RMM - Geometry Marathon 801 - 900

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801. Prove that:

$$\cos\frac{3\pi}{13}-\cos\frac{2\pi}{13}=\frac{\sqrt{13}}{3}\cos\left(\frac{1}{3}\arccos\left(-\frac{5}{2\sqrt{13}}\right)+\frac{\pi}{3}\right)+\frac{1}{6}$$

Proposed by Vasile Mircea Popa-Romania

Solution by Pham Duc Nam-Vietnam

$$\cos\frac{3\pi}{13}-\cos\frac{2\pi}{13}=\frac{\sqrt{13}}{3}\cos\left(\frac{1}{3}\arccos\left(-\frac{5}{2\sqrt{13}}\right)+\frac{\pi}{3}\right)+\frac{1}{6}$$

$$\Leftrightarrow \cos\frac{3\pi}{13}-\cos\frac{2\pi}{13}-\frac{1}{6}=\frac{\sqrt{13}}{3}\cos\left(\frac{1}{3}\arccos\left(-\frac{5}{2\sqrt{13}}\right)+\frac{\pi}{3}\right)$$

$*\frac{5}{2\sqrt{13}}<1$,apply: $\arccos(x)+\arccos(-x)=\pi\Rightarrow \arccos\left(-\frac{5}{2\sqrt{13}}\right)=\pi-\arccos\left(\frac{5}{2\sqrt{13}}\right)$

$$\Rightarrow\frac{\sqrt{13}}{3}\cos\left(\frac{1}{3}\arccos\left(-\frac{5}{2\sqrt{13}}\right)+\frac{\pi}{3}\right)=\frac{\sqrt{13}}{3}\cos\left(\frac{1}{3}\left(\pi-\arccos\left(\frac{5}{2\sqrt{13}}\right)\right)+\frac{\pi}{3}\right)$$

$$=\frac{\sqrt{13}}{3}\cos\left(-\frac{1}{3}\arccos\left(\frac{5}{2\sqrt{13}}\right)+\frac{2\pi}{3}\right)=\frac{\sqrt{13}}{3}\cos\left(\frac{1}{3}\arccos\left(\frac{5}{2\sqrt{13}}\right)-\frac{2\pi}{3}\right)$$

$\Rightarrow$ This expression has exactly form of roots of cubic equation which can be shown in trigonometric form: $2\sqrt{\frac{-p}{3}}\cos\left(\frac{1}{3}\arccos\left(\frac{3q}{2p}\sqrt{\frac{-3}{p}}\right)-\frac{2\pi k}{3}\right)(k=0,1,2)$

$$*\frac{\sqrt{13}}{3}\cos\left(\frac{1}{3}\arccos\left(\frac{5}{2\sqrt{13}}\right)-\frac{2\pi}{3}\right)=2\sqrt{\frac{-p}{3}}\cos\left(\frac{1}{3}\arccos\left(\frac{3q}{2p}\sqrt{\frac{-3}{p}}\right)-\frac{2\pi}{3}\right)(k=1)$$

$$\Rightarrow\begin{cases}\frac{\sqrt{13}}{3}=2\sqrt{\frac{-p}{3}}\\ \frac{5}{2\sqrt{13}}=\frac{3q}{2p}\sqrt{\frac{-3}{p}}\end{cases}\Rightarrow\begin{cases}p=-\frac{13}{12}\\ q=-\frac{65}{216}\end{cases}$$

$\Rightarrow\frac{\sqrt{13}}{3}\cos\left(\frac{1}{3}\arccos\left(-\frac{5}{2\sqrt{13}}\right)+\frac{\pi}{3}\right)$ is one root of equation: $t^3-\frac{13}{12}t-\frac{65}{216}=0$

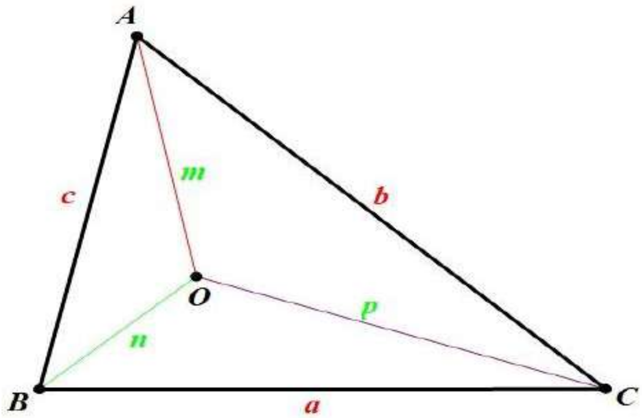
$\Leftrightarrow 216t^3-234t-65=0\Rightarrow$ All roots are: $t_k=\frac{\sqrt{13}}{3}\cos\left(\frac{1}{3}\arccos\left(\frac{5}{2\sqrt{13}}\right)-\frac{2k\pi}{3}\right)(k=0,1,2)$and $t_2<t_1<t_0$

$* f(t)=216t^3-234t-65\Rightarrow f'(t)=648t^2-234\Rightarrow f'(t)=0\Leftrightarrow t$
$=\pm\frac{\sqrt{13}}{6}, f\left(-\frac{\sqrt{13}}{6}\right)f\left(\frac{\sqrt{13}}{6}\right)<0\Rightarrow f(t)$
$=0$ has exactly one root (t_1)lies in $\left(-\frac{\sqrt{13}}{6},\frac{\sqrt{13}}{6}\right)$ and $f(t)$ is strictly decreasing in $\left(-\frac{\sqrt{13}}{6},\frac{\sqrt{13}}{6}\right)$

$$* \text{ Replace: } t=\cos\frac{3\pi}{13}-\cos\frac{2\pi}{13}-\frac{1}{6}$$
$$\Rightarrow 216\left(\cos\frac{3\pi}{13}-\cos\frac{2\pi}{13}-\frac{1}{6}\right)^3-234\left(\cos\frac{3\pi}{13}-\cos\frac{2\pi}{13}-\frac{1}{6}\right)-65=0$$
$\Rightarrow \cos\frac{3\pi}{13}-\cos\frac{2\pi}{13}-\frac{1}{6}$ is also one root of $216t^3-234t-65$
$=0$, so all roots can be shown as : $\cos\frac{7\pi}{13}-\cos\frac{4\pi}{13}-\frac{1}{6},\cos\frac{5\pi}{13}-\cos\frac{12\pi}{13}-\frac{1}{6}$ and $\left(\cos\frac{7\pi}{13}-\cos\frac{4\pi}{13}-\frac{1}{6}\right)<\left(\cos\frac{3\pi}{13}-\cos\frac{2\pi}{13}-\frac{1}{6}\right)$
$<\left(\cos\frac{5\pi}{13}-\cos\frac{12\pi}{13}-\frac{1}{6}\right)$

$*$ As we show above, in $\left(-\frac{\sqrt{13}}{6},\frac{\sqrt{13}}{6}\right)f(t)$ is strictly decreasing, $f(t)$
$=0$ has exactly one root (t_1)lies in $\left(-\frac{\sqrt{13}}{6},\frac{\sqrt{13}}{6}\right)\Rightarrow f\left(\cos\frac{3\pi}{13}-\cos\frac{2\pi}{13}-\frac{1}{6}\right)$
$=f(t_1)=f\left(\frac{\sqrt{13}}{3}\cos\left(\frac{1}{3}\arccos\left(\frac{5}{2\sqrt{13}}\right)-\frac{2\pi}{3}\right)\right)$ if and only:
$$\cos\frac{3\pi}{13}-\cos\frac{2\pi}{13}-\frac{1}{6}=\frac{\sqrt{13}}{3}\cos\left(\frac{1}{3}\arccos\left(\frac{5}{2\sqrt{13}}\right)-\frac{2\pi}{3}\right)$$
$$=\frac{\sqrt{13}}{3}\cos\left(\frac{1}{3}\arccos\left(-\frac{5}{2\sqrt{13}}\right)+\frac{\pi}{3}\right)\Rightarrow\cos\frac{3\pi}{13}-\cos\frac{2\pi}{13}$$
$$=\frac{\sqrt{13}}{3}\cos\left(\frac{1}{3}\arccos\left(-\frac{5}{2\sqrt{13}}\right)+\frac{\pi}{3}\right)+\frac{1}{6}$$

802.



In $\Delta ABC, O$ –Toricelli's point, $\sphericalangle A = 60^\circ$. Prove that:

$$a^2 = m^2 + n^2 + p^2$$

Proposed by Binh Luc-Vietnam

Solution by Jose Ferreira Queiroz-Olinda-Brazil

$$a^2 = n^2 + p^2 - 2np \cdot \cos 120^\circ = n^2 + p^2 + np$$

$$b^2 = m^2 + p^2 - 2mp \cdot \cos 120^\circ = m^2 + p^2 + mp$$

$$c^2 = m^2 + n^2 - 2mn \cdot \cos 120^\circ = m^2 + n^2 + mn$$

Then:

$$a^2 + b^2 + c^2 = 2m^2 + 2n^2 + 2p^2 + mn + np + pm; \qquad (1)$$

$$a^2 = b^2 + c^2 - 2bc \cdot \cos 60^\circ$$

$$a^2 = b^2 + c^2 - 2bc \cdot \frac{1}{2}$$

$$b^2 + c^2 = a^2 + bc; \qquad (2)$$

Now,

$$[OBC] = \frac{1}{2} NP \cdot \sin 120^\circ = \frac{\sqrt{3}}{4} np$$

$$[OAC] = \frac{\sqrt{3}}{4} mp$$

$$[OAB] = \frac{\sqrt{3}}{4} mn$$

$$[OAB] + [OBC] + [OCA] = [ABC] = \frac{1}{2} bc \cdot \sin 120^\circ$$

$$\frac{\sqrt{3}}{4}(mn+np+pm)=\frac{\sqrt{3}}{4}bc$$

$$bc=mn+np+pm; \qquad (3)$$

Using (1), (2) and (3), we get:

$$2a^2+bc=2m^2+2n^2+2p^2+bc$$

Therefore: $a^2=m^2+n^2+p^2$

803. In ΔABC the following relationship holds:

$$\sqrt{s^2+r^2+4Rr}\ge\sum_{cyc}\sqrt{2Rr\cdot\frac{b+c-a}{a}}$$

Proposed by Mihaly Bencze, Neculai Stanciu-Romania

Solution by Tapas Das-India

$$\sum_{cyc}\sqrt{2Rr\cdot\frac{b+c-a}{a}}=\sum_{cyc}\sqrt{2Rr\cdot\frac{2s-2a}{a}}=\sqrt{4Rr}\cdot\sum_{cyc}\sqrt{\frac{s-a}{a}}\overset{CBS}{\le}$$

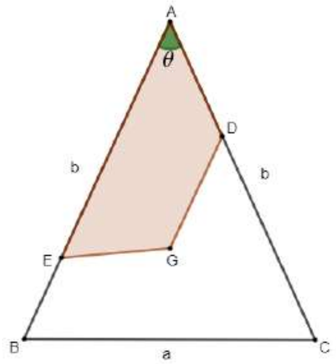
$$\le\sqrt{4Rr}\cdot\sqrt{(s-a+s-b+s-c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}=$$

$$=\sqrt{4Rr}\cdot\sqrt{(3s-2s)\left(\frac{ab+bc+ca}{abc}\right)}=\sqrt{4Rr}\cdot\sqrt{\frac{s(ab+bc+ca)}{4Rrs}}=$$

$=\sqrt{ab+bc+ca}=\sqrt{s^2+r^2+4Rr}$. Equality holds for: $a=b=c$.

