successively equivalent to:

$$\frac{1}{[(x+y)(x+z)]^2} + \frac{1}{[(y+z)(y+x)]^2} + \frac{1}{[(z+x)(z+y)]^2} \le \frac{9}{16xyz(x+y+z)}$$

$$16xyz(x+y+z)\left[(x+y)^2 + (y+z)^2 + (z+x)^2\right] \le 9(x+y)^2(y+z)^2(z+x)^2.$$

Using Ravi's substitution, $\exists \Delta ABC$ *such that:* $a = x+y,\ b = y+z,\ c = z+x.$

Let F, R *be the area, circumradius of* ΔABC *and* $s = \frac{a+b+c}{2} = x+y+z.$

By Heron's formula, we have: $F^2 = s(s-a)(s-b)(s-c) = xyz(x+y+z).$

So the given inequality is equivalent to:

$$16F^2(a^2+b^2+c^2) \le 9(abc)^2 = 9(4RF)^2 \Leftrightarrow a^2+b^2+c^2 \le 9R^2,$$

which is Leibniz's inequality.

So the proof is done. Equality holds if $x = y = z = \frac{1}{3}.$

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\frac{1}{(x+yz)^2} + \frac{1}{(y+zx)^2} + \frac{1}{(z+xy)^2} \le \frac{9}{16xyz}$$

$$\frac{1}{(x+y)^2(x+z)^2} + \frac{1}{(y+x)^2(y+z)^2} + \frac{1}{(z+x)^2(z+y)^2} \le \frac{9}{16xyz}; x+y+z = 1$$

$$(x+y)^2 + (y+z)^2 + (z+x)^2 \le \frac{9}{16xyz}(x+y)(y+z)(z+x)(x+y)(y+z)(z+x)$$

$$2(x^2+y^2+z^2+xy+yz+zx) \le$$

$$\le \frac{9}{16xyz}(x+y)(y+z)(z+x)\cdot\frac{(xy+yz+zx)(x+y+z)}{9}$$

$$4(x^2+y^2+z^2+xy+yz+zx) \le \left(\frac{x}{y}+\frac{y}{x}+\frac{y}{z}+\frac{z}{y}+\frac{x}{z}+\frac{z}{x}\right)(xy+yz+zx); (xyz=1)$$

$$2(x^2+y^2+z^2) \le \frac{xy^2}{z}+\frac{yz^2}{x}+\frac{zx^2}{y}+\frac{x^2y}{z}+\frac{y^2z}{x}+\frac{z^2x}{y}$$

$$2(x^3yz+y^3zx+z^3xy) \le x^2y^3+y^2z^3+x^2y^2+y^3z^2+z^3x^2 \text{ true because}:$$

$$x^2y^3+z^2y^3=y^3(x^2+z^2) \ge 2y^3zx \text{ and analogs.}$$

Solution 4 by Michael Sterghiou-Greece

$$\sum_{cyc}\frac{1}{(x+yz)^2} \le \frac{9}{16xyz}; \ (1)$$

$$\text{Let } (p,q,r)=(\textstyle\sum x, \sum xy, \prod x), p=1, q \le \frac{1}{3}, r \le \frac{1}{27}; \ (2)$$

$$\frac{1}{x+yz}=\frac{1}{x(x+y+z)+yz}=\frac{1}{(x+y)(x+z)}=\frac{y+z}{\prod(x+y)}$$

$$\text{So}, (1) \Rightarrow \frac{\sum(x+y)^2}{\prod(x+y)^2} \le \frac{9}{16r} \text{ or } \frac{2(1-q)}{(q-r)^2} \le \frac{9}{16r}$$

$$f(r)=-9q^2-14qr-9r^2+32r \le 0; (3)$$

$$f'(r)=2(16-7q-9r) \overset{(2)}{>} 0 \Rightarrow f(r) \nearrow$$

$$\text{So}, f(r) \text{ is maximal when } r \text{ is maximum.}$$

$$q=(2-3y)y, r=y^2(1-2y)$$

$$\text{From (3), we get } f(r)=-4(1-3y)^2y^2(1+y)^2 \le 0. \text{ We are done!}$$

$$\text{Equality holds for } y=\frac{1}{3}, z=\frac{1}{3} \text{ and } x=\frac{1}{3}.$$

1183. If $a, b, c \ge 0$ such that : $a+b+c=3$ then

$$\frac{a}{1+3b^2}+\frac{b}{1+3c^2}+\frac{c}{1+3a^2} \ge \frac{3}{4}$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{We have}: \frac{a}{1+3b^2}=a-\frac{3ab^2}{1+3b^2} \overset{AM-GM}{\ge} a-\frac{3ab^2}{4\sqrt[4]{(b^2)^3}}=a-\frac{3a\sqrt{b}}{4} \overset{AM-GM}{\ge} a-\frac{3(a+ab)}{8}.$$

$$\text{Then}: \frac{a}{1+3b^2} \ge \frac{5a-3ab}{8} \ (\text{and analogs})$$

$$\text{Therefore}, \ \sum_{cyc}\frac{a}{1+3b^2} \ge \sum_{cyc}\frac{5a-3ab}{8}=\frac{5.3}{8}-\frac{3}{8}\sum_{cyc}ab \ge \frac{15}{8}-\frac{(a+b+c)^2}{8}=\frac{15}{8}-\frac{9}{8}=\frac{3}{4}.$$

$$\text{Equality holds iff } (a,b,c)=(1,1,1).$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\frac{a}{1+3b^2}+\frac{b}{1+3c^2}+\frac{c}{1+3a^2}\geq\frac{3}{4}=\frac{a+b+c}{4}\Leftrightarrow\sum_{cyc}a\left(\frac{1}{1+3b^2}-\frac{1}{4}\right)\geq 0$$

$$\Leftrightarrow\frac{3}{4}\sum_{cyc}a\left(\frac{1-b^2}{1+3b^2}\right)\geq 0\Leftrightarrow\sum_{cyc}a\left(\frac{1-b^2}{1+3b^2}\right)\overset{(*)}{\geq}0$$

$$\text{Now}\frac{1-b^2}{1+3b^2}\overset{?}{\geq}\frac{1-b}{2}\Leftrightarrow(1-b)\left(\frac{1+b}{1+3b^2}-\frac{1}{2}\right)\overset{?}{\geq}0\Leftrightarrow\frac{(1-b)(1-2b-3b^2)}{2(1+3b^2)}\overset{?}{\geq}0$$

$$\Leftrightarrow\frac{(1-b)^2(1+3b)}{2(1+3b^2)}\overset{?}{\geq}0\to\text{true}\because b\geq 0$$

$$\therefore a\left(\frac{1-b^2}{1+3b^2}\right)\geq a\left(\frac{1-b}{2}\right)\ and\ analogs\ (\because a\geq 0\text{ etc})$$

$$\Rightarrow\sum_{cyc}a\left(\frac{1-b^2}{1+3b^2}\right)\geq\sum_{cyc}a\left(\frac{1-b}{2}\right)=\frac{1}{2}\sum_{cyc}a-\frac{1}{2}\sum_{cyc}ab$$

$$\geq\frac{1}{2}\sum_{cyc}a-\frac{1}{6}\left(\sum_{cyc}a\right)^2\overset{a+b+c=3}{=}\frac{3}{2}-\frac{9}{6}=0\Rightarrow(*)\text{ is true}$$

$$\therefore\frac{a}{1+3b^2}+\frac{b}{1+3c^2}+\frac{c}{1+3a^2}\geq\frac{3}{4}\ \forall a,b,c\geq 0\,|\,a+b+c=3,$$
$$\text{with equality iff } a=b=c=1\text{ (QED)}$$

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

$$1)\ If\ a=b=0,c=3,\frac{c}{1+3a^2}=\frac{3}{1}=3\geq\frac{3}{4}$$

$$2)\ If\ a=0,b+c=3,\frac{b}{1+3c^2}+\frac{c}{1}\geq\frac{3}{4}$$

$$Iff\ b+c+3c^3\geq\frac{3}{4}(1+3c^2)$$

$$Iff\ 4(b+c)+12c^3\geq 3+9c^2$$

$$9+12c^3\geq 9c^2$$

$$3+4c^3\geq 3c^2$$

$$2+1+c^3+c^3+2c^3\geq 3c^2\ which\ is\ true.$$

$$3)\ If\ a,b,c>0,\frac{a}{1+3b^2}+\frac{b}{1+3c^2}+\frac{c}{1+3a^2}$$

$$\geq\frac{a+b+c}{3}\left(\frac{1}{1+3a^2}+\frac{1}{1+3b^2}+\frac{1}{1+3c^2}\right)\geq\frac{3}{4}$$

$$\frac{1}{a\left(\frac{1}{a}+3a\right)}+\frac{1}{b\left(\frac{1}{b}+3b\right)}+\frac{1}{c\left(\frac{1}{c}+3c\right)}\geq\frac{3}{4}$$

$$\frac{\frac{1}{a^2}}{\frac{1}{a^2}+3}+\frac{\frac{1}{b^2}}{\frac{1}{b^2}+3}+\frac{\frac{1}{c^2}}{\frac{1}{c^2}+3}\ge\frac{3}{4}$$

$$\frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+9}\ge\frac{3}{4}$$

$$4\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+2\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)\right)\ge 3\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+9\right)$$

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\ge 3$$

$$\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\ge 3$$

$$\textit{Equality holds for } a=b=c=1.$$

1184. ***Let*** $a,b,c\ge 0: ab+bc+ca=1.$ ***Prove that:***

$$\frac{6a^2-a+bc}{2a+bc}+\frac{6b^2-b+ca}{2b+ca}+\frac{6c^2-c+ab}{2c+ab}\le\frac{3(a+b+c)^2}{2}.$$