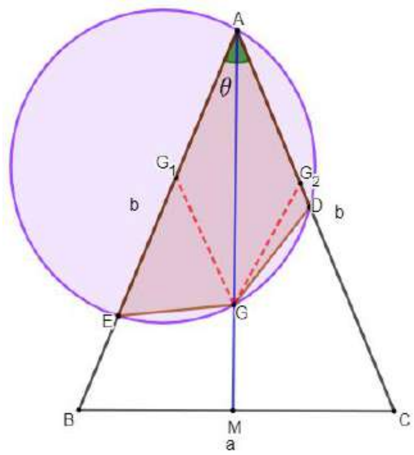
804. In $\Delta ABC, G$ –centroid, $\widehat{A}=\theta, BC=a, AB=AC=b$

$$\frac{[ADGE]}{[ABC]}=\left(\frac{2}{3}\right)^2-\left(\frac{a}{3b}\right)^2.\ \text{Find: } \measuredangle DGE=f(\theta).$$



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer



Let $GG_1 \parallel AC, GG_2 \parallel AB$. **Is** $AG_1 = AG_2 = \frac{b}{3}$.

$$\frac{[ADGE]}{[ABC]} = \left(\frac{2}{3}\right)^2 - \left(\frac{a}{3b}\right)^2; \qquad (1)$$

$$\begin{cases} [ADGE] = \frac{1}{2}\sin A\,(AE \cdot AG_1 + AD \cdot AG_2) = \frac{1}{2}\sin A \cdot \frac{b}{3}(AE + AD) \\ [ABC] = \frac{1}{2}\sin A \cdot b^2 \end{cases} (Gakopoulos'\ th)$$

$$\frac{[ADGE]}{[ABC]} = \frac{AE + AD}{3b} \overset{(1)}{=} \frac{4b^2 - a^2}{9b^2} \Rightarrow$$

$$\Rightarrow AD + AE = \frac{4b^2 - a^2}{3b}; \qquad (2)$$

Is: $AM^2 = b^2 - \frac{a^2}{4}; AG^2 = \frac{4}{9}AM^2 \Rightarrow AG^2 = \frac{1}{9}(4b^2 - a^2)$

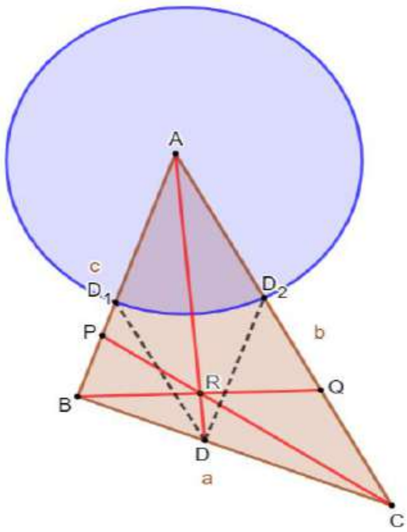
$$\text{Is: } AE\cdot AG_1+AD\cdot AG_2=\frac{b}{3}(AD+AE)\overset{(2)}{=}\frac{1}{9}(4b^2-a^2)$$

$$AG^2=AE\cdot AG_1+AD\cdot AG_2\Rightarrow ADGE-\text{cyclic } (Gakopoulos'th.),\text{ so}$$

$$\widehat{DGE}=180^\circ-\theta.$$

805. In $\Delta ABC, A$ −center of circle. Prove:

$$\frac{1}{AP}-\frac{1}{AB}=\frac{1}{AQ}-\frac{1}{AC}$$



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Rajarshi Chakraborty-India

$$AD_1=AD_2; AD_1\parallel DD_2; AD_2\parallel DD_1, so:\ AD_2DD_1\ is\ a\ rhombus, then:$$

$$\sphericalangle D_1AD=\sphericalangle DAD_2\Rightarrow\frac{AB}{AC}=\frac{BD}{DC};\quad (1)$$

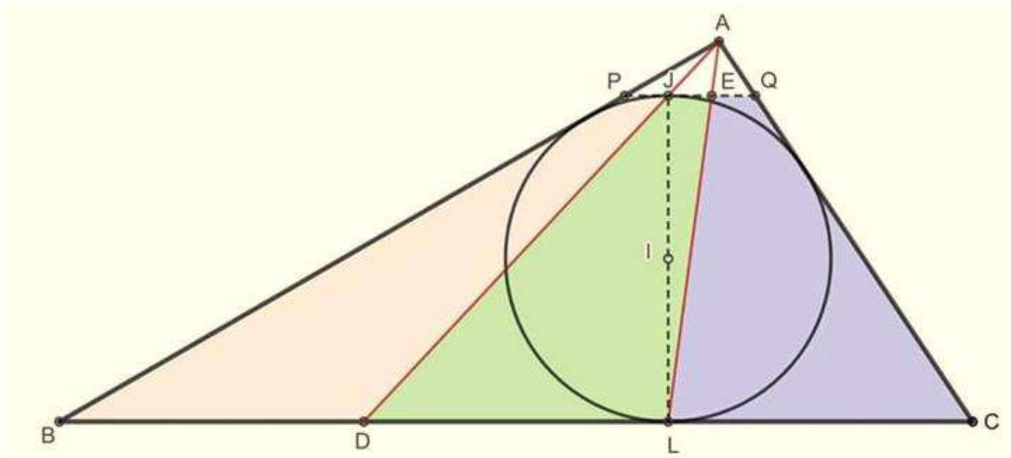
$$Now, by\ Ceva's\ theorem:\ \frac{AP}{PB}\cdot\frac{BD}{DC}\cdot\frac{CQ}{QA}=1;\quad (2)$$

$$\frac{AP}{PB}\cdot\frac{AB}{AC}\cdot\frac{CQ}{QA}=1\Rightarrow\frac{CQ}{AC\cdot QA}=\frac{PB}{AP\cdot AB}$$

$$\frac{AC-QA}{AC\cdot QA}=\frac{AB-AP}{AP\cdot AB}\Leftrightarrow\frac{1}{AQ}-\frac{1}{AC}=\frac{1}{AP}-\frac{1}{AB}$$

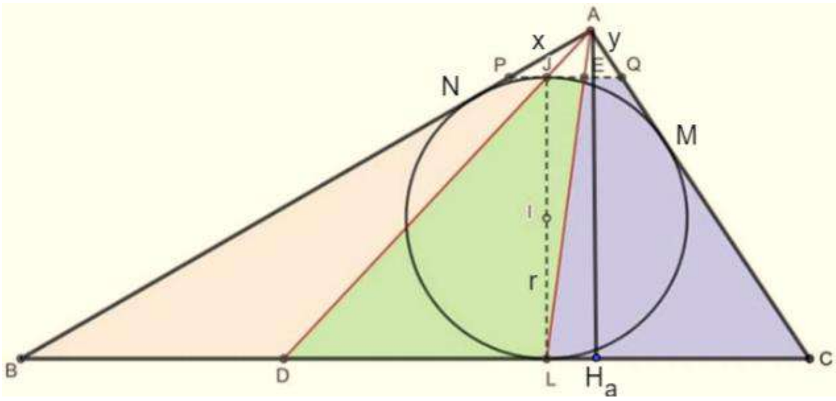
806. L, J −tangential points, $PQ\parallel BC$. Prove:

$$[BDJP]=[DJEJ]=[LCQE]\Leftrightarrow a=3(c-b)$$



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Jose Ferreira Queiroz-Olinda-Brazil



$$AP = x, AQ = y, LC = s - c, BL = s - b$$

$$AM = AN = s - a, JQ = QM = s - a - y$$

$$JP = PN = s - a - x, AH_a = h_a, AH = h_1$$

$$\Delta APQ \sim \Delta ABC \Rightarrow \frac{h_a}{h_1} = \frac{b}{y} = \frac{c}{x} \Rightarrow h_a = h_1 + 2r \Rightarrow \begin{cases} x = \frac{c}{s}(s-a) \\ y = \frac{b}{s}(s-a) \end{cases}$$

$$JP = PN = s - a - x = s - a - \frac{c}{x}(s-a) = \frac{(s-a)(s-c)}{s}$$

$$JQ = QM = s - a - y = s - a - \frac{b}{s}(s-a) = \frac{(s-a)(s-b)}{s}$$

$$PQ = JP + JQ = \frac{a(s-a)}{s}$$

$$\Delta AEQ \sim \Delta ALC \Rightarrow \frac{b}{y} = \frac{h_a}{h_1} = \frac{LC}{EQ} \Rightarrow EQ = \frac{LC \cdot y}{b} = \frac{s-c}{b} \cdot \frac{b}{s}(s-a)$$

$$EQ = \frac{(s-a)(s-c)}{s}$$

$$[LCQE] = [ALC] - [AEQ] = \frac{1}{2}(s-c)h_a - \frac{1}{2} \cdot \frac{(s-a)(s-c)}{s} h_1$$

$$[LCQE] = \frac{F(s-c)(2s-a)}{s^2}$$

$$\Delta APJ \sim \Delta ABD \Rightarrow \frac{c}{x} = \frac{h_a}{h_1} = \frac{BD}{JP} \Rightarrow BD = \frac{JP \cdot c}{x}$$

$$BD = \frac{c}{x} \cdot \frac{(s-a)(s-c)}{s} = \frac{c}{s}(s-a)(s-c) \cdot \frac{s}{c} \cdot \frac{1}{s-a} = s-c$$

In ΔABD and ΔAPJ: $BD = LC$ and $JP = ED$ then: $[BDJP] = [LCQE]$.

Now, $[DLEJ] = [DBC] - [APQ] - [BDJP] - [LCQE] =$

$$= \frac{1}{2}ah_a + \frac{1}{2}PQ \cdot h_1 - 2 \cdot \frac{F(s-c)(2s-a)}{s^2} =$$

$$= F - \frac{1}{2} \cdot \frac{a(s-a)}{s} \cdot (h_a - 2r) - \frac{2F(s-c)(2s-a)}{s^2} =$$

$$= F - \frac{1}{2} \cdot \frac{ah_a}{s} \cdot (h_a - 2r) - \frac{2F(s-c)(2s-a)}{s^2} =$$

$$= F - \frac{F}{s}(s-a) + \frac{ar(s-a)}{s} - \frac{2F(s-c)(2s-a)}{s^2} =$$

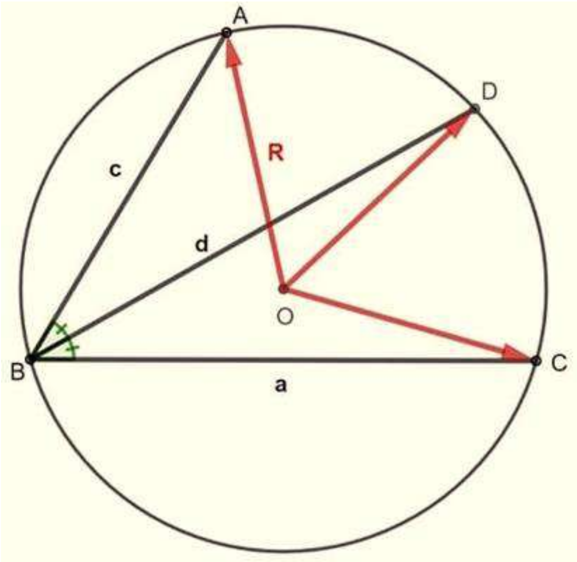
$$= \frac{F}{s^2}(b+c)(c-b)$$

$$\{DLEJ] = [BDJP] \Leftrightarrow \frac{F(s-c)(2s-a)}{s^2} = \frac{F}{s^2}(b+c)(c-b) \Leftrightarrow$$

$$\begin{cases} s-c = c-b \\ \dfrac{a+b-c}{2} = c-b \end{cases} \Leftrightarrow a = 3(c-b)$$

807. Prove that:

$$R^2 = \frac{d^2(d^2 - ac)}{(2d)^2 - (a+c)^2}$$



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Jose Ferreira Queiroz-Olinda-Brazil

$$\sphericalangle ABD = \sphericalangle DBC = \theta; \sphericalangle AOD = \sphericalangle DOC = 2\theta$$

Using Gakopoulos' Lemmas $NCCQ_3$:

$$ABCD - \text{cyclic} \Leftrightarrow BD = \frac{BA \cdot \sin\theta + BC \cdot \sin\theta}{\sin 2\theta}$$

$$d = \frac{a \cdot \sin\theta + c \cdot \sin\theta}{\sin 2\theta} \Rightarrow \cos\theta = \frac{a+c}{2d}$$

$$AC^2 = a^2 + c^2 - 2ac \cdot \cos 2\theta = R^2 + R^2 - 2R^2 \cdot \cos 4\theta$$

$$2R^2(1 - \cos 4\theta) = a^2 + c^2 - 2ac \cdot \cos 2\theta$$

$$2R^2(8\cos^2\theta - 8\cos^4\theta) = (a+c)^2 - 4ac \cdot \cos^2\theta$$

$$16R^2\left[\frac{(a+c)^2}{4d^2} - \frac{(a+c)^4}{16d^4}\right] = (a+c)^2 - 4ac \cdot \frac{(a+c)^2}{4d^2}$$

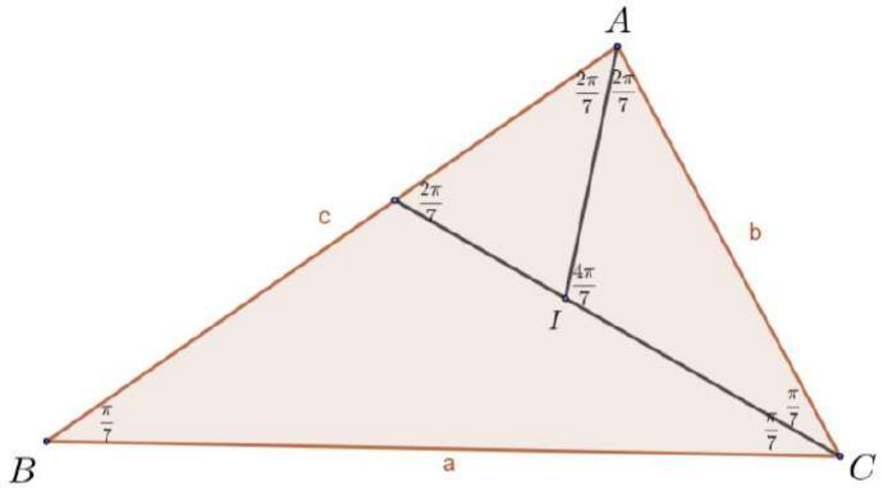
$$16R^2\left[1 - \frac{(a+c)^2}{4d^2}\right] = 4d^2 - 4ac$$

Therefore,

$$R^2 = \frac{d^2(d^2 - ac)}{(2d)^2 - (a+c)^2}$$

808. Let ABC be a heptagonal triangle, I −incenter of ABC. Prove that:

$$\frac{1}{AC}+\frac{1}{IC}=\frac{1}{AI}, \qquad AB+AI=BC, \qquad AC+CI=AB$$



Proposed by Juan Jose Isach Mayo-Valencia-Spain

Solution by Jose Ferreira Queiroz-Olinda-Brazil

ΔABC is heptagonal, then:

$$b(a+c)=ac, \qquad b^2=a^2-ac, \qquad c^2=b^2+ab, \qquad a^2=c^2+bc$$

$$\Delta AIC\sim\Delta ABC:$$

$$\frac{CI}{c}=\frac{b}{a}=\frac{AI}{b}\Rightarrow CI=\frac{bc}{a}\text{ and } AI=\frac{b^2}{a}$$

$$(I)\qquad \frac{1}{AC}+\frac{1}{IC}=\frac{1}{b}+\frac{1}{\frac{bc}{a}}=\frac{1}{b}+\frac{a}{bc}=\frac{1}{b}\cdot\frac{a+c}{c}=\frac{1}{b}\cdot\frac{a}{b}$$

$$\frac{1}{AC}+\frac{1}{IC}=\frac{a}{b^2}=\frac{1}{AI}$$

$$(II)\qquad AB+AI=c+\frac{b^2}{a}=\frac{ac+b^2}{a}=\frac{ac+a^2-ac}{a}=a$$

$$AB+AI=a=BC$$

$$(III)\qquad AC+CI=b+\frac{bc}{a}=b\cdot\frac{a+c}{a}=b\cdot\frac{c}{b}=c$$

$$AC+CI=c=AB$$

809.

$$If\ \alpha=\sum_{cyc}\frac{1}{m_a+m_b-m_c}, \beta=\sum_{cyc}\frac{1}{m_a}, \omega-Brocard's\ angle\ in\ \Delta ABC\ then:$$

$$\frac{1}{\sin\omega}\geq\sqrt{\frac{\alpha}{\beta}}\left(\sqrt{\frac{2(m_a+m_b)-m_c}{2(m_a+m_c)-m_b}}+\sqrt{\frac{2(m_a+m_c)-m_b}{2(m_a+m_b)-m_c}}\right)$$

Proposed by Bogdan Fuștei-Romania

Solutions 1,2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution 1 :

$$\textit{Lemma}:\ \textit{In any } \Delta ABC \textit{ we have}:\ \frac{\sqrt{a^2b^2+b^2c^2+c^2a^2}}{2F}\geq\frac{b}{c}+\frac{c}{b}\quad(*)$$

$$\textit{Proof}:\ (*)\ \Leftrightarrow\ 2bc\sqrt{a^2b^2+b^2c^2+c^2a^2}$$
$$\geq\left(b^2+c^2\right)\sqrt{2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)}$$

$$\overset{squaring}{\Leftrightarrow}\quad 4b^2c^2\left(a^2b^2+b^2c^2+c^2a^2\right)$$
$$\geq\left(2b^2c^2+b^4+c^4\right)\left[2\left(a^2b^2+b^2c^2+c^2a^2\right)-\left(a^4+b^4+c^4\right)\right]$$

$$\Leftrightarrow\ 0\geq -a^4\left(b^2+c^2\right)^2+2\left(b^4+c^4\right)\left(a^2b^2+c^2a^2\right)-\left(b^4+c^4\right)^2$$
$$=-\left[a^2\left(b^2+c^2\right)-\left(b^4+c^4\right)\right]^2$$

Which is true and the proof of the lemma is done.

Now, m_a,m_b,m_c *– can be the sides of a triangle then* $a'=\sqrt{m_a},b'=\sqrt{m_b},c'=\sqrt{m_c}$ *–*

can also be the sides of a triangle with area $F'=\frac{1}{4}\sqrt{2\sum m_bm_c-\sum m_a{}^2}$.

Then $m_{a'},m_{b'},m_{c'}$ *– can be the sides of a triangle with area* $S=\frac{3}{4}F'$,

$$m_{a'}=\sqrt{\frac{2(m_b+m_c)-m_a}{4}}\quad(\textit{and analogs})\textit{ and}$$

$$\sum_{cyc}m_{b'}{}^2m_{c'}{}^2=\frac{9}{16}\sum_{cyc}b'^2c'^2=\frac{9}{16}\sum_{cyc}m_bm_c.$$

Using the lemma in $\Delta m_{a'}m_{b'}m_{c'}$ *we get* : $\frac{m_{b'}}{m_{c'}}+\frac{m_{c'}}{m_{b'}}\leq\frac{\sqrt{\sum m_{b'}{}^2m_{c'}{}^2}}{2S}$

$$or\ \sqrt{\frac{2(m_a+m_b)-m_c}{2(m_a+m_c)-m_b}}+\sqrt{\frac{2(m_a+m_c)-m_b}{2(m_a+m_b)-m_c}}\le\frac{2\sqrt{\sum m_bm_c}}{\sqrt{2\sum m_bm_c-\sum m_a{}^2}}\quad(1)$$

$$Now\ we\ have:\ \frac{\alpha}{\beta}=\frac{\sum(m_a+m_b-m_c)(m_a+m_c-m_b)}{\prod(m_a+m_b-m_c)}\cdot\frac{m_am_bm_c}{\sum m_bm_c}=$$

$$=\frac{2\sum m_bm_c-\sum m_a{}^2}{\left(4F_{\Delta m_am_bm_c}\right)^2}\cdot\frac{m_am_bm_c\sum m_a}{\sum m_bm_c}\overset{AM-GM}{\le}\frac{2\sum m_bm_c-\sum m_a{}^2}{\sum m_bm_c}\cdot\frac{\sum m_b{}^2m_c{}^2}{(3F)^2}=$$

$$=\frac{2\sum m_bm_c-\sum m_a{}^2}{\sum m_bm_c}\cdot\frac{\frac{9}{16}\sum b^2c^2}{9F^2}=\frac{2\sum m_bm_c-\sum m_a{}^2}{\sum m_bm_c}\cdot\frac{1}{4\sin^2\omega}\quad(2)$$

$$From\ (1)\ and\ (2)\ we\ get:$$

$$\sqrt{\frac{\alpha}{\beta}}\left(\sqrt{\frac{2(m_a+m_b)-m_c}{2(m_a+m_c)-m_b}}+\sqrt{\frac{2(m_a+m_c)-m_b}{2(m_a+m_b)-m_c}}\right)\le\frac{1}{\sin\omega},\quad as\ desired.$$

<u>***Solution 2 :***</u>

$$We\ have:\quad \sin\omega=\frac{2F}{\sqrt{\sum b^2c^2}}=\frac{\frac{8}{3}F_m}{\sqrt{\frac{16}{9}\sum m_b{}^2m_c{}^2}}=\frac{2F_m}{\sqrt{\sum m_b{}^2m_c{}^2}}=\sin\omega_m,$$

$$where\ F_m\ and\ \omega_m\ are\ the\ area\ and\ the\ Brocard's\ angle\ of\ \Delta m_am_bm_c.$$

$$So\ the\ desired\ inequality\ can\ be\ rewritten\ as\ follows:$$

$$\frac{1}{\sin\omega_m}\ge\sqrt{\frac{\alpha}{\beta}}\left(\sqrt{\frac{2(m_a+m_b)-m_c}{2(m_a+m_c)-m_b}}+\sqrt{\frac{2(m_a+m_c)-m_b}{2(m_a+m_b)-m_c}}\right)$$

$$So\ it\ suffices\ to\ prove\ that:$$

$$\frac{1}{\sin\omega}\ge\sqrt{\frac{\alpha'}{\beta'}}\left(\sqrt{\frac{2(a+b)-c}{2(a+c)-b}}+\sqrt{\frac{2(a+c)-b}{2(a+b)-c}}\right),\quad\forall\Delta ABC,\ \ where:$$

$$\alpha'=\sum_{cyc}\frac{1}{a+b-c}=\frac{4R+r}{2F}\ \ and\ \ \beta'=\sum_{cyc}\frac{1}{a}=\frac{ab+bc+ca}{4RF}.$$

$$Since\ \sqrt{a},\sqrt{b},\sqrt{c}-can\ be\ the\ sides\ of\ a\ triangle\ with\ area$$

$$F' = \frac{\sqrt{2\sum bc - \sum a^2}}{4} = \frac{\sqrt{r(4R+r)}}{2},$$

then $m_{\sqrt{a}}, m_{\sqrt{b}}, m_{\sqrt{c}}$ *– can be the sides of a triangle with area:*

$$S = \frac{3}{4}F' = \frac{3\sqrt{r(4R+r)}}{8} \text{ and } m_{\sqrt{a}} = \sqrt{\frac{2(b+c)-a}{4}} \text{ (and analogs).}$$

Using the lemma proved in the first solution in $\Delta m_{\sqrt{a}} m_{\sqrt{b}} m_{\sqrt{c}}$, *we get:*

$$\frac{m_{\sqrt{b}}}{m_{\sqrt{c}}} + \frac{m_{\sqrt{c}}}{m_{\sqrt{b}}} \le \frac{\sqrt{\sum m_{\sqrt{b}}{}^2 m_{\sqrt{c}}{}^2}}{2S}.$$

$$Or \sqrt{\frac{2(a+b)-c}{2(a+c)-b}} + \sqrt{\frac{2(a+c)-b}{2(a+b)-c}} \le \frac{\sqrt{\frac{9}{16}\sum \sqrt{b}^2 \sqrt{c}^2}}{2.\frac{3\sqrt{r(4R+r)}}{8}} = \sqrt{\frac{ab+bc+ca}{r(4R+r)}}$$

$$Therefore, \sqrt{\frac{\alpha'}{\beta'}}\left(\sqrt{\frac{2(a+b)-c}{2(a+c)-b}} + \sqrt{\frac{2(a+c)-b}{2(a+b)-c}}\right) \le \sqrt{\frac{2R(4R+r)}{ab+bc+ca}}.\sqrt{\frac{ab+bc+ca}{r(4R+r)}} =$$

$$= \sqrt{\frac{2R}{r}} = \frac{\sqrt{abc(a+b+c)}}{2F} \overset{AM-GM}{\le} \frac{\sqrt{a^2b^2+b^2c^2+c^2a^2}}{2F} = \frac{1}{\sin\omega}, \text{ as desired.}$$

810. ***In ΔABC holds:***

$$144r^2 \le \sum_{cyc}(b^2+c^2)\csc\frac{A}{2} \le \frac{9R(3R^2-4r^2)}{r}$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have: \sum_{cyc}(b^2+c^2)\csc\frac{A}{2} \overset{Tereshin}{\le} \sum_{cyc} 4Rm_a.\csc\frac{A}{2} \overset{CBS}{\le} 4R\sqrt{\sum_{cyc} m_a{}^2.\sum_{cyc}\csc^2\frac{A}{2}} =$$

$$= 4R\sqrt{\frac{3(s^2-r^2-4Rr)}{2}\cdot\frac{s^2+r^2-8Rr}{r^2}} \overset{Gerretsen}{\leq} \frac{4R}{r}\sqrt{\frac{3(4R^2+2r^2)(4R^2-4Rr+4r^2)}{2}} =$$

$$= \frac{4R.2\sqrt{(2R^2+r^2).3(R^2-Rr+r^2)}}{r} \overset{AM-GM}{\leq} \frac{4R[(2R^2+r^2)+3(R^2-Rr+r^2)]}{r} =$$

$$= \frac{4R(5R^2-3Rr+4r^2)}{r} = \frac{9R(3R^2-4r^2)-R(R-2r)(7R+26r)}{r} \overset{Euler}{\leq} \frac{9R(3R^2-4r^2)}{r}.$$

$$Now\ we\ have: \sum_{cyc}(b^2+c^2)\csc\frac{A}{2} \overset{CBS}{\geq} \sum_{cyc}\frac{(b+c)^2}{2\sin\frac{A}{2}} \overset{CBS}{\geq} \frac{4(a+b+c)^2}{2\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\right)} \geq$$

$$\overset{Jensen}{\geq} \frac{2.4s^2}{3\sin\frac{\pi}{6}} \overset{Mitrinovic}{\geq} \frac{8.27r^2}{\frac{3}{2}} = 144r^2.$$

So the proof is completed. Equality holds iff ΔABC is equilateral.