Proposed by Nguyen Thuong-Vietnam,

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\frac{6a^2-a+bc}{2a+bc}\le\frac{3}{2}(a+b+c)^2$$

$$\sum_{cyc}\frac{4a^2+bc}{2a+bc}\le a^2+b^2+c^2+\sqrt{3}+\sum_{cyc}(a+b)(a+c)\ \textit{true because}:$$

$$\frac{4a^2+bc}{2a+bc}\le a^2+1$$

$$4a^2+bc\le 2a^3+a^2bc+2a+bc$$

$$4a^2\le 2a^2+2a+a^2bc;\ a^2bc\ge 0$$

$$\frac{4b^2+ca}{2b+ca}\le b^2+1,\qquad 4b^2+ca\le 2b^3+b^2ca+2b+ca$$

$$\frac{4c^2+ab}{2c+ab}\le c^2+1,\qquad 4c^2+ab\le 2c^3+c^2ab+2c+ab$$

$$4c^2 \le 2c^3 + 2c + c^2ab$$

Solution 2 by Sanonog Huayrerai-Nakon Pathom-Thailand

$$We\ have: \frac{6a^2 - a + bc}{2a + bc} = \frac{1}{2}\left(3.\frac{4a^2 + bc}{2a + bc} - 1\right) \overset{AM-GM}{\le} \frac{1}{2}\left(3.\frac{2a^3 + 2a + bc}{2a + bc} - 1\right) =$$

$$= \frac{1}{2}\left(3.\frac{(2a + bc)(a^2 + 1) - a^2bc}{2a + bc} - 1\right) \le \frac{1}{2}[3(a^2 + 1) - 1]$$

$$= \frac{3a^2 + 2}{2}\ \ (and\ analogs)$$

$$Therefore,\ \sum_{cyc} \frac{6a^2 - a + bc}{2a + bc} \le \sum_{cyc} \frac{3a^2 + 2}{2} = \frac{3(a^2 + b^2 + c^2 + 2)}{2} = \frac{3(a + b + c)^2}{2}.$$

$$Equality\ holds\ iff\ (a, b, c) = (1, 1, 0)\ and\ permutation.$$

1185. **If $a, b, c > 0$ then:**

$$\sum_{cyc} \frac{a^5 + b^5}{a^2b^2(a^3 + b^3)} \ge \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$$

Proposed by Zaza Mzavanadze-Georgia

Solution 1 by Tapas Das-India

$$\frac{a^5 + b^5}{a^2b^2(a^3 + b^3)} - \frac{1}{2}\left(\frac{1}{a^2} + \frac{1}{b^2}\right) = \frac{a^5 + b^5}{a^2b^2(a^3 + b^3)} - \frac{a^2 + b^2}{2a^2b^2} =$$

$$= \frac{2(a^5 + b^5) - (a^2 + b^2)(a^3 + b^3)}{2a^2b^2(a^3 + b^3)} = \frac{a^5 - a^2b^3 - a^3b^2 + b^5}{2a^2b^2(a^3 + b^3)} =$$

$$= \frac{a^2(a^3 - b^3) - b^2(a^3 - b^3)}{2a^2b^2(a^3 + b^3)} = \frac{(a + b)(a - b)(a - b)(a^2 + ab + b^2)}{2a^2b^2(a^3 + b^3)} > 0$$

$$\frac{a^5 + b^5}{a^2b^2(a^3 + b^3)} \ge \frac{1}{2}\left(\frac{1}{a^2} + \frac{1}{b^2}\right)\ (and\ analogs)$$

$$Therefore,$$

$$\sum_{cyc} \frac{a^5 + b^5}{a^2b^2(a^3 + b^3)} \ge \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{By Chebyshev's inequality, we have: } a^5+b^5 \geq \frac{1}{2}(a^3+b^3)(a^2+b^2)$$

$$\text{Then: } \frac{a^5+b^5}{a^2b^2(a^3+b^3)} \geq \frac{a^2+b^2}{2a^2b^2} = \frac{1}{2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right).$$

Similarly, we have:

$$\frac{b^5+c^5}{b^2c^2(b^3+c^3)} \geq \frac{1}{2}\left(\frac{1}{b^2}+\frac{1}{c^2}\right) \text{ and } \frac{c^5+a^5}{c^2a^2(c^3+a^3)} \geq \frac{1}{2}\left(\frac{1}{c^2}+\frac{1}{a^2}\right).$$

Adding these inequalities yields the desired result.

Equality holds iff $a=b=c$.

Solution 3 by Daoudi Abdessatar-Tunisia

$$\sum_{cyc}\frac{a^5+b^5}{a^2b^2(a^3+b^3)} = \frac{1}{2}\sum_{cyc}\frac{\left(\sqrt[5]{\frac{a^5+b^5}{2}}\right)^5}{\left(\sqrt{\frac{a^2+b^2}{2}}\right)^2\left(\sqrt[3]{\frac{a^3+b^3}{2}}\right)^3}\left(\frac{1}{a^2}+\frac{1}{b^2}\right) \overset{\text{Power Means}}{\geq}$$

$$\geq \frac{1}{2}\sum_{cyc}\left(\frac{1}{a^2}+\frac{1}{b^2}\right) = \sum_{cyc}\frac{1}{a^2}$$

1186. Let $a,b,c \geq 0$ such that $a+b+c=3$. Prove that:

$$\frac{1}{36+ba(a+b)}+\frac{1}{36+bc(b+c)}+\frac{1}{36+ca(c+a)} \geq \frac{3}{38}$$

Proposed by Tran Quoc Thinh-Vietnam

Solution 1 by Samir Zaakouni-Morocco

$$S = \frac{1}{36+ba(a+b)}+\frac{1}{36+bc(b+c)}+\frac{1}{36+ca(c+a)}$$

$$S \geq \frac{3}{38} \Leftrightarrow \frac{c}{36c+abc(a+b)}+\frac{a}{36a+abc(b+c)}+\frac{b}{36b+abc(c+a)} \geq \frac{3}{38}$$

$$\frac{a+b+c}{3} \geq \sqrt[3]{abc} \Rightarrow 0 \leq abc \leq 1$$

$$36c+abc(a+b) \leq a+b+36c$$

$$36+abc(a+b)\le 35c+3$$

$$\frac{c}{36+abc(a+b)}\ge\frac{c}{35c+3}$$

$$\frac{a}{36a+abc(b+c)}\ge\frac{a}{35a+3}\ and\ \frac{b}{36b+abc(c+a)}\ge\frac{b}{35b+3}$$

$$S\ge\frac{a}{35a+3}+\frac{b}{35b+3}+\frac{c}{35c+3}$$

$$Let: f(a,b,c)=\frac{a}{35a+3}+\frac{b}{35b+3}+\frac{c}{35c+3}, a+b+c=3,(a,b,c)\in\mathbb{R}^*.$$

$$\begin{cases}\min f(a,b,c)=\frac{a}{35a+3}+\frac{b}{35b+3}+\frac{c}{35c+3}\\ a+b+c=3,(a,b,c)\in\mathbb{R}^*\end{cases}$$

$$L(a,b,c,\lambda)=\frac{a}{35a+3}+\frac{b}{35b+3}+\frac{c}{35c+3}-\lambda(a+b+c-3)$$

$$\begin{cases}\frac{dL(a,b,c,\lambda)}{da}=0\Leftrightarrow\frac{3}{(35a+3)^2}-\lambda=0\Leftrightarrow\lambda=\frac{3}{(35a+3)^2};(1)\\ \frac{dL(a,b,c,\lambda)}{db}=0\Leftrightarrow\frac{3}{(35b+3)^2}-\lambda=0\Leftrightarrow\lambda=\frac{3}{(35b+3)^2};(2)\\ \frac{dL(a,b,c,\lambda)}{dc}=0\Leftrightarrow\frac{3}{(35c+3)^2}-\lambda=0\Leftrightarrow\lambda=\frac{3}{(35c+3)^2};(3)\\ \frac{dL(a,b,c,\lambda)}{d\lambda}=0\Leftrightarrow 3=a+b+c;\ \ (*)\end{cases}$$

$$\begin{cases}\frac{3}{(35b+3)^2}=\frac{3}{(35a+3)^2}\\ \frac{3}{(35c+3)^2}=\frac{3}{(35a+3)^2}\end{cases}\Rightarrow\begin{cases}b=a\\ c=a\end{cases}\Rightarrow a=b=c.$$

$$3-3a=0\Rightarrow a=b=c=1.$$

$$\min_{a+b+c=3}\{f(a,b,c)\}=\frac{1}{38}+\frac{1}{38}+\frac{1}{38}=\frac{3}{38}.$$

$$Therefore, S\ge\frac{3}{38}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let\ x=ab(a+b),y=bc(b+c),z=ca(c+a),\ \ p=a+b+c=3,q=ab+bc+ca,r=abc.$$

$$We\ have:\ (*)\ \Leftrightarrow \sum_{cyc}\frac{1}{36+x}\ge\frac{3}{38}\Leftrightarrow 38\sum_{cyc}(36+y)(36+y)\ge 3\prod_{cyc}(36+x)$$

$$\Leftrightarrow 7776\ge 3xyz+1152(x+y+z)+70(xy+yz+zx),\ \ with:$$

$$xyz=(abc)^2\prod_{cyc}(b+c)=r^2(pq-r)=r^2(3q-r),\ \ x+y+z=pq-3r$$
$$=3(q-r)$$

$$xy+yz+zx=abc\sum_{cyc}a(a+b)(a+c)=r(p^3-2pq+3r)=3r(9-2q+r).$$

So the given inequality is equivalent to:

$$f(r)=r^3-(70+3q)r^2+(522+140q)r+2592-1152q\ge 0\ \ (1)$$

We have:

$$f'(r)=3r^2+140q+[522-2(70+3q)r]\overset{q\le 3\ \&\ r\le 1}{\ge}0, then\ f\ is\ increasing.$$

$$If\ q\le\frac{9}{4}, we\ have:$$

$$f(r)=r^3+140qr+[522-(70+3q)r]r+1152\left(\frac{9}{4}-q\right)\overset{r\le 1}{\ge}0.$$

$$If\ \frac{9}{4}\le q\le 3,\ \ by\ fourth\ degree\ Schur's\ inequality, we\ have:$$

$$r\ge\frac{5p^2q-p^4-4q^2}{6p}=\frac{-4q^2+45q-81}{18},\ \ then:$$

$$f(r)\ge f\left(\frac{-4q^2+45q-81}{18}\right)=$$

$$=\frac{(3-q)(4q-9)[16q^4+3q^2(1953-80q)+243(1125-182q)]}{18^3}\ge 0.$$

$$Equality\ holds\ iff\ (a,b,c)=(1,1,1)\ or\ \left(\frac{3}{2},\frac{3}{2},0\right)\ and\ permutation.$$

1187. If $a, b > 0$ and $n \in \mathbb{N}$ then:

$$\left(\frac{2a^3}{b\sqrt{b^4+3a^4}}\right)^n + \left(\frac{2b^3}{a\sqrt{a^4+3b^4}}\right)^n \geq 2$$

Proposed by Marin Chirciu-Romania

Solution by Rin Huynh-Vietnam

$$\left(\frac{2a^3}{b\sqrt{b^4+3a^4}}\right)^n + \left(\frac{2b^3}{a\sqrt{a^4+3b^4}}\right)^n \overset{Power\ Means;CBS}{\geq} \frac{1}{2^{n-1}}\left(\frac{2a^3}{b\sqrt{b^4+3a^4}} + \frac{2b^3}{a\sqrt{a^4+3b^4}}\right)^n$$

$$= 2\left(\frac{2a^3}{b\sqrt{b^4+3a^4}} + \frac{2b^3}{a\sqrt{a^4+3b^4}}\right) \geq \frac{2(a^2+b^2)^2}{ab\sqrt{b^4+3a^4}+ab\sqrt{a^4+3b^4}} \geq$$

$$\geq \frac{2(a^4+2a^2b^2+b^4)}{ab\sqrt{2(b^4+3a^4+a^4+3b^4)}} \geq \frac{4\sqrt{2a^2b^2(a^4+b^4)}}{ab\sqrt{8(a^4+b^4)}} = 2$$

Equality holds for $a = b$.

1188. If $0 < a_1, a_2, \ldots, a_n, a_1 + a_2 + \cdots + a_n = 3, n \in \mathbb{N}$ then:

$$\sqrt[3]{na_1+1} + \sqrt[3]{na_2+1} + \cdots + \sqrt[3]{na_n+1} < n\sqrt[3]{4}$$

Proposed by Sebastian Ilinca-Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$$x \to \sqrt[3]{x}\ is\ a\ concave\ function\ and\ \sum_{i=1}^{n} a_i = 3(given)$$

$$\therefore \sum_{i=1}^{n} \sqrt[3]{na_i+1} \leq n\sqrt[3]{\frac{1}{n}\sum_{i=1}^{n}(na_i+1)} = n\sqrt[3]{\frac{1}{n}\left\{n\sum_{i=1}^{n} a_i + n\right\}} = n\sqrt[3]{\frac{3n+n}{n}} = n\sqrt[3]{4}$$

$$equality\ at\ a_1 = a_2 = \cdots = a_n = \frac{3}{n}$$

Solution 2 by Tapas Das-India

$$\sqrt[3]{na_1+1} + \sqrt[3]{na_2+1} + \cdots + \sqrt[3]{na_n+1} \overset{CBS}{\leq}$$

$$\leq n\sqrt[3]{\frac{(na_1+1)+(na_2+1)+\cdots+(na_n+1)}{n}}$$

$$= n\sqrt[3]{(a_1+a_2+\cdots+a_n)+1} = n\sqrt[3]{3+1} = n\sqrt[3]{4}$$

$$\because \frac{x_1^m + x_2^m + \cdots + x_n^m}{n} \le \left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)^m, when\ 0 < m < 1.$$

1189. If $a, b, c > 0, ab + bc + ca = 3$ and $\lambda \ge 0$ then:

$$\frac{bc}{1+\lambda a^2} + \frac{ca}{1+\lambda b^2} + \frac{ab}{1+\lambda c^2} \ge \frac{3}{\lambda + 1}$$

Proposed by Marin Chirciu-Romania

Solution by Tran Quoc Anh-Vietnam

$$We\ have:$$

$$(ab+bc+ca)^2 \ge 3abc(a+b+c) \Rightarrow 3 \ge abc(a+b+c); (1)$$

$$Alternatively:$$

$$\sum_{cyc} \frac{ab}{1+\lambda c^2} = \sum_{cyc} \frac{(ab)^2}{ab + \lambda abc^2} \overset{Radon}{\ge} \frac{(ab+bc+bc)^2}{ab+bc+ca+\lambda abc(a+b+c)} \overset{(1)}{\ge}$$

$$\ge \frac{3^2}{3+3\lambda} = \frac{3}{1+\lambda}$$

$$Equality\ holds\ iff\ a = b = c = 1.$$

1190. *Given $a, b, c > 0 : abc = 1$. Prove that :*

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \ge \frac{(a+b)(2-ab) + (b+c)(2-bc) + (c+a)(2-ca)}{4}$$

Proposed by Nguyen Thuong-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$The\ given\ inequality\ can\ be\ rewritten\ as\ follows:$$

$$\left(\frac{1}{a+b} + \frac{c^2(a+b)}{4}\right) + \left(\frac{1}{b+c} + \frac{a^2(b+c)}{4}\right) + \left(\frac{1}{c+a} + \frac{b^2(c+a)}{4}\right) \ge a+b+c.$$

$$By\ AM-GM\ inequality, we\ have: \frac{1}{a+b} + \frac{c^2(a+b)}{4} \ge c.$$

$$Adding\ this\ inequality\ with\ similar\ ones\ yields\ the\ desired\ result.$$

$$Equality\ holds\ iff\ a = b = c = 1.$$

Solution 2 by Michael Sterghiou-Greece

$$\sum_{cyc} \frac{1}{a+b} \ge \frac{1}{4} \sum_{cyc} (a+b)(2-ab); \ (1)$$

$$Let\ (p,q,r)=\left(\sum a,\sum ab,\prod a\right),r=1,p,q\geq 3,then\ (1)\ reduces\ to:$$

$$\frac{\sum(a+b)(b+c)}{\prod(a+b)}\geq\frac{1}{4}\left[4\left(\sum a\right)-\sum\left(a^2b+ab^2\right)\right]\ or\ \frac{p^2+q}{pq-1}\geq\frac{4p-pq+3}{4}$$

$$\sum ab(a+b)=pq-3r=pq-3;\ \prod(a+b)=pq-r=pq-1\ or$$

$$f(q)=4\left(p^2+q\right)-(pq-1)(4p-pq+3)\geq 0;\ \ (2)$$

$$f'(q)=2p^2(q-2)-4p+4=2[p(p(q-2)-2)+2]\geq 0\ as\ p\geq 3\ and$$

$$q-2\geq 1,hence\ f-increasing\ function.$$

$$Assume\ p=3t^2,t\geq 1\ then\ as\ q^2\geq 3pr=3p=9t^2\ or\ q\geq 3t\ and$$

$$f(q)\geq f(3t)=\cdots=(t-1)\left[(t-1)\left(27t^4+54t^3+45t^2+24t+19\right)+18\right]\geq 0.Done!$$

$$Equality\ holds\ for\ t=1,p,q=3,a=b=c=1.$$

1191. If $x,y,z>0,x+y+z=1$, then:

$$\frac{x}{xyz+x^2+2}+\frac{y}{xyz+y^2+2}+\frac{z}{xyz+z^2+2}\leq\frac{27}{58}$$

Proposed by Tuan Kiet-Vietnam

Solution by Michael Stergiou-Greece

$$\frac{x}{xyz+x^2+2}+\frac{y}{xyz+y^2+2}+\frac{z}{xyz+z^2+2}\leq\frac{27}{58};\ \ (1)$$

$$Let\ (p,q,r)=\left(\sum x,\sum xy,\prod x\right):p=1,q\leq 1,r\leq\frac{1}{27}.$$

$$\sum_{cyc}xy(x+y)=pq-3r=q-3r,$$

$$\sum_{cyc}x^2y^2=q^2-2pr=q^2-2r$$

$$\sum_{cyc}x^2=1-2q$$

$$\sum_{cyc}x(r^2+y^2+2)(r+z^2+2)=\cdots=2(qr+q-r^2-r+2)=A$$

$$\prod_{cyc}(r+x^2+2)=\cdots=q^2r+2q^2-2qr^2-8qr-8q+r^3+6r^2+12r+12=B$$

$$We\ need\ to\ show: \frac{A}{B} \leq \frac{57}{58}.$$

$$Let\ f(q,r) = -27q^2r - 54q^2 + 54qr^2 + 332qr + 332q - 27r^3 - 278r^2 - 440r - 92 \leq 0$$

$$But\ \frac{df}{dq} = -54qr - 108q + 54r^2 + 332r + 332 > 0\ as\ the\ term + 332$$

$$out\ weigths\ all\ negative\ terms, hence\ f \nearrow and\ f(q) \leq f\left(\frac{1+9r}{4}\right)$$

$$as\ by\ Schur's\ 3^{rd}\ inequality\ q \leq \frac{1+9r}{4}.$$

$$This\ leads\ to\ \frac{1}{16}(1-27r)(25r^2 - 105r - 198) \leq 0. We\ are\ done!$$

$$Equality\ holds\ for\ r = \frac{1}{27}, q = \frac{1}{3}\ and\ x = y = z = \frac{1}{3}.$$

1192.