811. *In ΔABC the following relationship holds:*

$$\sum_{cyc}\sqrt{a(2a+2b-c)} \geq 3\sqrt{2R(h_a+h_b+h_c)}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

The given inequality can be rewritten as follows:

$$\sum_{cyc}\sqrt{a(2a+2b-c)} \geq 3\sqrt{ab+bc+ca}.$$

Using Ravi's substitution: $a=y+z, b=z+x, c=x+y,\ \ x,y,z>0$. We have:

$$\sum_{cyc}\sqrt{a(2a+2b-c)} = \sum_{cyc}\sqrt{(y+z)(x+y+4z)} = \sum_{cyc}\sqrt{(y+2z)^2+\sqrt{xy+yz+zx}^2} \geq$$

$$\overset{Minkowski}{\geq} \sqrt{\left[\sum(y+2z)\right]^2+\left(\sum\sqrt{xy+yz+zx}\right)^2} = \sqrt{9\sum(z+x)(x+y)} = 3\sqrt{ab+bc+ca}.$$

So the proof is completed. Equality holds iff ΔABC is equilateral.

812. In ΔABC the following relationship holds:

$$\left(\sum_{cyc} a\sqrt{bc}\right)\left(\sum_{cyc} \sqrt[3]{ab}\left(\sqrt[3]{a}+\sqrt[3]{b}\right)\right) \le 54\sqrt{3}\cdot R^3$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Alex Szoros-Romania

$$(\forall)x,y \ge 0:\ x^3+y^3 \ge xy(x+y).$$

$$Let: x=\sqrt[3]{a}\ and\ y=\sqrt[3]{b}\ then: a+b \ge \sqrt[3]{ab}\left(\sqrt[3]{a}+\sqrt[3]{b}\right)$$

$$\sum_{cyc} \sqrt[3]{ab}\left(\sqrt[3]{a}+\sqrt[3]{b}\right) \le \sum_{cyc}(a+b) = 2\sum_{cyc} a = 4s; \qquad (1)$$

$$\sum_{cyc} a\sqrt{bc} \le \sum_{cyc} a\left(\frac{b+c}{2}\right) = \frac{1}{2}\sum_{cyc}(ab+ac) = \sum_{cyc} ab \le \sum_{cyc} a^2 \le 9R^2; \qquad (2)$$

$$But\ s \le \frac{3\sqrt{3}}{3}R\ (Mitrinovic) \Rightarrow 4s \le 6\sqrt{3}\cdot R; \qquad (3)$$

$$From\ (1),(2)\ and\ (3):$$

$$\left(\sum_{cyc} a\sqrt{bc}\right)\left(\sum_{cyc} \sqrt[3]{ab}\left(\sqrt[3]{a}+\sqrt[3]{b}\right)\right) \le 4s\cdot 9R^2 \le 6\sqrt{3}R\cdot 9R^2 = 54\sqrt{3}\cdot R^3$$

Solution 2 by Tapas Das-India

$$\sqrt[3]{ab}\left(\sqrt[3]{a}+\sqrt[3]{b}\right) = \left(\sqrt[3]{a^2b}+\sqrt[3]{ab^2}\right) = \sqrt[3]{a\cdot a\cdot b}+\sqrt[3]{a\cdot b\cdot b} \overset{AM-GM}{\le}$$

$$\le \frac{a+a+b}{3}+\frac{a+b+b}{3} = \frac{3(a+b)}{3} = a+b$$

$$\sum_{cyc} \sqrt[3]{ab}\left(\sqrt[3]{a}+\sqrt[3]{b}\right) \le \sum_{cyc}(a+b) = 4s$$

$$\sum_{cyc} a\sqrt{bc} = \sum_{cyc} \sqrt{ab\cdot ac} \le \frac{1}{2}\sum_{cyc}(ab+ac) = ab+bc+ca$$

$$\left(\sum_{cyc} a\sqrt{bc}\right)\left(\sum_{cyc} \sqrt[3]{ab}\left(\sqrt[3]{a}+\sqrt[3]{b}\right)\right) \le 4s(ab+bc+ca) \le$$

$$\le \frac{(a+b+c)^2}{3}\cdot 4s = \frac{4s^2\cdot 4s}{3} \le \frac{4}{3}\cdot\frac{27R^2}{4}\cdot 4\cdot\frac{3\sqrt{3}}{2}R = 54\sqrt{3}\cdot R^3$$

813. *In ΔABC the following relationship holds:*

$$2R \le \sum_{cyc} \frac{AI}{\sin A(\sin B + \sin C)} \le \frac{R^2}{r}$$

Proposed by Ertan Yildirim-Izmir-Turkiye

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$AI = \frac{r}{\sin\frac{A}{2}} = \frac{4Rr\cos\frac{A}{2}}{a} = \frac{bc\cos\frac{A}{2}}{s} = \frac{(b+c)w_a}{2s} \quad (and\ analogs)$$

$$\sum_{cyc} \frac{AI}{\sin A(\sin B + \sin C)} = \sum_{cyc} \frac{(b+c)w_a}{2s} \cdot \frac{(2R)^2}{a(b+c)} = \frac{2R^2}{s}\sum_{cyc}\frac{w_a}{a}.$$

$$Now, we\ know\ that: \ h_a \le w_a \le \sqrt{s(s-a)} \ (and\ analogs), then:$$

$$\sum_{cyc} \frac{AI}{\sin A(\sin B + \sin C)} \ge \frac{2R^2}{s}\sum_{cyc}\frac{h_a}{a} = 4R^2r\sum_{cyc}\frac{1}{a^2} \overset{AM-GM}{\ge} 4R^2r\sum_{cyc}\frac{1}{bc} = \frac{4R^2r}{2Rr} = 2R.$$

$$And: \sum_{cyc} \frac{AI}{\sin A(\sin B + \sin C)} = \frac{2R^2}{s}\sum_{cyc}\frac{w_a}{a} = \frac{R^2}{s^2r}\sum_{cyc} w_a h_a \le \frac{R^2}{s^2r}\sum_{cyc} s(s-a) = \frac{R^2}{r}.$$

So the proof is completed. Equality holds iff ΔABC is equilateral.

814. *In ΔABC the following relationship holds:*

$$\frac{a^3}{b^3} + \frac{b^3}{c^3} + \frac{c^3}{a^3} + \frac{R^3}{r^3} \ge 8 + \frac{b^3}{a^3} + \frac{c^3}{b^3} + \frac{a^3}{c^3}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

The given inequality can be rewritten as follows:

$$\frac{R^3}{r^3} - 8 \ge \frac{(a^3-b^3)(a^3-c^3)(b^3-c^3)}{a^3b^3c^3} \quad (1)$$

$$WLOG, we\ may\ assume\ that\ c = min\{a,b,c\}.$$

$$If\ b \ge a \ge c\ we\ have:\ LHS_{(1)} \overset{Euler}{\ge} 0 \ge RHS_{(1)}.$$

$$Now, we\ assume\ that\ a \ge b \ge c.\ We\ have:$$

$$\frac{a^3}{b^3}+\frac{b^3}{c^3}+\frac{c^3}{a^3}+\frac{R^3}{r^3} \overset{Bandila}{\ge} \frac{a^3}{b^3}+\frac{b^3}{c^3}+\frac{c^3}{a^3}+\left(\frac{a}{c}+\frac{c}{a}\right)^3 \overset{?}{\ge} 8+\frac{b^3}{a^3}+\frac{c^3}{b^3}+\frac{a^3}{c^3}$$

$$\Leftrightarrow \left(1+\frac{c^3}{a^3}-\frac{b^3}{a^3}-\frac{c^3}{b^3}\right)+\left(\frac{a^3}{b^3}+\frac{b^3}{c^3}+\frac{c^3}{a^3}-3\right)+3\left(\frac{a}{c}+\frac{c}{a}-2\right) \ge 0.$$

$$\Leftrightarrow \frac{(a^3-b^3)(b^3-c^3)}{a^3b^3}+\left(\frac{a^3}{b^3}+\frac{b^3}{c^3}+\frac{c^3}{a^3}-3\right)+\frac{3(c-a)^2}{ca} \ge 0.$$

$$Which\ is\ true\ by\ AM-GM\ \left(\therefore \frac{a^3}{b^3}+\frac{b^3}{c^3}+\frac{c^3}{a^3} \ge 3\right)\ and\ a \ge b \ge c.$$

$$So\ the\ proof\ is\ done.\ Equality\ holds\ iff\ \Delta ABC\ is\ equilateral.$$

815. In any Δ ABC holds:

$$\frac{a^4}{b^4}+\frac{b^4}{c^4}+\frac{c^4}{a^4}+\frac{R^4}{r^4} \ge 16+\frac{b^3}{a^3}+\frac{c^3}{b^3}+\frac{a^3}{c^3}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Lemma:\ \ In\ any\ \Delta ABC\ we\ have:\ \frac{b}{a}+\frac{c}{b}+\frac{a}{c} \le \frac{R}{r}+1.$$

$$Proof:\ \ We\ assume\ that\ c = \min\{a,b,c\}.\ If\ a \ge b \ge c\ \ we\ have:$$

$$\frac{R}{r}+1 \overset{Bandila}{\ge} \frac{a}{c}+\frac{c}{a}+1 = \frac{b}{a}+\frac{c}{b}+\frac{a}{c}+\frac{(a-b)(b-c)}{ab} \ge \frac{b}{a}+\frac{c}{b}+\frac{a}{c}.$$

$$If\ b \ge a \ge c\ we\ have:$$

$$\frac{R}{r}+1 \overset{Bandila}{\ge} \frac{b}{c}+\frac{c}{b}+1 = \frac{b}{a}+\frac{c}{b}+\frac{a}{c}+\frac{(b-a)(a-c)}{ca} \ge \frac{b}{a}+\frac{c}{b}+\frac{a}{c}.$$

$$So\ the\ proof\ of\ the\ lemma\ is\ complete.$$

$$Now,\ 16+\frac{b^3}{a^3}+\frac{c^3}{b^3}+\frac{a^3}{c^3}=16+\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)^3-3\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+3\le$$

$$\overset{Lemma\ \&\ AM-GM}{\overbrace{\le}}\ 16+\left(\frac{R}{r}+1\right)^3-3.3.3+3=3+\frac{R^4}{r^4}-\left(\frac{R}{r}-2\right)\left(\frac{R^3}{r^3}+\frac{R^2}{r^2}-\frac{R}{r}-5\right)\le$$

$$\overset{Euler}{\overbrace{\le}}\ 3+\frac{R^4}{r^4}\overset{AM-GM}{\overbrace{\le}}\frac{a^4}{b^4}+\frac{b^4}{c^4}+\frac{c^4}{a^4}+\frac{R^4}{r^4},\ as\ desired.$$

Equality holds iff ΔABC is equilateral.

816. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{s_a^4+(s_b+s_c)^4}{s_bs_c}\ge\frac{918r^3}{R}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{s_a^4+(s_b+s_c)^4}{s_bs_c}\overset{(x+y)^4\ge 8xy(x^2+y^2)}{\ge}\sum_{cyc}\frac{s_a^4+8s_bs_c(s_b^2+s_c^2)}{s_bs_c}$$

$$=\sum_{cyc}\frac{s_a^4}{s_bs_c}+8\sum_{cyc}(s_b^2+s_c^2)\overset{Bergstrom}{\ge}\frac{\left(\sum_{cyc}s_a^2\right)^2}{\sum_{cyc}s_bs_c}+16\sum_{cyc}s_a^2$$

$$\ge\frac{\left(\sum_{cyc}s_a^2\right)^2}{\sum_{cyc}s_a^2}+16\sum_{cyc}s_a^2=17\sum_{cyc}s_a^2\ge 17\sum_{cyc}h_a^2\ge\frac{17}{3}\left(\sum_{cyc}h_a\right)^2$$

$$\ge\frac{17}{3}(9r)^2=\frac{459r^3}{r}\overset{Bergstrom}{\ge}\frac{459r^3}{\frac{R}{2}}=\frac{918r^3}{R}\ (QED),$$

equality iff Δ ABC is equilateral

Solution 2 by Tapas Das-India

$$\sum_{cyc}\frac{s_a^4+(s_b+s_c)^4}{s_bs_c}=\sum_{cyc}\frac{s_a^4}{s_bs_c}+\sum_{cyc}\frac{(s_b+s_c)^4}{s_bs_c}\overset{Holder}{\ge}$$

$$\ge\frac{(s_a+s_b+s_c)^4}{3^2(s_as_b+s_bs_c+s_cs_a)}+\frac{(2s_a+2s_b+2s_c)^4}{3^2(s_as_b+s_bs_c+s_cs_a)}\overset{(1)}{\ge}$$

$$\geq \frac{(s_a+s_b+s_c)^4}{9\cdot\frac{(s_a+s_b+s_c)^2}{3}}+\frac{16(s_a+s_b+s_c)^4}{9\cdot\frac{(s_a+s_b+s_c)^2}{3}}=\frac{17}{3}(s_a+s_b+_c)^2\geq$$

$$\geq\frac{17}{3}(h_a+h_b+h_c)^2=\frac{17}{3}\cdot(9r)^2=\frac{17}{3}\cdot 81r^2=459r^2$$

We need to prove:

$$459r^2\geq\frac{918r^2}{R}\Leftrightarrow R\geq 2r\ (Euler).$$

$$(1)\Leftrightarrow(x+y+z)^2\geq 3(xy+yz+zx)$$

817. In ΔABC the following relationship holds:

$$8s_as_bs_c\leq\left(\frac{R}{2r}\right)^2(h_a+h_b)(h_b+h_c)(h_c+h_a)$$

Proposed by Marin Chirciu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$m_a^2m_b^2m_c^2=\frac{1}{64}(2b^2+2c^2-a^2)(2c^2+2a^2-b^2)(2a^2+2b^2-c^2)$$

$$\overset{(1)}{=}\frac{1}{64}\left\{-4\sum_{cyc}a^6+6\left(\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4\right)+3a^2b^2c^2\right\}$$

$$\text{Now,}\sum_{cyc}a^6=\left(\sum_{cyc}a^2\right)^3-3(a^2+b^2)(b^2+c^2)(c^2+a^2)$$

$$=\left(\sum_{cyc}a^2\right)^3-3\left(2a^2b^2c^2+\sum_{cyc}\left(a^2b^2\left(\sum_{cyc}a^2-c^2\right)\right)\right)$$

$$=\left(\sum_{cyc}a^2\right)^3+3a^2b^2c^2-3\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)$$

$$\therefore\sum_{cyc}a^6\overset{(2)}{=}\left(\sum_{cyc}a^2\right)^3+3a^2b^2c^2-3\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)$$

$$\text{Also,}\sum_{cyc} a^4b^2+\sum_{cyc} a^2b^4=\sum_{cyc}\left(a^2b^2\left(\sum_{cyc} a^2-c^2\right)\right)$$

$$\stackrel{(3)}{=}\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right)-3a^2b^2c^2 \therefore (1),(2),(3)\Rightarrow m_a^2m_b^2m_c^2$$

$$=\frac{1}{64}\left(-4\left(\sum_{cyc} a^2\right)^3-12a^2b^2c^2+12\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right)+6\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right)\right.$$

$$\left.-18a^2b^2c^2+3a^2b^2c^2\right)$$

Solution 2 by Tapas Das-India

$$\left(\frac{R}{2r}\right)^2(h_a+h_b)(h_b+h_c)(h_c+h_a)=$$

$$=\left(\frac{R}{2r}\right)^2(2F)^3\left(\frac{1}{a}+\frac{1}{b}\right)\left(\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{c}+\frac{1}{a}\right)=\left(\frac{R}{2r}\right)^2(2F)^3\cdot\frac{(a+b)(b+c)(c+a)}{a^2b^2c^2}=$$

$$=\left(\frac{R}{2r}\right)^2\cdot 8F^3\cdot\frac{(a+b+c)(ab+bc+ca)-abc}{(4Rrs)^2}=$$

$$=\frac{8r^3s^3}{16R^2r^2s^2}\cdot\left(\frac{R}{2r}\right)^2[2s(^2+r^2+4Rr)-4Rrs]=$$

$$=\frac{rs}{2R^2}\cdot\left(\frac{R}{2r}\right)^2\cdot 2s(s^2+r^2+2Rr)=\frac{rs}{R^2}\cdot\left(\frac{R}{2r}\right)^2\cdot s(s^2+r^2+2Rr)$$

$$s_as_bs_c\le m_am_bm_c\le\frac{Rs^2}{2},\ \ 8s_as_bs_c\le 4Rs^2$$

We need to show:

$$8\cdot\frac{Rs^2}{2}\le\frac{rs^2}{R^2}(s^2+r^2+2Rr)\left(\frac{R}{2r}\right)^2\Leftrightarrow 8\cdot\frac{Rs^2}{2}\le\frac{rs^2}{R^2}(s^2+r^2+2Rr)\cdot\frac{R^2}{4r^2}\Leftrightarrow$$

$$16Rr\le s^2+r^2+2Rr\Leftrightarrow s^2+r^2\ge 14Rr$$

$$14Rr\le 16Rr-5r^2+r^2(Gerretsen)\Leftrightarrow 2Rr\ge 4r^2$$

$$R\ge 2r\ (Euler)$$

Solution 3 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$s_a=\frac{bc.2m_a}{b^2+c^2}=\frac{bc\sqrt{(2b^2+2c^2-a^2)(a^2+b^2+c^2)}}{(b^2+c^2)\sqrt{a^2+b^2+c^2}}\overset{AM-GM}{\leq}$$

$$\leq\frac{bc[(2b^2+2c^2-a^2)+(a^2+b^2+c^2)]}{2(b^2+c^2)\sqrt{a^2+b^2+c^2}}\leq$$

$$\overset{AM-QM}{\leq}\frac{bc.3(b^2+c^2).\sqrt{3}}{2(b^2+c^2)(a+b+c)}=\frac{3\sqrt{3}bc}{4s}\quad (and\ analogs)$$

$$Then:\ 8s_as_bs_c\leq 8.\frac{\left(3\sqrt{3}\right)^3(abc)^2}{(4s)^3}=\frac{81\sqrt{3}.(4Rsr)^2}{8s^3}$$

$$=\frac{162\sqrt{3}R^2r^2}{s}\overset{Mitrinovic}{\leq}54R^2r\ \ (1)$$

$$Now\ we\ have:\ \left(\frac{R}{2r}\right)^2\prod_{cyc}(h_b+h_c)=\left(\frac{R}{2r}\right)^2.\frac{(2sr)^3}{(4Rsr)^2}\prod_{cyc}(b+c)$$

$$=\frac{s}{8r}.2s(s^2+r^2+2Rr)\geq$$

$$\overset{Gerretsen}{\geq}\frac{s^2(18Rr-4r^2)}{4r}\overset{Cosnita\ \&\ Turtoiu}{\geq}\frac{27Rr}{2}.\frac{9R-2r}{2}\overset{Euler}{\geq}\frac{27Rr}{2}.4R$$

$$=54R^2r\overset{(1)}{\geq}8s_as_bs_c.$$

$$So\ the\ proof\ is\ done.\ Equality\ holds\ iff\ \Delta ABC\ is\ equilateral.$$

818. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{r_a-h_a}{r_a+h_a}\geq\frac{(a-b)^2+(b-c)^2+(c-a)^2}{2(a^2+b^2+c^2)}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\frac{r_a-h_a}{r_a+h_a}=\frac{\frac{F}{s-a}-\frac{2F}{a}}{\frac{F}{s-a}+\frac{2F}{a}}=\frac{\frac{1}{s-a}-\frac{2}{a}}{\frac{1}{s-a}+\frac{2}{a}}=\frac{3a-2s}{2s-a}=\frac{2a-(b+c)}{b+c}$$

$$\sum_{cyc}\frac{r_a-h_a}{r_a+h_a}=\sum_{cyc}\frac{2a-(b+c)}{b+c}=\sum_{cyc}\left(\frac{2a}{b+c}-1\right)=2\sum_{cyc}\frac{a}{b+c}-3=$$

$$= 2\sum_{cyc}\frac{a^2}{ab+ac} - 3 \geq \frac{2(a+b+c)^2}{2(ab+bc+ca)} - 3 =$$

$$= \frac{(a+b+c)^2 - 3(ab+bc+ca)}{ab+bc+ca} =$$

$$= \frac{2a^2+2b^2+2c^2-2ab-2bc-2ca}{ab+bc+ca} \geq \frac{(a-b)^2+(b-c)^2+(c-a)^2}{2(a^2+b^2+c^2)}$$

Equality holds for $a = b = c$.

819. *In ΔABC the following relationship holds:*

$$160\sum_{cyc}\left(\frac{A}{\pi}\right)^2 - 27\sum_{cyc}\left(1-\frac{2A}{\pi}\right)^5 \geq 53$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$Let\ x := \frac{A}{\pi},\ y := \frac{B}{\pi},\ z := \frac{C}{\pi}\ and\ p := x+y+z = 1,\ q := xy+yz+zx,\ r := xyz.$

$$We\ have:$$

$$\sum_{cyc}\left(\frac{A}{\pi}\right)^2 = \sum_{cyc}x^2 = 1-2q\ and\ using\ the\ following\ identity:$$

$$u^5+v^5+w^5 = (u+v+w)^5 - 5(u+v)(v+w)(w+u)\left(u^2+v^2+w^2+uv+vw+wu\right),$$

$$we\ get:$$

$$\sum_{cyc}(1-2x)^5 =$$

$$= \left(3-2\sum x\right)^5 - 5\prod[2-2(x+y)].\left(\sum(1-2x)^2 + \sum(1-2x)(1-2y)\right) =$$

$$= 1 - 5.8xyz\left(4\sum x^2 + 4\sum yz - 2\right) = 1 - 40r(2-4q) = 1 - 80r(1-2q).$$

$$So\ the\ inequality\ becomes:$$

$$160(1-2q) - 27[1-80r(1-2q)] \geq 53$$

$$or\ \ 1-4q+27r(1-2q)\geq 0\ \ or\ \ 9r\geq\frac{4q-1}{3(1-2q)}.\ \left(\because q\leq\frac{p^2}{3}=\frac{1}{3}\right)$$

$$The\ inequality\ is\ true\ for\ 4q\leq 1.$$

$$If\ 4q\geq 1\ we\ have:\ 9r\overset{Schur}{\geq}4pq-p^3=4q-1\overset{q\leq\frac{1}{3}}{\geq}\frac{4q-1}{3(1-2q)}.$$

$$So\ the\ proof\ is\ completed.\ Equality\ holds\ iff\ \Delta ABC\ is\ equilateral.$$