$$\text{If } a,b,c > 0 \text{ such that} : a^5b^5 + b^5c^5 + c^5a^5 + a^5b^5c^5 = 4, \text{then} :$$

$$\sqrt[3]{a^5+2} + \sqrt[3]{b^5+2} + \sqrt[3]{c^5+2} \geq 3\sqrt[3]{3}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$The\ given\ condition\ can\ be\ rewritten\ as\ follows:$$

$$\frac{1}{a^5+2} + \frac{1}{b^5+2} + \frac{1}{c^5+2} = 1.$$

$$By\ Hölder's\ inequality, we\ have:$$

$$\left(\sum_{cyc} \sqrt[3]{a^5+2}\right)^3 \left(\sum_{cyc} \frac{1}{a^5+2}\right) \geq (1+1+1)^4 = 81.$$

$$Therefore,\ \sqrt[3]{a^5+2} + \sqrt[3]{b^5+2} + \sqrt[3]{c^5+2} \geq \sqrt[3]{81} = 3\sqrt[3]{3}.$$

$$Equality\ holds\ iff\ a = b = c = 1.$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$a^5=x,\mathbf{b}^5=\mathbf{y},\mathbf{c}^5=\mathbf{z}\,(say)\Rightarrow a^5\mathbf{b}^5+\mathbf{b}^5\mathbf{c}^5+\mathbf{c}^5a^5+a^5\mathbf{b}^5\mathbf{c}^5=4$$

$$\Rightarrow\sum_{cyc}xy+xyz=4\Rightarrow 4-xyz=\sum_{cyc}xy\overset{A-G}{\geq}3\sqrt[3]{x^2y^2z^2}$$

$$\Rightarrow\alpha^3+3\alpha^2-4\leq 0\ \left(\alpha=\sqrt[3]{xyz}\right)\Rightarrow(\alpha-1)(\alpha+2)^2\leq 0\Rightarrow\alpha=\sqrt[3]{xyz}\leq 1$$

$$\Rightarrow xyz\overset{(*)}{\leq}1\Rightarrow 4-xyz\geq 4-1\Rightarrow\sum_{cyc}xy\geq 3$$

$$\therefore\left(\sum_{cyc}x\right)^2\geq 3\sum_{cyc}xy\geq 9\Rightarrow\sum_{cyc}x\overset{(**)}{\geq}3$$

$$\text{Now, }\sqrt[3]{a^5+2}+\sqrt[3]{b^5+2}+\sqrt[3]{c^5+2}=\sum_{cyc}\sqrt[3]{x+2}$$

$$\overset{A-G}{\geq}3\sqrt[9]{(x+2)(y+2)(z+2)}=3\sqrt[9]{2\sum_{cyc}xy+4\sum_{cyc}x+8+xyz}$$

$$\overset{\sum_{cyc}xy=4-xyz}{=}3\sqrt[9]{8-xyz+4\sum_{cyc}x+8}\overset{\text{via }(*),(**)}{\geq}3\sqrt[9]{16-1+4(3)}=3\sqrt[9]{3^3}=3\sqrt[3]{3}$$

$$\therefore\sqrt[3]{a^5+2}+\sqrt[3]{b^5+2}+\sqrt[3]{c^5+2}\geq 3\sqrt[3]{3}$$

$$\forall\,a,b,c>0\,\middle|\,a^5b^5+b^5c^5+c^5a^5+a^5b^5c^5=4,''=''\text{ iff }a=b=c=1\text{ (QED)}$$

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

$$For\ a,b,c>0,a^5b^5+b^5c^5+c^5a^5+(abc)^5=4,we\ have$$

$$\sqrt[3]{a^5+2}+\sqrt[3]{b^5+2}+\sqrt[3]{c^5+2}\geq 3\sqrt[3]{3},\qquad 3\sqrt[3]{(a^5+2)(b^5+2)(c^5+2)}\geq\sqrt[3]{3}$$

$$(a^5+2)(b^5+2)(c^5+2)\geq 3^3$$

$$2a^5b^5+2b^5c^5+2c^5a^5+4a^5+4b^5+4c^5+(abc)^5+8$$

$$\geq 3^3\ which\ is\ true,because$$

$$a^5b^5+b^5c^5+c^5a^5+(abc)^5=4;\ \ (abc)^5\leq 1$$

$$4(a^5+b^5+c^5)\geq 12\Leftrightarrow a^5b^5+b^5c^5+c^5a^5\geq 3$$

$$(ab)^5+(bc)^5+(ca)^5+(abc)^5=10$$

$$4\sqrt[4]{(abc)^{15}}\leq 4\Leftrightarrow(abc)^{\frac{15}{4}}\leq 1\Leftrightarrow abc\leq 1$$

$$(ab)^5+(bc)^5+(ca)^5\ge 3 \Leftrightarrow 3(ab)^5+(bc)^5+(ca)^5\ge 9$$

$$\left(a^5+b^5+c^5\right)^2\ge 9\Rightarrow a^5+b^5+c^5\ge 3.$$

1193. *If $a,b,c>0,\ \ 2abc+3(ab+bc+ca)=27$ then :*

$$\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\le\frac{9}{2}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$The\ given\ condition\ can\ be\ written\ in\ the\ following\ equivalent\ way:$$

$$\frac{3}{a+3}+\frac{3}{b+3}+\frac{3}{c+3}=2\ \ or\ \ \frac{a}{a+3}+\frac{b}{b+3}+\frac{c}{c+3}=1.$$

$$By\ CBS\ inequality, we\ have:$$

$$1=\sum_{cyc}\frac{a}{a+3}\ge\frac{\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2}{a+b+c+9}\ \Leftrightarrow\ a+b+c+9\ge\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2.$$

$$Therefore,\ \ \sqrt{ab}+\sqrt{bc}+\sqrt{ca}\le\tfrac{9}{2}.\ Equality\ holds\ for\ a=b=c=\tfrac{3}{2}.$$

1194. If $a,b,c>0$ *such that* $ab+bc+ca=3$, then :

$$\frac{a}{(b+c)^2}+\frac{b}{(c+a)^2}+\frac{c}{(a+b)^2}\ge\frac{3(abc-2)}{3abc-7}$$

Proposed by Nguyen Thuong-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{a}{(b+c)^2}+\frac{b}{(c+a)^2}+\frac{c}{(a+b)^2}=\sum_{cyc}\frac{a^3}{(ab+ac)^2}$$

$$\overset{\text{Radon}}{\ge}\frac{\left(\sum_{cyc}a\right)^3}{\left(2\sum_{cyc}ab\right)^2}\overset{?}{\ge}\frac{3}{4}-\frac{3(abc-1)}{16}\Leftrightarrow\frac{4\left(\sum_{cyc}a\right)^3}{\left(\sum_{cyc}ab\right)^2}\overset{?}{\ge}15-3abc$$

$$\overset{ab+bc+ca=3}{\Leftrightarrow}\Leftrightarrow\frac{4\left(\sum_{cyc}a\right)^3}{9}+3abc\overset{?}{\ge}\frac{15}{3.\sqrt{3}}\left(\sum_{cyc}ab\right).\sqrt{\sum_{cyc}ab}$$

$$\Leftrightarrow \left(4\left(\sum_{cyc} a\right)^3 + 27abc\right)^2 \underset{(*)}{\overset{?}{\geq}} 675\left(\sum_{cyc} ab\right)^3$$

Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0$,
$y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y$
$\Rightarrow x,y,z$ form sides of atriangle with semiperimeter, circumradius and inradius
$= s, R, r$ (say) yielding $2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \to (1)$

$$\Rightarrow a = s-x, b = s-y, c = s-z$$

Via such substitutions, $\sum_{cyc} ab = \sum_{cyc}(s-x)(s-y) = 4Rr + r^2 \to (2)$

and $abc = (s-x)(s-y)(s-y) = r^2 s \to (3)$

Via $(1),(2),(3),(*) \Leftrightarrow s^2(4s^2+27r^2)^2 \overset{(**)}{\geq} 675r^3(4R+r)^3$ and

$$\because s^2 \overset{\text{Gerretsen}}{\geq} 16Rr - 5r^2 \overset{\text{Euler}}{\geq} 16Rr - \frac{5Rr}{2} = \frac{27Rr}{2} \text{ and}$$

$$\because (4s^2+27r^2)^2 \overset{\text{Gerretsen}}{\geq} (4(16Rr-5r^2)+27r^2)^2$$

$\therefore$ in order to prove $(**)$, it suffices to prove :

$$\frac{27Rr}{2}(4(16Rr-5r^2)+27r^2)^2 \geq 675r^3(4R+r)^3$$

$$\Leftrightarrow 896R^3 - 1504R^2r - 551Rr^2 - 50r^3 \geq 0$$

$$\Leftrightarrow (R-2r)(896R^2+288Rr+25r^2) \geq 0$$

$$\to \text{true} \because R \overset{\text{Euler}}{\geq} 2r \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{a}{(b+c)^2} + \frac{b}{(c+a)^2} + \frac{c}{(a+b)^2} \geq \frac{3}{4} - \frac{3(abc-1)}{16} \overset{?}{\geq} \frac{3(abc-2)}{3abc-7}$$

$$\Leftrightarrow \frac{1}{4} - \frac{t-1}{16} - \frac{t-2}{3t-7} \overset{?}{\geq} 0 \ (t = abc)$$

$$\Leftrightarrow \frac{4(3t-7)-(t-1)(3t-7)-16(t-2)}{16(3t-7)} \overset{?}{\geq} 0 \Leftrightarrow \frac{-3(t-1)^2}{16(3t-7)} \overset{?}{\geq} 0$$

$$\to \text{true} \because 3 = \sum_{cyc} ab \overset{A-G}{\geq} 3\sqrt[3]{a^2b^2c^2} \Rightarrow \sqrt[3]{t^2} \leq 1 \Rightarrow t \leq 1$$

$$\Rightarrow 3t \leq 3 < 7 \Rightarrow 3t-7 < 0 \Rightarrow \frac{-3(t-1)^2}{16(3t-7)} \geq 0$$

$$\therefore \frac{a}{(b+c)^2} + \frac{b}{(c+a)^2} + \frac{c}{(a+b)^2} \geq \frac{3(abc-2)}{3abc-7}$$

$\forall\, a,b,c > 0 \mid ab+bc+ca = 3, " = "$ iff $a=b=c=1$ (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $x = \frac{1}{a},\ y = \frac{1}{b},\ z = \frac{1}{c}$ *and* $p = x+y+z,\ q = xy+yz+zx,\ r = xyz.$

$$\textit{From the given condition, we have}: \ p = 3r.$$

$$\textit{From the well known inequality}: \ q^2 \ge 3pr, \ \textit{we have}: \ q \ge 3r.$$

$$\textit{The problem becomes to prove}: \ \sum_{cyc} \frac{(yz)^2}{x(y+z)^2} \ge \frac{3(2xyz-1)}{7xyz-3}.$$

$$\textit{We have}: \ \sum_{cyc} \frac{(yz)^2}{x(y+z)^2} \overset{CBS}{\ge} \frac{q^2}{pq+3r} \overset{p=3r}{=} \frac{q}{3r\left(1+\frac{1}{q}\right)} \overset{q\ge 3r}{\ge} \frac{3r}{3r+1} \overset{?}{\ge} \frac{3(2r-1)}{7r-3}$$

$$\Leftrightarrow \frac{3(r-1)^2}{(3r+1)(7r-3)} \ge 0, \textit{which is true because}: 3r = p \overset{AM-GM}{\ge} 3\sqrt[3]{r} \Rightarrow r \ge 1.$$

$$\textit{So the proof is done. Equality holds iff } a = b = c = 1.$$

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\textit{For } a, b, c > 0, ab + bc + ca = 3, \textit{we have}:$$

$$\frac{a}{(b+c)^2} + \frac{b}{(c+a)^2} + \frac{c}{(a+b)^2} \ge \frac{3(abc-2)}{3abc-7} \Leftrightarrow$$

$$\frac{\left(\frac{a}{(b+c)}\right)^2}{a} + \frac{\left(\frac{b}{c+a}\right)^2}{b} + \frac{\left(\frac{c}{a+b}\right)^2}{c} \ge \frac{3}{4} \Leftrightarrow$$

$$\frac{\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right)^2}{a+b+c} \ge \frac{3}{4} \Leftrightarrow \frac{\left(\frac{(a+b+c)^2}{2(ab+bc+ca)}\right)^2}{a+b+c} \ge \frac{3}{4}$$

$$\frac{(a+b+c)^4}{6^2(a+b+c)} \ge \frac{3}{4} \Leftrightarrow (a+b+c)^3 \ge 3^3 \textit{ which is true, because } 3(ab+bc+ca) = 9$$

$$\Leftrightarrow$$

$$(a+b+c)^2 \ge 3(ab+bc+ca) = 9 \Leftrightarrow a+b+c \ge 3 \textit{ true.}$$

Solution 4 by Daoudi Abdessattar-Tunisia

$$\sum ab = 3, r = abc, p = \sum a \ge 3$$

$$\frac{3(r-2)}{3(r-2)-1} \le -\frac{3r}{16} + \frac{15}{16}$$

$$\textit{Let } f(x) = \frac{x}{x-1} - \textit{concave function, then } f(x) \le f'(-1)(x+3) + f(-3)$$

$$P = \sum_{cyc} \frac{a}{(b+c)^2} = \sum_{cyc} \frac{a^2}{2r + a(b^2+c^2)} \ge \frac{p^2}{3p+3r}$$

$$16P \ge 3r \ge 15 \Leftrightarrow 16p^2 + (3p+3r)3r \ge 45(p+r) \Leftrightarrow$$

$$16p^2 + 9pr + 9r^2 - 45p - 45r \ge 0 \Leftrightarrow$$

$$15p(p-3)+(p^2+9pr+9r^2-45r)\geq 0 \text{ true, because } p\geq 3 \text{ and}$$
$$p^2+9pr+9r^2\overset{AM-GM}{\geq}45r$$

1195. If $a,b,c>0$ such that : $a+b+c\leq 3$, then :

$$\frac{9}{abc}-\left(\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}\right)\geq 6$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\ abc.\sum_{cyc}\frac{a+b}{b+c}\overset{AM-HM}{\leq}abc.\sum_{cyc}\frac{a+b}{4}\left(\frac{1}{b}+\frac{1}{c}\right)=\frac{1}{4}\sum_{cyc}a(a+b)(b+c)=$$

$$=\frac{1}{4}\sum_{cyc}a^2b+\frac{1}{2}\sum_{cyc}ab^2+\frac{3abc}{4}\overset{AM-GM}{\leq}\frac{1}{4}\sum_{cyc}a^2b+\frac{1}{2}\sum_{cyc}ab^2+\frac{1}{4}\sum_{cyc}a^2b=\frac{1}{2}\sum_{cyc}c(a^2+b^2).$$

$$Then:\ \frac{9}{abc}-\left(\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}\right)\geq\frac{(a+b+c)^3}{3abc}-\frac{1}{2}\sum_{cyc}\frac{c(a^2+b^2)}{abc}=$$

$$=\frac{a^3+b^3+c^3}{3abc}+\frac{1}{2}\sum_{cyc}\frac{c(a^2+b^2)}{abc}+2\overset{AM-GM}{\geq}\frac{3abc}{3abc}+\frac{1}{2}\sum_{cyc}\frac{c.2ab}{abc}+2=6.$$

So the proof is completed. Equality holds iff $a=b=c=1$.

Solution 2 by Soumava Chakraborty-Kolkata-India

Assigning $b+c=\alpha, c+a=\beta, a+b=\gamma\Rightarrow\alpha+\beta-\gamma=2c>0$,
$\beta+\gamma-\alpha=2a>0$ and $\gamma+\alpha-\beta=2b>0\Rightarrow\alpha+\beta>\gamma,\beta+\gamma>\alpha,\gamma+\alpha>\beta$
$\Rightarrow\alpha,\beta,\gamma$ form sides of a triangle with semiperimeter, circumradius and inradius
$=s,R,r$ (say) yielding $2\sum_{cyc}a=\sum_{cyc}\alpha=2s\Rightarrow\sum_{cyc}a=s\to(1)$

$$\Rightarrow a=s-\alpha,b=s-\beta,c=s-\gamma\therefore abc=(s-\alpha)(s-\beta)(s-\gamma)=r^2s\to(2)$$

$$\therefore\frac{9}{abc}-\left(\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}\right)\overset{3\geq a+b+c\Rightarrow 9\geq\frac{(a+b+c)^3}{3}}{\geq}\frac{\left(\sum_{cyc}a\right)^3}{3abc}-\sum_{cyc}\frac{\alpha}{\beta}$$

$$\overset{via\ (1),(2)}{=}\frac{s^3}{3r^2s}-\sum_{cyc}\frac{\alpha}{\beta}\overset{?}{\geq}6\Leftrightarrow\sum_{cyc}\frac{\alpha}{\beta}\underset{(\blacksquare)}{\overset{?}{\leq}}\frac{s^2-18r^2}{3r^2}$$

$$\text{Now}, \frac{s^2}{r^2}=\frac{s^4}{\Delta^2}=\frac{s^4}{s(s-\alpha)(s-\beta)(s-\gamma)}\overset{(\bullet)}{=}\frac{\left(\sum_{cyc}a\right)^3}{abc}$$

$$\text{and } 1+\frac{4R}{r}=1+\frac{4s\alpha\beta\gamma}{4s(s-\alpha)(s-\beta)(s-\gamma)}=1+\frac{\prod_{cyc}(b+c)}{abc}$$

$$\Rightarrow 1+\frac{4R}{r}\overset{(\bullet\bullet)}{=}\frac{abc+\prod_{cyc}(b+c)}{abc}$$

$$\text{Now}, \sum_{cyc}\frac{\alpha}{\beta}=\sum\frac{b+c}{c+a}\Rightarrow\sum_{cyc}\frac{\alpha}{\beta}\overset{(\bullet\bullet\bullet)}{=}\frac{\sum_{cyc}(a+b)(b+c)^2}{\prod_{cyc}(b+c)}$$

$$\therefore (\bullet),(\bullet\bullet),(\bullet\bullet\bullet)\Rightarrow\frac{s^2}{r^2}\ge\left(\sum_{cyc}\frac{\alpha}{\beta}\right)\left(1+\frac{4R}{r}\right)$$

$$\Leftrightarrow\frac{\left(\sum_{cyc}a\right)^3}{abc}\ge\left(\frac{abc+\prod_{cyc}(b+c)}{abc}\right)\left(\frac{\sum_{cyc}(a+b)(b+c)^2}{\prod_{cyc}(b+c)}\right)$$

$$\Leftrightarrow\left(\prod_{cyc}(b+c)\right)\left(\sum_{cyc}a\right)^3\ge\left(abc+\prod_{cyc}(b+c)\right)\left(\sum_{cyc}(a+b)(b+c)^2\right)$$

$$\Leftrightarrow\sum_{cyc}a^2b^4+\sum_{cyc}a^3b^3\overset{(i)}{\ge}abc\left(\sum_{cyc}a^2b\right)+3a^2b^2c^2$$

$$\text{Now}, \forall\, u,v,w>0, v^3+v^3+u^3\overset{A-G}{\ge}3v^2u, w^3+w^3+v^3\overset{A-G}{\ge}3w^2v$$

$$\text{and } u^3+u^3+w^3\overset{A-G}{\ge}3u^2w \text{ and summing up}: \sum_{cyc}u^3\ge\sum_{cyc}uv^2$$

$$\text{and choosing } u=ab, v=bc \text{ and } w=ca,$$

$$\sum_{cyc}a^3b^3\overset{(*)}{\ge}abc\left(\sum_{cyc}a^2b\right) \text{ and } \sum_{cyc}a^2b^4\underbrace{\overset{A-G}{\ge}}_{(**)}3a^2b^2c^2 \therefore (*)+(**)\Rightarrow(i) \text{ is true}$$

$$\Rightarrow\frac{s^2}{r^2}\ge\left(\sum_{cyc}\frac{\alpha}{\beta}\right)\left(1+\frac{4R}{r}\right)\Rightarrow\sum_{cyc}\frac{\alpha}{\beta}\le\frac{s^2}{r(4R+r)}\overset{?}{\le}\frac{s^2-18r^2}{3r^2}$$

$$\Leftrightarrow(4R-2r)s^2\underbrace{\overset{?}{\ge}}_{(\blacksquare\blacksquare)}18r^2(4R+r) \text{ and } \because s^2\overset{\text{Mitrinovic}}{\ge}27r^2$$

$$\therefore \text{ in order to prove } (\blacksquare\blacksquare), \text{it suffices to prove}: 3(4R-2r)\ge2(4R+r)$$

$$\Leftrightarrow 4R\ge 8r\to \text{true via Euler}\Rightarrow(\blacksquare\blacksquare)\Rightarrow(\blacksquare) \text{ is true}$$

$$\Rightarrow\frac{9}{abc}-\left(\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}\right)\ge6$$

$$\forall\, a,b,c>0\,\middle|\, a+b+c\le3, ''='' \text{ iff } a=b=c \text{ (QED)}$$