820. ***If ω is Brocard's angle in ΔABC then***

$$\cos A+\frac{m_c}{h_a}\cos B+\frac{m_b}{h_a}\cos C\leq\frac{1}{2}\left(\frac{1}{\sin\omega}+\frac{m_bm_c}{{h_a}^2}\right)$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Lemma:\ \ In\ any\ \Delta ABC\ we\ have:\ \frac{1}{\sin\omega}\geq\frac{b}{c}+\frac{c}{b}\ \ (*)$$

$$Proof:\ \ Since\ \ \sin\omega=\frac{2F}{\sqrt{a^2b^2+b^2c^2+c^2a^2}}\ \ and\ 4F$$
$$=\sqrt{2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)},$$

$$then:\ (*)\ \Leftrightarrow\ 2bc\sqrt{a^2b^2+b^2c^2+c^2a^2}$$
$$\geq(b^2+c^2)\sqrt{2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)}$$

$$\overset{squaring}{\Leftrightarrow}\ \ 4b^2c^2(a^2b^2+b^2c^2+c^2a^2)$$
$$\geq(2b^2c^2+b^4+c^4)[2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)]$$

$$\Leftrightarrow\ 0\geq -a^4(b^2+c^2)^2+2(b^4+c^4)(a^2b^2+c^2a^2)-(b^4+c^4)^2$$
$$=-[a^2(b^2+c^2)-(b^4+c^4)]^2$$

$$Which\ is\ true\ and\ the\ proof\ of\ the\ lemma\ is\ complete.$$

$$In\ \Delta m_am_bm_c\ we\ get:\ \frac{m_b}{m_c}+\frac{m_c}{m_b}\overset{Lemma}{\leq}\frac{1}{\sin\omega_m}=\frac{\sqrt{\sum{m_b}^2{m_c}^2}}{2F_m}=\frac{\sqrt{\frac{9}{16}\sum b^2c^2}}{\frac{3F}{2}}=\frac{1}{\sin\omega}.$$

$$So\ it\ suffices\ to\ prove:\quad \cos A+\frac{m_c}{h_a}\cos B+\frac{m_b}{h_a}\cos C\le\frac{1}{2}\left(\frac{m_b}{m_c}+\frac{m_c}{m_b}+\frac{m_bm_c}{{h_a}^2}\right)$$

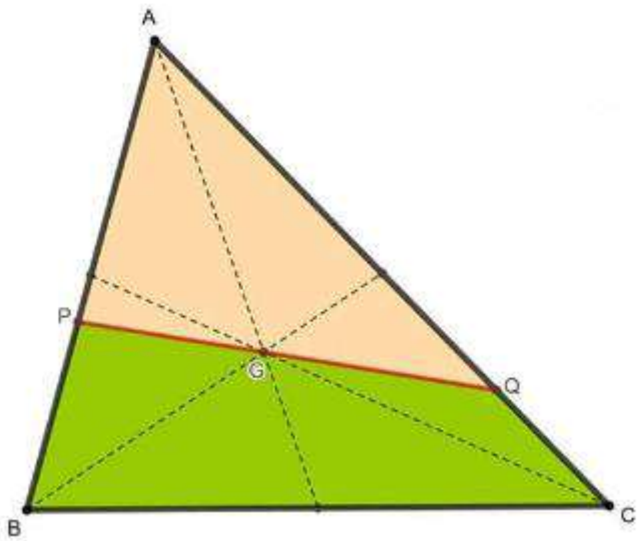
$$\overset{A=\pi-(B+C)}{\Leftrightarrow}\quad -(\cos B\cos C-\sin B\sin C)+\frac{m_c}{h_a}\cos B+\frac{m_b}{h_a}\cos C\le\frac{1}{2}\left(\frac{m_b}{m_c}+\frac{m_c}{m_b}+\frac{m_bm_c}{{h_a}^2}\right)$$

$$\Leftrightarrow\frac{1}{2}\left(\sqrt{\frac{m_c}{m_b}}\cos B+\sqrt{\frac{m_b}{m_c}}\cos C-\sqrt{\frac{m_bm_c}{{h_a}^2}}\right)^2+\frac{1}{2}\left(\sqrt{\frac{m_c}{m_b}}\sin B-\sqrt{\frac{m_b}{m_c}}\sin C\right)^2\ge 0.$$

$$Which\ is\ true\ and\ the\ proof\ is\ completed.$$

821. In $\Delta ABC, G$ −centroid, $\frac{AP}{AB}=p$ the following relationship holds:

$$\frac{PG}{GQ}=3p-1$$



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution 1 by Miguel Angel Perez Garcia Ortega-Mexico

Let $P(1-p;p;0)$ barycentric coordinates of ΔABC, then:

$$PG=px+(p-1)y+(1-2p)z=0$$

$$Q = PG \cap CA = (2p-1; 0; p) = \left(\frac{2p-1}{3p-1}; 0; \frac{p}{3p-1}\right)$$

$$G = P + (3p-1)Q$$

Hence:

$$\frac{PG}{GQ} = 3p-1$$

Solution 2 by Jose Ferreira Queiroz-Olinda-Brazil

Using Gakopoulos' Lemmas, we have:

$$(1)\quad \frac{AB}{AP} + \frac{AC}{AQ} = 3 \Rightarrow \frac{AC}{AQ} = 3 - \frac{1}{p}$$

$$(2)\quad \frac{PG}{GQ} = \frac{BM}{MC} \cdot \frac{AP}{AB} \cdot \frac{AC}{AQ}$$

$$\frac{PG}{GQ} = p\left(3 - \frac{1}{p}\right) \Leftrightarrow \frac{PG}{GQ} = 3p-1$$

822. In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{\left(\csc\frac{B}{2} + \csc\frac{C}{2}\right)^{n+1}}{\left(\csc\frac{A}{2} + \sqrt{\csc\frac{B}{2}\csc\frac{C}{2}}\right)^n} \geq 12;\ n \in \mathbb{N}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sqrt{\csc\frac{B}{2}\csc\frac{C}{2}} \overset{AM-GM}{\leq} \frac{1}{2}\left(\csc\frac{B}{2} + \csc\frac{C}{2}\right)$$

$$\csc\frac{A}{2} + \sqrt{\csc\frac{B}{2}\csc\frac{C}{2}} \leq \csc\frac{A}{2} + \frac{1}{2}\left(\csc\frac{B}{2} + \csc\frac{C}{2}\right)$$

Analogous:

$$\csc\frac{B}{2} + \sqrt{\csc\frac{C}{2}\csc\frac{A}{2}} \leq \csc\frac{B}{2} + \frac{1}{2}\left(\csc\frac{C}{2} + \csc\frac{A}{2}\right) \text{ and}$$

$$\csc\frac{C}{2}+\sqrt{\csc\frac{B}{2}+\csc\frac{A}{2}}\le\csc\frac{C}{2}+\frac{1}{2}\left(\csc\frac{A}{2}+\csc\frac{B}{2}\right)$$

$$\sum_{cyc}\left(\csc\frac{A}{2}+\sqrt{\csc\frac{B}{2}\csc\frac{C}{2}}\right)\le 2\sum_{cyc}\csc\frac{A}{2};\qquad(1)$$

Now,

$$\sum_{cyc}\frac{\left(\csc\frac{B}{2}+\csc\frac{C}{2}\right)^{n+1}}{\left(\csc\frac{A}{2}+\sqrt{\csc\frac{B}{2}\csc\frac{C}{2}}\right)^{n}}\overset{Radon}{\ge}\frac{2^{n+1}\left(\sum\csc\frac{A}{2}\right)^{n+1}}{\sum\left(\csc\frac{A}{2}+\sqrt{\csc\frac{B}{2}\csc\frac{C}{2}}\right)^{n}}\overset{(1)}{\ge}$$

$$\ge\frac{2^{n+1}\left(\sum\csc\frac{A}{2}\right)^{n+1}}{2^{n}\left(\sum\csc\frac{A}{2}\right)^{n}}=2\sum_{cyc}\csc\frac{A}{2}\overset{\substack{Jensen\\ x\to\csc\frac{x}{2}\ is\ convex}}{\ge}3\csc\frac{\pi}{6}=6$$

823. If $x\in\mathbb{R}$ then in ΔABC holds:

$$(1-x)\cos A+\left(1+\frac{r}{R}\right)x\le 1+\frac{x^2}{2}$$

Proposed by Bogdan Fuştei-Romania

Solution 1 by Tapas Das-India

$$1+\frac{r}{R}=\cos A+\cos B+\cos C$$

Case 1: If $x>0$, we need to show:

$$(1-x)\cos A+\left(1+\frac{r}{R}\right)x\le 1+\frac{x^2}{2}$$

$$(1-x)\cos A+(\cos A+\cos B+\cos C)x\le 1+\frac{x^2}{2}$$

$$\cos A+(\cos B+\cos C)x\le 1+\frac{x^2}{2}$$

$$\cos A+2\cos\frac{B+C}{2}\cos\frac{B-C}{2}\cdot x\le 1+\frac{x^2}{2}$$

$$1-2\sin^2\frac{A}{2}+2\sin\frac{A}{2}\cos\frac{B-C}{2}\cdot x\le 1+\frac{x^2}{2}$$

$$-2\sin^2\frac{A}{2}+2\sin\frac{A}{2}\cos\frac{B-C}{2}\cdot x\le\frac{x^2}{2}$$

$$x^2-4\sin\frac{x}{2}\cos\frac{B-C}{2}\cdot x+4\sin^2\frac{x}{2}\ge 0;\qquad (1)$$

$$a>0;(\text{coefficient of } x^2);$$

$$\Delta=\left(4\sin\frac{A}{2}\cos\frac{B-C}{2}\right)^2-4\cdot 4\cdot\sin^2\frac{A}{2}=-16\sin^2\frac{A}{2}\left(1-\cos^2\frac{B-C}{2}\right)=$$

$$=-16\sin^2\frac{A}{2}\sin^2\frac{B-C}{2}<0\Rightarrow(1)\text{ is true.}$$

Similarly, if we take $x<0$, then from (1), we get the similar result.

$$(1-x)\cos A+\left(1+\frac{r}{R}\right)x\le 1+\frac{x^2}{2};x\in\mathbb{R}$$

Solution 2 by Ravi Prakash-New Delhi-India

$$1+\frac{r}{R}=\cos A+\cos B+\cos C$$

$$x^2-2x(\cos B+\cos C)+2(1-\cos A)\ge 0;\qquad (1)$$

$$\text{Now, }(\cos B+\cos C)^2\le 4\cos^2\frac{B+C}{2}\cos^2\frac{B-C}{2}-4\sin^2\frac{A}{2}=$$

$$=4\sin^2\frac{A}{2}\left(\cos^2\frac{B-C}{2}-1\right)=-4\sin^2\frac{A}{2}\cos^2\frac{B-C}{2}\le 1.$$

Thus, (1) its true.

824. In ΔABC the following relationship holds:

$$\frac{9R}{4r}\ge\left(\sum_{cyc}\sqrt{\frac{m_a}{m_b+m_c}}\right)^2\ge\frac{9r}{R}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$(1):\ \sum_{cyc}\frac{m_b+m_c}{m_a}=\sum_{cyc}\left(\frac{m_b}{m_a}+\frac{m_c}{m_a}\right)\overset{CBS}{\le}2\sqrt{\left(\sum_{cyc}m_b^2\right)\left(\sum_{cyc}\frac{1}{m_a^2}\right)}\overset{m_a\ge\sqrt{s(s-a)}}{\le}$$

$$\le 2\sqrt{\left(\frac{3}{4}\sum_{cyc}a^2\right)\left(\sum_{cyc}\frac{1}{s(s-a)}\right)} \overset{Leibniz}{\le} \sqrt{3\cdot 9R^2\cdot\frac{4R+r}{s^2r}} \overset{Doucet}{\le}$$

$$\le 3R\cdot\frac{\sqrt{\frac{3(4R+r)}{3r(4R+r)r}}\,3R}{r}$$

Hence, we have:

$$\sum_{cyc}\sqrt{\frac{m_a}{m_b+m_c}} \overset{CBS}{\le} \sqrt{3\sum_{cyc}\frac{m_a}{m_b+m_c}}$$

$$\left(\sum_{cyc}\sqrt{\frac{m_a}{m_b+m_c}}\right)^2 \le 3\sum_{cyc}\frac{m_a}{m_b+m_c} \overset{AM-HM}{\le} \frac{3}{4}\sum_{cyc}\frac{m_b+m_c}{m_a} =$$

$$= \frac{3}{4}\sum_{cyc}\left(\frac{m_b}{m_a}+\frac{m_c}{m_a}\right) \le \frac{3}{4}\cdot\frac{3R}{r} = \frac{9R}{4r}$$

$$\text{Let: } \frac{m_a}{m_b+m_c} = x; \frac{m_b}{m_c+m_a} = y; \frac{m_c}{m_a+m_b} = z$$

$$\left(\sum_{cyc}\sqrt{\frac{m_a}{m_b+m_c}}\right)^2 = \left(\sum_{cyc}\sqrt{x}\right)^2 \overset{AM-GM}{\ge} \left[3\sqrt[6]{xyz}\right]^2 = 9\cdot\sqrt[3]{xyz} \overset{GM-HM}{\ge}$$

$$\ge 9\cdot\frac{3}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}} = 9\cdot\frac{3}{\sum\frac{m_a}{m_b+m_c}} \ge \frac{27}{\sum\frac{m_a}{m_b+m_c}} \ge \frac{27}{\frac{3R}{r}} = \frac{9r}{R}$$