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\textit{For } a,b,c>0, a+b+c\le3, \textit{then}$$

$$\frac{9}{abc}-\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}\ge6\Leftrightarrow$$

$$\frac{9}{abc}\geq 6+\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}\Leftrightarrow$$

$$\frac{9}{abc}\geq 6+\frac{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{a}{c}+\frac{c}{b}+\frac{b}{a}}{2}\Leftrightarrow$$

$$18\geq 12abc+a^2c+c^2b+b^2a+a^2b+b^2c+c^2a\Leftrightarrow$$

$$18\geq\frac{18(a+b+c)^3}{27}\Leftrightarrow 27\geq(a+b+c)^3;(a+b+c\leq 3)\ true.$$

$$\frac{18}{27}(a+b+c)^3\geq 12abc+a^2c+c^2b+b^2a+a^2b+b^2c+c^2a\Leftrightarrow$$

$$2(a+b+c)^3\geq 3(12abc+a^2c+c^2b+b^2a+a^2b+b^2c+c^2a)\Leftrightarrow$$

$$2(a^3+b^3+c^3)+6(a^2c+c^2b+b^2a+a^2b+b^2c+c^2a)+12abc\geq$$

$$\geq 36abc+3(a^2c+c^2b+b^2a+a^2b+b^2c+c^2a)\Leftrightarrow$$

$$2(a^3+b^3+c^3)+3(a^2c+c^2b+b^2a+a^2b+b^2c+c^2a)\geq 24abc\ true.$$

1196. **If $a,b,c\geq 0$ such that : $a+b+c=2$, then :**

$$\sqrt{a+\frac{a}{b+1}}+\sqrt{b+\frac{b}{c+1}}+\sqrt{c+\frac{c}{a+1}}\geq 2(abc+1)$$

Proposed by Phan Ngoc Chau-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$\boxed{\text{Case 1}}$ **Exactly 1 variable = 0 and WLOG we may assume $a=0$ with**

$$b,c>0;b+c=2\ \text{and then}:\sqrt{a+\frac{a}{b+1}}+\sqrt{b+\frac{b}{c+1}}+\sqrt{c+\frac{c}{a+1}}$$

$$=\sqrt{b+\frac{b}{3-b}}+\sqrt{2(2-b)}\geq 2(abc+1)=2$$

$$\Leftrightarrow b+\frac{b}{3-b}+2(2-b)+2\sqrt{2(2-b)\left(b+\frac{b}{3-b}\right)}\geq 4$$

$$\Leftrightarrow 2\sqrt{\frac{2b(2-b)(4-b)}{3-b}}\geq b-\frac{b}{3-b}=\frac{b(2-b)}{3-b}\Leftrightarrow 2\sqrt{2(4-b)}\geq\sqrt{\frac{b(2-b)}{3-b}}$$

$$\Leftrightarrow 8(4-b)\geq\frac{b(2-b)}{3-b}\Leftrightarrow 8(4-b)(3-b)\geq b(2-b)\Leftrightarrow 9b^2-58b+96\geq 0$$

$$\rightarrow\text{true}\ \forall\ b,\because\Delta=58^2-36.96=-92<0$$

$$\therefore\sqrt{a+\frac{a}{b+1}}+\sqrt{b+\frac{b}{c+1}}+\sqrt{c+\frac{c}{a+1}}>2(abc+1),$$

when exactly 1 variable = 0

$\boxed{\text{Case 2}}$ *Exactly* 2 *variables* = 0 *and* WLOG *we may assume* b = c = 0 *with*

$a = 2$ *and then* : $\text{LHS} = \sqrt{2.2} = 2 = 2(abc+1)$

$$\therefore \sqrt{a+\frac{a}{b+1}}+\sqrt{b+\frac{b}{c+1}}+\sqrt{c+\frac{c}{a+1}} \geq 2(abc+1),$$

when *exactly* 2 *variables* = 0

$\boxed{\text{Case 3}}$ $a, b, c > 0$ *and then* : $\text{LHS}^2 > \text{RHS}^2$

$$\Leftrightarrow \sum_{cyc} a+\sum_{cyc}\frac{a}{b+1}+2\sum_{cyc}\left(\sqrt{a+\frac{a}{b+1}}.\sqrt{b+\frac{b}{c+1}}\right) \overset{(*)}{>} 4(abc+1)^2$$

$$\text{Now}, \frac{a}{b+1} \overset{b<2}{>} \frac{a}{3} \Rightarrow a+\frac{a}{b+1} > \frac{4a}{3} \textit{ and analogs}$$

$$\therefore 2\sum_{cyc}\left(\sqrt{a+\frac{a}{b+1}}.\sqrt{b+\frac{b}{c+1}}\right) > 2.\frac{4}{3}\sum_{cyc}\sqrt{ab} > \frac{8}{3}.\sqrt{\sum_{cyc} ab}$$

$$\Rightarrow 2\sum_{cyc}\left(\sqrt{a+\frac{a}{b+1}}.\sqrt{b+\frac{b}{c+1}}\right) \overset{(\bullet)}{>} \frac{8}{3}.\sqrt{\sum_{cyc} ab}$$

$$\text{Again}, \frac{a}{b+1} = \frac{a(b+1-b)}{b+1} = a-\frac{ab}{b+1} \overset{A-G}{\geq} a-\frac{ab}{2\sqrt{b}}$$

$$\therefore \frac{a}{b+1} \geq a-\frac{1}{2}a\sqrt{b} \textit{ and analogs} \Rightarrow$$

$$\sum_{cyc}\frac{a}{b+1} \geq \sum_{cyc} a-\frac{1}{2}\sum_{cyc}\left(\sqrt{a}\sqrt{ab}\right) \overset{CBS}{\geq} 2-\frac{1}{2}.\sqrt{\sum_{cyc} a}.\sqrt{\sum_{cyc} ab}$$

$$\overset{a+b+c=2}{=} 2-\frac{1}{\sqrt{2}}.\sqrt{\sum_{cyc} ab} \overset{\frac{3}{4}>\frac{1}{\sqrt{2}}}{>} 2-\frac{3}{4}.\sqrt{\sum_{cyc} ab} \therefore \sum_{cyc} a+\sum_{cyc}\frac{a}{b+1}$$

$$> 2+2-\frac{3}{4}.\sqrt{\sum_{cyc} ab} \Rightarrow \text{LHS of } (*) \overset{\text{via } (\bullet)}{>}$$

$$4+\left(\frac{8}{3}-\frac{3}{4}\right).\sqrt{\sum_{cyc} ab} \overset{A-G}{>} 4+\frac{23}{4}.\sqrt[3]{\sqrt{ab}.\sqrt{bc}.\sqrt{ca}}$$

$$\overset{?}{>} 4(abc+1)^2 \Leftrightarrow 4+\frac{23t}{4} \overset{?}{>} 4(t^3+1)^2 \left(t = \sqrt[3]{abc}\right) \Leftrightarrow 16+23t \overset{?}{>} 16(t^3+1)^2$$

$$\Leftrightarrow 48t^5+96t^2-69 \overset{?}{<} 0$$

$$\Leftrightarrow (3t-2)\left(16t^4+\frac{32}{3}t^3+\frac{64}{9}t^2+\frac{992}{27}t+\frac{1984}{81}\right)-\frac{1621}{81} \overset{?}{<} 0$$

$$\rightarrow \text{true} \because 2 = \sum_{cyc} a \overset{A-G}{\geq} 3\sqrt[3]{abc} \Rightarrow 3t-2 \leq 0 \textit{ and } t > 0 \Rightarrow (*) \text{ is true}$$

$$\therefore \sqrt{a+\frac{a}{b+1}}+\sqrt{b+\frac{b}{c+1}}+\sqrt{c+\frac{c}{a+1}}>2(abc+1)$$

$\forall\, a,b,c>0 \mid a+b+c=2 \therefore$ **combining** all **3 cases,**

$$\sqrt{a+\frac{a}{b+1}}+\sqrt{b+\frac{b}{c+1}}+\sqrt{c+\frac{c}{a+1}}\geq 2(abc+1)\ \forall\, a,b,c>0 \mid a+b+c=2,$$

$'' ='' $ iff $\{a=b=0, c=2\}$ or $\{b=c=0, a=2\}$ or $\{c=a=0, b=2\}$ (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

After squaring, the given inequality is equivalent to:

$$\sum_{cyc}\frac{a}{b+1}+2\sum_{cyc}\sqrt{\left(c+\frac{c}{a+1}\right)\left(a+\frac{a}{b+1}\right)}\geq 4(abc)^2+8abc+2\quad (1)$$

$$Now,\ \sum_{cyc}\frac{a}{b+1}=\sum_{cyc}\left(a-\frac{ab}{b+1}\right)=2-\sum_{cyc}\frac{2ab}{a+3b+c}\overset{AM-GM}{\geq}2-\sum_{cyc}\frac{2ab}{2\sqrt{3ab}}$$

$$=2-\frac{\sqrt{3}}{3}\sum_{cyc}\sqrt{ab}.$$

$$Then:\ \ LHS_{(1)}\geq 2-\frac{\sqrt{3}}{3}\sum_{cyc}\sqrt{ab}+2\sum_{cyc}\sqrt{ca}=2+\left(2-\frac{\sqrt{3}}{3}\right)\sum_{cyc}\sqrt{ab}\geq$$

$$\overset{AM-GM}{\geq}\ 2+\left(2-\frac{\sqrt{3}}{3}\right).3\sqrt[3]{abc}\overset{?}{\geq}RHS_{(1)}\ \overset{x:=\sqrt[3]{abc}}{\Leftrightarrow}\ 3\left(2-\frac{\sqrt{3}}{3}\right)x\geq 4x^6+8x^3.$$

$$By\ AM-GM\ inequality, we\ have:\ \ x\leq\frac{a+b+c}{3}=\frac{2}{3}.$$

$$Then:\ \ 4x^6+8x^3=4x^3(x^3+2)\leq x.4\left(\frac{2}{3}\right)^2\left(\left(\frac{2}{3}\right)^3+2\right)\leq 3\left(2-\frac{\sqrt{3}}{3}\right)x.$$

So the proof is done. Equality holds iff $(a,b,c)=(2,0,0)$ *and permutation.*

1197. **For** all **non − negative** x, y, z **such that:**

$$xy+yz+zx=x+y+z>0,$$

the following relationship holds:

$$\sqrt{\frac{x^2+y}{z+xy}}+\sqrt{\frac{y^2+z}{x+yz}}+\sqrt{\frac{z^2+x}{y+zx}}\geq 3$$