825. *Let ABC be a triangle with its inradius r and circumradius R.*

$$\textit{Prove that}: \quad \cos\frac{A+B}{4}\cos\frac{B+C}{4}\cos\frac{C+A}{4} \ge \left(\frac{\sqrt{3}}{2}\right)^3\sqrt[3]{\frac{2r}{R}}$$

Proposed by Kunihiko Chikaya-Tokyo-Japan

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Lemma :

$$\textit{If } x+y+z=\pi \textit{ then we have}: \; 4\cos\frac{x}{2}\cos\frac{y}{2}\cos\frac{z}{2} = \sin x + \sin y + \sin z.$$

$$Proof: We\ have: LHS = 2\left(\cos\frac{x+y}{2}+\cos\frac{x-y}{2}\right)\cos\frac{z}{2} = 2\left(\sin\frac{z}{2}+\cos\frac{x-y}{2}\right)\cos\frac{z}{2} =$$

$$= \sin z + 2\cos\frac{x-y}{2}\cos\frac{z}{2} = \sin z + \cos\frac{x-y+z}{2} + \cos\frac{x-y-z}{2} =$$

$$= \sin z + \sin y + \sin x = RHS.$$

$$For\ x = \frac{A+B}{2} = \frac{\pi}{2} - \frac{C}{2},\ \ y = \frac{B+C}{2},\ \ z = \frac{C+A}{2}, we\ get:$$

$$\cos\frac{A+B}{4}\cos\frac{B+C}{4}\cos\frac{C+A}{4} = \frac{1}{4}\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right) \overset{AM-GM}{\geq} \frac{3}{4}\sqrt[3]{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}.$$

$$Also\ we\ have:\ \ \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{s}{4R}\ and\ s \geq 3\sqrt{3}r\ (Mitrinovic's\ inequality)$$

$$Therefore,\ \ \cos\frac{A+B}{4}\cos\frac{B+C}{4}\cos\frac{C+A}{4} \geq \frac{3}{4}\sqrt[3]{\frac{3\sqrt{3}r}{4R}} = \left(\frac{\sqrt{3}}{2}\right)^3\sqrt[3]{\frac{2r}{R}}.$$

$$Equality\ holds\ iff\ \Delta ABC\ is\ equilateral.$$

826. In ΔABC the following relationship holds:

$$\sum_{cyc} a^4 + 6\left(\sum_{cyc} ab\right)^2 + 4a\sum_{cyc} ab(a^2+b^2) \geq 11664r^4$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Tapas Das-India

$$(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$$

$$(a+b+c)^4 = \left(\sum_{cyc} a^2\right)^2 + 4\left(\sum_{cyc} ab\right)^2 + 4\sum_{cyc} a^2 \cdot \sum_{cyc} ab =$$

$$= \sum_{cyc} a^4 + 4\left(\sum_{cyc} ab\right)^2 + 2[\sum_{cyc} a^2b^2 + 2\sum_{cyc} a^2bc + 4\sum_{cyc} ab(a^2+b^2) =$$

$$=\sum_{cyc}a^4+4\left(\sum_{cyc}ab\right)^2+2\left(\sum_{cyc}ab\right)^2+4\sum_{cyc}ab(a^2+b^2)=$$

$$=(a+b+c)^4=(2s)^4=16(s^2)^2\ge 16(27r^2)^2=11664r^4$$

Solution 2 by Tapas Das-India

$$\sum_{cyc}a^4+6\left(\sum_{cyc}ab\right)^2+4\sum_{cyc}ab(a^2+b^2)\overset{AGM}{\ge}3(abc)^{\frac{4}{3}}+54(abc)^{\frac{4}{3}}+4\sum_{cyc}ab\cdot 2ab=$$

$$=2(abc)^{\frac{4}{3}}+54(abc)^{\frac{4}{3}}+8\sum_{cyc}a^2b^2\overset{AM-GM}{\ge}57(abc)^{\frac{4}{3}}+24(abc)^{\frac{4}{3}}=81(abc)^{\frac{4}{3}}\ge$$

$$\ge 81(4R\cdot F)^{\frac{4}{3}}\ge 81\left(8r^2\cdot 3\sqrt{3}r\right)^{\frac{4}{3}}=81\left(2^3\cdot 3^{\frac{3}{2}}\cdot r^3\right)^{\frac{4}{3}}=$$

$$=81\cdot 2^4\cdot 3^2\cdot r^4=11664r^4$$

827. *In ΔABC the following relationship holds:*

$$\sum_{cyc}m_a\sqrt{5a^2+8m_bm_c-b^2-c^2}\ge 3F\sum_{cyc}\left(\sqrt{\frac{m_b}{m_c}}+\sqrt{\frac{m_c}{m_b}}\right)$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{We know that: } w_a\ge h_a,$$

$$\text{with: } w_a=\frac{\sqrt{bc}.\sqrt{(a+b+c)(-a+b+c)}}{b+c}\ \text{ and }\ h_a=\frac{2F}{a}.$$

$$\text{Then: } a\sqrt{b^2+2bc+c^2-a^2}\ge 2F\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\quad (1)$$

$$m_a,m_b,m_c-\text{can be the sides of a triangle with area } F_m.$$

$$\text{Applying (1) in } \Delta m_am_bm_c, \text{we have:}$$

$$m_a\sqrt{{m_b}^2+2m_bm_c+{m_c}^2-{m_a}^2}\ge 2F_m\left(\sqrt{\frac{m_b}{m_c}}+\sqrt{\frac{m_c}{m_b}}\right),\ with:$$

$${m_a}^2=\frac{2b^2+2c^2-a^2}{4}\ (and\ analogs)\ and\ F_m=\frac{3F}{4}.$$

$$Then:\ m_a\sqrt{5a^2+8m_bm_c-b^2-c^2}\ge 3F\left(\sqrt{\frac{m_b}{m_c}}+\sqrt{\frac{m_c}{m_b}}\right)\ (and\ analogs)$$

$$Therefore,\ \sum_{cyc}m_a\sqrt{5a^2+8m_bm_c-b^2-c^2}\ge 3F\sum_{cyc}\left(\sqrt{\frac{m_b}{m_c}}+\sqrt{\frac{m_c}{m_b}}\right).$$

828. In ΔABC the following relationship holds:

$$\frac{1}{3r^2}\le\sum_{cyc}\frac{1}{w_a^2\cos^2\frac{B-C}{2}}\le\frac{R}{6r^3}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Tapas Das-India

$$\cos\frac{A-B}{2}=\frac{a+b}{c}\sin\frac{C}{2}\text{ and }w_a^2=\frac{4bcs(s-a)}{(b+c)^2}$$

$$w_a^2\cos^2\frac{B-C}{2}=\frac{4bcs(s-a)}{(b+c)^2}\cdot\frac{(b+c)^2}{a^2}\cdot\sin^2\frac{A}{2}=$$

$$=\frac{4bcs(s-a)}{a^2}\cdot\frac{(s-b)(s-c)}{bc}=\frac{4s(s-a)(s-b)(s-c)}{a^2}=$$

$$=\frac{4s\cdot sr^2}{a^2}=\frac{4s^2r^2}{a^2}\ (and\ analogs)$$

$$\sum_{cyc}\frac{1}{w_a^2\cos^2\frac{B-C}{2}}=\frac{1}{4s^2r^2}\sum_{cyc}a^2=\frac{1}{4s^2r^2}\cdot 2(s^2+r^2-4Rr)\overset{Gerretsen}{\le}$$

$$\le\frac{4R^2+4Rr+3r^2-r^2-4Rr}{2r^2(16Rr-5r^2)}=\frac{4R^2+2r^2}{2r^2(16Rr-5r^2)}=\frac{2R^2+r^2}{r^2(16Rr-5r^2)}$$

Now, we need to show:

$$\frac{2R^2+r^2}{r^2(16Rr-5r^2)}\le\frac{R}{6r^2}\Leftrightarrow(2R^2+r^2)6r\le R(16Rr-5r^2)\Leftrightarrow$$

$4R^2r-5Rr^2-6r^3\ge 0\Leftrightarrow (R-2r)(4R+3r)\ge 0$ which is true from

$R\ge 2r\ (Euler)$.

$$\sum_{cyc}\frac{1}{w_a^2\cos^2\frac{B-C}{2}}=\frac{1}{4s^2r^2}\sum_{cyc}a^2$$

We need to show:

$$\frac{1}{4s^2r^2}\sum_{cyc}a^2\ge\frac{1}{3r^2}\Leftrightarrow\frac{1}{4s^2}\sum_{cyc}a^2\ge\frac{1}{3}\Leftrightarrow 3\sum_{cyc}a^2\ge 4s^2$$

$$3(a^2+b^2+c^2)\ge(a+b+c)^2\Leftrightarrow(a-b)^2+(b-c)^2+(c-a)^2\ge 0$$

Solution 2 by Ertan Yildirim-Turkiye

(1) $$\cos\frac{B-C}{2}=\frac{h_a}{w_a}$$

(2) $$\sum_{cyc}a^2\le 9R^2(Leibniz)$$

(3) $$\sum_{cyc}\frac{1}{h_a}=\frac{1}{r}$$

(4) $$27Rr\le 2s^2$$

$$\sum_{cyc}\frac{1}{w_a^2\cos^2\frac{B-C}{2}}\overset{Lemma\ 1}{=}\sum_{cyc}\frac{1}{w_a^2\cdot\frac{h_a^2}{w_a^2}}=\sum_{cyc}\frac{1}{h_a^2}$$

Since: $3(x^2+y^2+y^2)\ge(x+y+z)^2$

$$3\sum_{cyc}\frac{1}{h_a^2}\ge\left(\sum_{cyc}\frac{1}{h_a}\right)^2\overset{Lemma\ 3}{=}\frac{1}{r^2}\Rightarrow\sum_{cyc}\frac{1}{h_a^2}=\frac{1}{3r^2}$$

$$\sum_{cyc}\frac{1}{w_a^2\cos^2\frac{B-C}{2}}\overset{Lemma\ 1}{=}\sum_{cyc}\frac{1}{w_a^2\cdot\frac{h_a^2}{w_a^2}}=\sum_{cyc}\frac{1}{h_a^2}=\sum_{cyc}\frac{4R^2}{(bc)^2}=$$

$$=\frac{4R^2}{(abc)^2}\cdot\sum_{cyc}a^2=\frac{4R^2}{16s^2r^2R^2}\cdot\sum_{cyc}a^2\Rightarrow$$

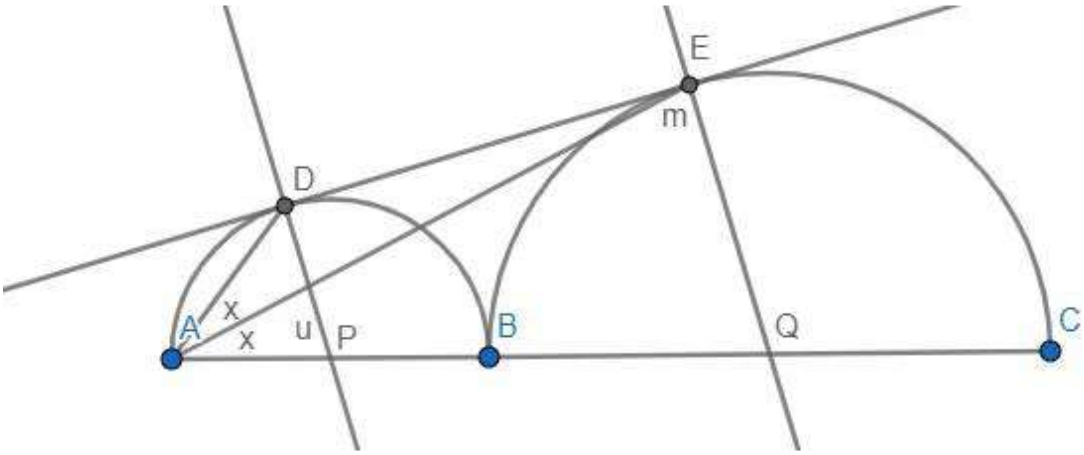
$$\frac{1}{4s^2r^2}\cdot\sum_{cyc}a^2\overset{Lemma\ 2}{\le}\frac{1}{4s^2r^2}\cdot 9R^2\overset{Lemma\ 4}{\le}\frac{1}{2\cdot 27Rr\cdot r^2}\cdot 9R^2$$

829. Prove that:

$$|AE^2|\left(\frac{1}{r^2}-\frac{1}{R^2}\right)=8 \qquad R,r-radies$$

Proposed by Murat Oz-Turkiye

Solution by Soumava Chakraborty-Kolkata-India



$$\Delta\, PAD \Rightarrow m(\measuredangle PAD)+m(\measuredangle PDA)+u=180^\circ$$

$$\Rightarrow 2x+2x+u=180^\circ\ (\because PA=PD=r)\Rightarrow u\overset{(*)}{=}180^\circ-4x$$

$$DP\,||\,EQ\Rightarrow m(\measuredangle EQA)=u\ \therefore \text{via } \Delta\, EQA,\ u+m+x=180^\circ$$

$$\overset{\text{via }(*)}{\Rightarrow} 180^\circ-4x+m+x=180^\circ\Rightarrow m\overset{(**)}{=}3x$$

$$\textit{Cosine law}\Rightarrow AD^2=PA^2+PD^2-2PA.PD.\cos u$$

$$=r^2+r^2-2r^2\cos u\overset{\text{via }(*)}{=}2r^2-2r^2\cos(180^\circ-4x)$$

$$=2r^2(1+\cos 4x)=4r^2\cos^2 2x\Rightarrow AD\overset{(***)}{=}2r.\cos 2x$$

$$\text{Now, }\textit{Sine law}\text{ on }\Delta\, ADE\Rightarrow\frac{AE}{\sin(90^\circ+2x)}=\frac{AD}{\sin(90^\circ-m)}$$

$$\overset{\text{via }(**),(***)}{\Rightarrow}\frac{AE}{\cos 2x}=\frac{2r.\cos 2x}{\cos 3x}\Rightarrow\frac{AE}{r}\overset{(\bullet)}{=}\frac{2\cos^2 2x}{\cos 3x}$$

$$\text{Again, sine }\textit{law}\text{ on }\Delta\, AEQ\Rightarrow\frac{AE}{\sin u}=\frac{EQ}{\sin x}\overset{\text{via }(*)}{\Rightarrow}\frac{AE}{\sin(180^\circ-4x)}=\frac{R}{\sin x}$$

$$\Rightarrow\frac{AE}{R}\overset{(\bullet\bullet)}{=}\frac{\sin 4x}{\sin x}$$

$$\text{Also, sine }\textit{law}\text{ on }\Delta\, AEQ\Rightarrow\frac{AE}{\sin u}=\frac{AQ}{\sin m}\overset{\text{via }(*),(**)}{\Rightarrow}\frac{AE}{\sin 4x}=\frac{R+2r}{\sin 3x}$$

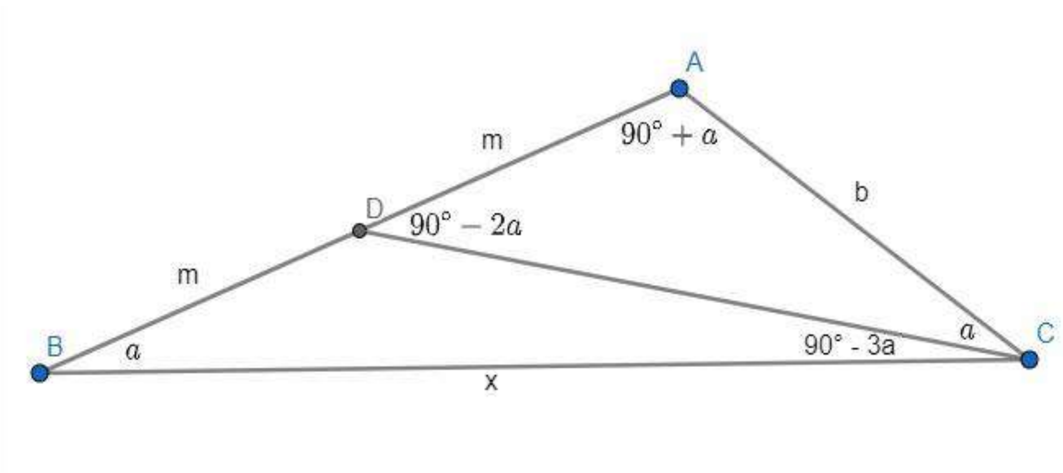
$$\Rightarrow\frac{R+2r}{AE}\overset{(\blacksquare)}{=}\frac{\sin 3x}{\sin 4x}$$

$$\text{Now, }(\bullet),(\bullet\bullet)\Rightarrow\frac{2r}{AE}+\frac{R}{AE}=\frac{\cos 3x}{\cos^2 2x}+\frac{\sin x}{\sin 4x}$$

$$\Rightarrow\frac{R+2r}{AE}\overset{(\blacksquare\blacksquare)}{=}\frac{\cos 3x.\sin 4x+\sin x.\cos^2 2x}{\sin 4x.\cos^2 2x}$$

$$\therefore (\blacksquare),(\blacksquare\blacksquare) \Rightarrow \frac{\sin 3x}{\sin 4x} = \frac{\cos 3x.\sin 4x + \sin x.\cos^2 2x}{\sin 4x.\cos^2 2x}$$
$$\Rightarrow \cos^2 2x.(\sin 3x - \sin x) = \cos 3x.\sin 4x$$
$$\Rightarrow 2\cos^2 2x.\cos 2x.\sin x = 4\cos 3x.\cos 2x.\sin x.\cos x$$
$$\Rightarrow 2\cos^2 2x = 4\cos 3x.\cos x\ (\because 180^\circ - 4x > 0 \Rightarrow x < 45^\circ \Rightarrow \cos 2x \neq 0)$$
$$\Rightarrow 1 + \cos 4x = 2(\cos 4x + \cos 2x) \Rightarrow \cos 4x + 2\cos 2x - 1 = 0$$
$$\Rightarrow \cos^2 2x + \cos 2x - 1 = 0 \Rightarrow \cos 2x = \frac{-1 \pm \sqrt{5}}{2} \Rightarrow \cos 2x \overset{(*)}{=} \frac{\sqrt{5}-1}{2}$$
$$\left(\because 0 < 2x < \frac{\pi}{2} \Rightarrow \cos 2x > 0\right) \therefore 2\cos^2 x = \frac{\sqrt{5}+1}{2} \Rightarrow \cos^2 x \overset{(**)}{=} \frac{\sqrt{5}+1}{4}$$
$$\frac{AE^2}{r^2} \overset{\text{via }(\bullet)}{=} \frac{(2\cos^2 2x)^2}{\cos^2 x\,(4\cos^2 x - 3)^2} \overset{\text{via }(*),(**)}{=} \frac{4\left(\frac{\sqrt{5}-1}{2}\right)^4}{\left(\frac{\sqrt{5}+1}{4}\right)(\sqrt{5}-2)^2} = \frac{(14-6\sqrt{5}).4}{5\sqrt{5}-11}$$
$$= \frac{(14-6\sqrt{5})(5\sqrt{5}-11)(5\sqrt{5}+11)}{5\sqrt{5}-11} = 70\sqrt{5} - 66\sqrt{5} + 154 - 150$$
$$\therefore \frac{AE^2}{r^2} \overset{(i)}{=} 4\sqrt{5} + 4$$
$$\frac{AE^2}{R^2} \overset{\text{via }(\bullet\bullet)}{=} \left(\frac{\sin 4x}{\sin x}\right)^2 = 16\cos^2 x.\cos^2 2x \overset{\text{via }(*),(**)}{=} 4(\sqrt{5}+1).\frac{6-2\sqrt{5}}{4}$$
$$\therefore \frac{AE^2}{R^2} \overset{(ii)}{=} 4\sqrt{5} - 4 \therefore (i) - (ii) \Rightarrow |AE^2|\left(\frac{1}{r^2} - \frac{1}{R^2}\right) = 4\sqrt{5} + 4 - 4\sqrt{5} + 4 = 8\ (ans)$$

830.



Prove that:

$$\frac{BC}{AC} = \sqrt{\varphi^3}$$

Proposed by Murat Oz-Turkiye

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Sine } law \text{ on } \Delta\, ACD \Rightarrow \frac{b}{\sin(90^\circ - 2a)} = \frac{m}{\sin a} \Rightarrow m \overset{(*)}{=} \frac{b\sin a}{\cos 2a}$$

$$and \text{ sine } law \text{ on } \Delta\, ABC \Rightarrow \frac{x}{\sin(90^\circ + a)} = \frac{b}{\sin a} = \frac{2m}{\cos(90^\circ - 2a)} \to (1)$$

$$and\ (1) \Rightarrow m \overset{(**)}{=} \frac{b\cos 2a}{2\sin a}$$

$$(*), (**) \Rightarrow \frac{b\sin a}{\cos 2a} = \frac{b\cos 2a}{2\sin a} \Rightarrow \cos^2 2a = 2\sin^2 a = 1 - \cos 2a$$

$$\Rightarrow \cos^2 2a + \cos 2a - 1 = 0 \Rightarrow \cos 2a \overset{(***)}{=} \frac{\sqrt{5}-1}{2}$$

$$(\because 90^\circ - 3a > 0 \Rightarrow 0 < 2a < 60^\circ)$$

$$\text{Now}, \frac{BC}{AC} = \frac{x}{b} = \frac{\frac{b\cos a}{\sin a}}{b} \left(\because \frac{x}{\sin(90^\circ + a)} = \frac{b}{\sin a} \text{ from } (1)\right) \Rightarrow \frac{BC}{AC} \overset{(****)}{=} \cot a$$

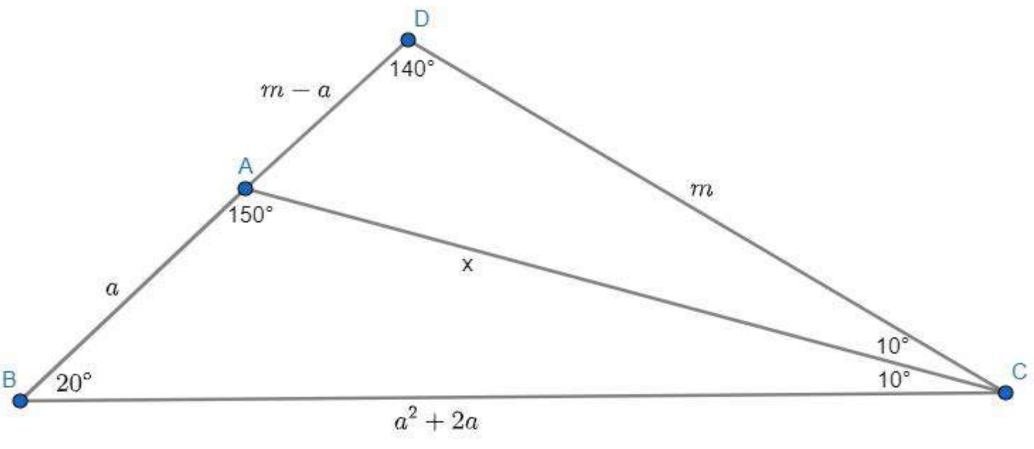
$$\text{Via } (***), 1 - 2\sin^2 a = \frac{\sqrt{5}-1}{2} \Rightarrow \sin^2 a = \frac{3-\sqrt{5}}{4} \Rightarrow 1 + \cot^2 a = \frac{4}{3-\sqrt{5}}$$

$$\Rightarrow \cot^2 a = \frac{4}{3-\sqrt{5}} - 1 = \frac{2(\sqrt{5}+1)(6+2\sqrt{5})}{(6-2\sqrt{5})(6+2\sqrt{5})} = \frac{2(\sqrt{5}+1)^3}{16} = \left(\frac{\sqrt{5}+1}{2}\right)^3$$

$$\overset{\text{via } (****)}{\Rightarrow} \left(\frac{BC}{AC}\right)^2 = \varphi^3 \Rightarrow \frac{BC}{AC} = \sqrt{\varphi^3} \text{ (QED)}$$

831.

Let ABC be a triangle with $\angle ABC = 20^\circ$ and $\angle ACB = 10^\circ$; $|AB| = a$, $|BC| = a^2