Proposed by Nguyen Thuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$x,y,z\geq 0\,\big|\,xy+yz+zx=x+y+z>0\Rightarrow \text{at most one variable}=0$$

$$\boxed{\text{Case 1}}\ \text{Exactly one variable}=0\text{ and WLOG we may assume }x=0$$

$$\text{with } y,z>0;\ yz=y+z\text{ and then : LHS}=\sqrt{\frac{y}{z}}+\sqrt{\frac{y^2+z}{yz}}+\sqrt{\frac{z^2}{y}}$$

$$\overset{A-G}{\geq}3\sqrt[3]{\sqrt{\frac{y^2+z}{y}}}\overset{?}{>}3\Leftrightarrow\frac{y^2+z}{y}\overset{?}{>}1\Leftrightarrow y-1+\frac{1}{y-1}\overset{?}{>}0$$

$$\left(\because yz=y+z\Rightarrow z(y-1)=y\Rightarrow\frac{z}{y}=\frac{1}{y-1}\right)\rightarrow\text{true}\because y-1+\frac{1}{y-1}\overset{A-G}{\geq}2>0$$

$$\therefore\sqrt{\frac{x^2+y}{z+xy}}+\sqrt{\frac{y^2+z}{x+yz}}+\sqrt{\frac{z^2+x}{y+zx}}>3$$

$$\boxed{\text{Case 2}}\ x,y,z>0\text{ and }\sqrt{\frac{x^2+y}{z+xy}}+\sqrt{\frac{y^2+z}{x+yz}}+\sqrt{\frac{z^2+x}{y+zx}}$$

$$\overset{A-G}{\geq}3\sqrt[3]{\sqrt{\frac{x^2+y}{z+xy}\cdot\frac{y^2+z}{x+yz}\cdot\frac{z^2+x}{y+zx}}}\overset{?}{\geq}3$$

$$\Leftrightarrow(x^2+y)(y^2+z)(z^2+x)\overset{?}{\geq}(z+xy)(x+yz)(y+zx)$$

$$\Leftrightarrow\sum_{cyc}x^3y^2+\sum_{cyc}x^2y^3\geq\sum_{cyc}x^2y^2+xyz\sum_{cyc}x^2$$

$$\overset{\because 1=\frac{xy+yz+zx}{x+y+z}}{\Leftrightarrow}\sum_{cyc}x^3y^2+\sum_{cyc}x^2y^3\geq\left(\sum_{cyc}x^2y^2\right)\left(\frac{\sum_{cyc}xy}{\sum_{cyc}x}\right)+xyz\sum_{cyc}x^2$$

$$\Leftrightarrow\left(\sum_{cyc}x\right)\left(\sum_{cyc}\left(x^2y^2\left(\sum_{cyc}x-z\right)\right)\right)$$

$$\geq\left(\sum_{cyc}x^2y^2\right)\left(\sum_{cyc}xy\right)+xyz\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x\right)$$

$$\Leftrightarrow\left(\sum_{cyc}x\right)\left(\left(\sum_{cyc}x\right)\left(\sum_{cyc}x^2y^2\right)-xyz\left(\sum_{cyc}xy\right)\right)$$

$$\geq\left(\sum_{cyc}x^2y^2\right)\left(\sum_{cyc}xy\right)+xyz\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x\right)$$

$$\Leftrightarrow\left(\sum_{cyc}x^2y^2\right)\left(\sum_{cyc}x\right)^2-xyz\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x\right)$$

$$\geq\left(\sum_{cyc}x^2y^2\right)\left(\sum_{cyc}xy\right)+xyz\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x\right)$$

$$\Leftrightarrow\left(\sum_{cyc}x^2y^2\right)\left(\left(\sum_{cyc}x\right)^2-\sum_{cyc}xy\right)\geq xyz\left(\sum_{cyc}x\right)\left(\sum_{cyc}x^2+\sum_{cyc}xy\right)$$

$$\Leftrightarrow\left(\sum_{cyc}x^2y^2\right)\left(\sum_{cyc}x^2+\sum_{cyc}xy\right)\geq xyz\left(\sum_{cyc}x\right)\left(\sum_{cyc}x^2+\sum_{cyc}xy\right)$$

$$\Leftrightarrow\sum_{cyc}x^2y^2\geq xyz\left(\sum_{cyc}x\right)\to\text{true}\therefore\sqrt{\frac{x^2+y}{z+xy}}+\sqrt{\frac{y^2+z}{x+yz}}+\sqrt{\frac{z^2+x}{y+zx}}\geq 3$$

$$\therefore\text{combining both cases},\sqrt{\frac{x^2+y}{z+xy}}+\sqrt{\frac{y^2+z}{x+yz}}+\sqrt{\frac{z^2+x}{y+zx}}\geq 3$$

$$\forall\, x,y,z\geq 0\,|\,xy+yz+zx=x+y+z>0,''=''\text{ iff } x=y=z=1\text{ (QED)}$$

1198. If $a,b,c>0$ such that : $a+b+c=1$, then :

$$\sum_{cyc}\frac{a}{\sqrt[4]{a^2(a+4b)+b^2(4b+6a)+12abc}}\geq 1$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\sum_{cyc}\frac{a}{\sqrt[4]{a^2(a+4b)+b^2(4b+6a)+12abc}}\geq 1$$

$$\sum_{cyc}\sqrt[4]{\frac{a^5}{a^4+4a^3b+3ab^3+6a^2b^2+12a^2bc}}\geq 1$$

$$\frac{\sqrt[4]{(\sum a)^5}}{\sqrt[4]{\sum a^4+4\sum(a^3b+ab^3)+6a^2b^2+12a^2bc+ab^2c+abc}}\geq 1$$

$$\left(\sum_{cyc} a\right)^5 \geq \sum a^4+4\sum(a^3b+ab^3)+6a^2b^2+12\sum_{cyc} a^2bc$$

$$+abc \text{ which is true, because}$$

$$\left(\sum_{cyc} a\right)^5 \geq \sum_{cyc}(a^4+a^3b)+3\sum_{cyc}(a^3b+ab^3)+3\sum_{cyc} a^2b^2+a^3c+c^3b+b^3a$$

$$+12\sum_{cyc} a^2bc$$

$$\left(\sum_{cyc} a\right)^5 \geq \sum_{cyc} a^3+3\sum_{cyc}(a^2b+ab^2)+6\sum_{cyc} a^2bc$$

$$\left(\sum_{cyc} a\right)^5 \geq \sum_{cyc} a^3+3\sum_{cyc}(a^2b+ab^2)+6abc$$

$$(a+b+c)^5 \geq (a+b+c)^3 \Leftrightarrow (a+b+c)^2 \geq a+b+c \text{ true because}$$

$$a+b+c=1.$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum_{\text{cyc}} \frac{a}{\sqrt[4]{a^2(a+4\text{b})+\text{b}^2(4\text{b}+6a)+12abc}}$$

$$=\sum_{\text{cyc}} \frac{a^{\frac{5}{4}}}{\sqrt[4]{a^3(a+4\text{b})+ab^2(4\text{b}+6a)+12a^2\text{bc}}}$$

$$\overset{\text{Radon}}{\geq} \frac{\left(\sum_{\text{cyc}} a\right)^{\frac{5}{4}}}{\left(\sum_{\text{cyc}}(a^3(a+4\text{b})+ab^2(4\text{b}+6a)+12a^2\text{bc})\right)^{\frac{1}{4}}}$$

$$\overset{a+\text{b}+\text{c}=1}{=} \frac{1}{\left(\sum_{\text{cyc}} a^4+4\sum_{\text{cyc}} a^3\text{b}+4\sum_{\text{cyc}} ab^3+6\sum_{\text{cyc}} a^2\text{b}^2+12abc\left(\sum_{\text{cyc}} a\right)\right)^{\frac{1}{4}}}$$

$$=\frac{1}{\left(\sum_{\text{cyc}} a^4+4a\text{b}\left(\sum_{\text{cyc}} a^2-\text{c}^2\right)+6\sum_{\text{cyc}} a^2\text{b}^2+12a\text{bc}\left(\sum_{\text{cyc}} a\right)\right)^{\frac{1}{4}}}$$

$$=\frac{1}{\left(\left(\sum_{\text{cyc}} a^4+2\sum_{\text{cyc}} a^2\text{b}^2\right)+4\left(\sum_{\text{cyc}} ab\right)\left(\sum_{\text{cyc}} a^2\right)+\left(4\sum_{\text{cyc}} a^2\text{b}^2+8abc\left(\sum_{\text{cyc}} a\right)\right)\right)^{\frac{1}{4}}}$$

$$=\frac{1}{\left(\left(\sum_{cyc}a^2\right)^2+4\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2\right)+4\left(\sum_{cyc}ab\right)^2\right)^{\frac{1}{4}}}=\frac{1}{\left(\sum_{cyc}a^2+2\left(\sum_{cyc}ab\right)\right)^{\frac{1}{4}}}$$

$$=\frac{1}{\left(\left(\sum_{cyc}a\right)^2\right)^{\frac{1}{4}}}\overset{a+b+c=1}{=}1,''='' \text{ iff } a=b=c=\frac{1}{3}\ (\text{QED})$$

1199. If $a,b,c\geq 0$ such that $a^2+b^2+c^2=3$, then :

$$\frac{c}{1+ab}+\frac{a}{1+bc}+\frac{b}{1+ca}\geq\frac{3}{2}$$

Proposed by Nguyen Thuong-BenTre-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$\boxed{\text{Case 1}}$ Exactly 2 variables $=0$ and WLOG we may assume $b=c=0$

$$\text{and } a>0\ \therefore\ \frac{c}{1+ab}+\frac{a}{1+bc}+\frac{b}{1+ca}=a=\sqrt{3}>\frac{3}{2}$$

$\boxed{\text{Case 2}}$ Exactly 1 variable $=0$ and WLOG we may assume $a=0$ and $b,c>0$

$$\therefore\ \frac{c}{1+ab}+\frac{a}{1+bc}+\frac{b}{1+ca}=c+b=\sqrt{b^2+c^2+2bc}>\sqrt{b^2+c^2}=\sqrt{3}>\frac{3}{2}$$

$$\boxed{\text{Case 3}}\ a,b,c>0\ \therefore\ \frac{c}{1+ab}+\frac{a}{1+bc}+\frac{b}{1+ca}=\sum_{cyc}\frac{c^4}{c^3+abc^3}$$

$$\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{cyc}a^2\right)^2}{\sum_{cyc}a^3+abc\left(\sum_{cyc}a^2\right)}\overset{a^2+b^2+c^2=3}{=}\frac{\left(\sum_{cyc}a^2\right)^2}{\sum_{cyc}a^3+3abc}\overset{?}{\geq}\frac{3}{2}\overset{a^2+b^2+c^2=3}{=}\frac{\sqrt{3}}{2}\sqrt{\sum_{cyc}a^2}$$

$$\Leftrightarrow\frac{\left(\sum_{cyc}a^2\right)^4}{\left(\sum_{cyc}a^3+3abc\right)^2}\overset{?}{\geq}\frac{3\left(\sum_{cyc}a^2\right)}{4}$$

$$\Leftrightarrow\sum_{cyc}a^6+12\left(\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4\right)\underbrace{\overset{?}{\geq}}_{(*)}6\sum_{cyc}a^3b^3+18abc\sum_{cyc}a^3+3a^2b^2c^2$$

$$\text{Now, }3\left(\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4\right)=3\sum_{cyc}\left(a^4b^2+a^2b^4\right)\overset{A-G}{\geq}6\sum_{cyc}a^3b^3$$

$$\therefore\ 3\left(\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4\right)\overset{(i)}{\geq}6\sum_{cyc}a^3b^3$$

$$\text{Also, }9\left(\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4\right)=9\sum_{cyc}\left(a^4b^2+a^4c^2\right)\overset{A-G}{\geq}18\sum_{cyc}a^4bc$$

$$\therefore 9\left(\sum_{cyc} a^4b^2 + \sum_{cyc} a^2b^4\right) \overset{(ii)}{\geq} 18abc\sum_{cyc} a^3 \text{ and } \sum_{cyc} a^6 \underbrace{\overset{A-G}{\geq}}_{(iii)} 3a^2b^2c^2$$

$$\therefore (i)+(ii)+(iii) \Rightarrow (*) \text{ is true} \Rightarrow \frac{c}{1+ab}+\frac{a}{1+bc}+\frac{b}{1+ca} \geq \frac{3}{2}$$

$$\text{and hence, combining all cases, } \frac{c}{1+ab}+\frac{a}{1+bc}+\frac{b}{1+ca} \geq \frac{3}{2}$$

$$\forall a,b,c \geq 0 \,\middle|\, a^2+b^2+c^2=3, '' ='' \text{ iff } a=b=c=1 \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{Let } p := a+b+c,\ \ q := a+bc+ca,\ \ r := abc.$$

$$\text{From the given condition, we have: } \ p^2-2q=3 \ \ (1)$$

$$\text{By AM}-\text{GM inequality, we have:}$$

$$(a+b+c)(ab+bc+ca) \geq 9abc \ \Leftrightarrow\ pq \geq 9r \ \ (2)$$

$$\text{Then: } \sum_{cyc}\frac{a}{1+bc} \overset{CBS}{\overbrace{\geq}} \frac{(a+b+c)^2}{a+b+c+3abc} = \frac{3p^2}{3p+9r} \overset{(2)}{\overbrace{\geq}} \frac{3p^2}{3p+pq} =$$

$$\overset{(1)}{\overbrace{=}} \frac{6p}{6+(p^2-3)} = \frac{3}{2}+\frac{3(3-p)(p-1)}{2(p^2+3)} \geq \frac{3}{2}, \ \ \text{because:}$$

$$3=\sqrt{3(a^2+b^2+c^2)} \overset{CBS}{\overbrace{\geq}} p \geq \sqrt{a^2+b^2+c^2} > 1.$$

$$\text{So the proof is done. Equality holds iff } a=b=c=1.$$

Solution 3 by Khaled Abd Imouti-Damascus-Syria

$$\frac{c}{1+ab}+\frac{a}{1+bc}+\frac{b}{1+ca} = \frac{c^2}{c+abc}+\frac{a^2}{a+abc}+\frac{b^2}{b+abc} \overset{G=\sqrt[3]{abc}}{=}$$

$$= \frac{c^2}{c+G^3}+\frac{a^2}{a+G^3}+\frac{b^2}{b+G^3}$$

$$\text{Let be } f(x)=\frac{x^2}{x+G^3}, \text{then } f'(x)=\frac{x^2+2xG^3}{(x+G^3)^2} \text{ and } f''(x)=\frac{2G^3}{(x+G^3)^3} > 0.$$

$$\text{So, } f-\text{convex function and}$$

$$\frac{c^2}{c+G^3}+\frac{a^2}{a+G^3}+\frac{b^2}{b+G^3}\ge 3\cdot\frac{\left(\frac{a+b+c}{3}\right)^2}{\left(\frac{a+b+c}{3}\right)+G^3}=\frac{s^2}{s+3G^3}\overset{?}{\ge}\frac{3}{2}\Leftrightarrow$$

$$2s^2\ge 3s+9G^3\Leftrightarrow 2s^2-3s-9G^3\ge 0$$

$$\Delta=9(1+8G^3)>0$$

$$s_1=\frac{3}{4}\left(1-\sqrt{1+8G^3}\right)\ impossible.$$

$$s_2=\frac{3}{4}(1+\sqrt{1+8G^3}>\frac{3}{2},but\ s^2=(a+b+c)^2=3+2(ab+bc+ca)\le$$

$$\le 3+2(a^2+b^2+c^2)=9\Rightarrow s^2\le 9\Rightarrow x\le 3\ and\ s>\frac{3}{2}\Leftrightarrow$$

$$(a+b+c)^2\ge\frac{9}{4}\Leftrightarrow 3+2(ab+bc+ca)\ge\frac{9}{4}\ true.$$

$$So,\frac{c}{1+ab}+\frac{b}{1+ac}+\ +\frac{a}{1+bc}\ge\frac{3}{2}\ its\ true.$$

Solution 4 by Daoudi Abdessattar-Tunisia

$$a^2+b^2+c^2=3,\frac{c}{1+ab}-\frac{3c^2}{3c+q}\ge 0\Leftrightarrow\frac{c(q-3r)}{(1+ab)(3c+q)}\ge 0\ true.$$

$$\sum_{cyc}\frac{c}{1+ab}\ge 3\sum_{cyc}\frac{a^2}{3a+q}\ge\frac{3p^2}{3(p+q)}\ge\frac{3}{2}\Leftrightarrow$$

$$2p^2-3p-3q\ge 0\Leftrightarrow 6+q-3\sqrt{3+2q}\ge 0\Leftrightarrow (q-3)^2\ge 0$$

1200. If $a,b,c\ge 0$ such that $ab+bc+ca>0$, then :

$$\sqrt{\frac{ab}{a^2+b^2}}+\sqrt{\frac{bc}{b^2+c^2}}+\sqrt{\frac{ca}{c^2+a^2}}\ge\sqrt{\frac{ab+bc+ca}{a^2+b^2+c^2}}$$

Proposed by Nguyen Thuong-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have :

$$\sum_{cyc}\sqrt{\frac{ab}{a^2+b^2}}\overset{abc\ge 0}{\ge}\sum_{cyc}\sqrt{\frac{ab}{a^2+b^2+c^2}}=\frac{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}}{\sqrt{a^2+b^2+c^2}}=$$

$$=\sqrt{\frac{ab+bc+ca+2\sqrt{abc}(\sqrt{a}+\sqrt{b}+\sqrt{c})}{a^2+b^2+c^2}}\ge\sqrt{\frac{ab+bc+ca}{a^2+b^2+c^2}}.$$

$$So\ the\ proof\ is\ done.\ Equality\ holds\ iff\ (a=0,b,c>0)\ and\ permutation.$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\text{LHS}^2-\frac{ab+bc+ca}{a^2+b^2+c^2}$$
$$=\left(\frac{ab}{a^2+b^2}-\frac{ab}{a^2+b^2+c^2}\right)+\left(\frac{bc}{b^2+c^2}-\frac{bc}{a^2+b^2+c^2}\right)$$
$$+\left(\frac{ca}{c^2+a^2}-\frac{ca}{a^2+b^2+c^2}\right)+2\sum_{cyc}\left(\sqrt{\frac{ab}{a^2+b^2}}\cdot\sqrt{\frac{bc}{b^2+c^2}}\right)$$
$$\geq\frac{ab(a^2+b^2+c^2-a^2-b^2)}{(a^2+b^2)(a^2+b^2+c^2)}+\frac{bc(a^2+b^2+c^2-b^2-c^2)}{(b^2+c^2)(a^2+b^2+c^2)}$$
$$+\frac{ca(a^2+b^2+c^2-c^2-a^2)}{(b^2+c^2)(a^2+b^2+c^2)}+2.(0+0+0)$$
$$=\frac{abc}{a^2+b^2+c^2}\sum_{cyc}\frac{c}{a^2+b^2}\geq 0\ (\because a,b,c\geq 0)\Rightarrow \text{LHS}^2\geq\frac{ab+bc+ca}{a^2+b^2+c^2}$$
$$\Rightarrow\sqrt{\frac{ab}{a^2+b^2}}+\sqrt{\frac{bc}{b^2+c^2}}+\sqrt{\frac{ca}{c^2+a^2}}\geq\sqrt{\frac{ab+bc+ca}{a^2+b^2+c^2}}\ \forall\, a,b,c>0\,\Big|\sum_{cyc}ab>0,$$
$$''=''\ \text{iff}\ \{a=0,b,c>0\}\ \text{or}\ \{b=0,c,a>0\}\ \text{or}\ \{c=0,a,b>0\}\ (\text{QED})$$

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\sum_{cyc}\sqrt{\frac{ab}{a^2+b^2}}\geq\sqrt{\frac{ab+bc+ca}{a^2+b^2+c^2}}$$
$$\sum_{cyc}\sqrt{\frac{ab(a^2+b^2+c^2)}{a^2+b^2}}\geq\sqrt{ab+bc+ca}$$
$$\sum_{cyc}\sqrt{ab+\frac{abc^2}{a^2+b^2}}\geq\sqrt{ab+bc+ca}$$
$$\left(\sum_{cyc}\sqrt{ab+\frac{abc^2}{a^2+b^2}}\right)^2\geq ab+bc+ca$$
$$\sum_{cyc}\left(ab+\frac{abc^2}{a^2+b^2}\right)+2\sum_{cyc}\sqrt{\left(ab+\frac{abc^2}{a^2+b^2}\right)\left(bc+\frac{bca^2}{b^2+c^2}\right)}\geq ab+bc+ca$$
$$which\ is\ true\ because\ a,b,c>0\ and\ ab+bc+ca>0.$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru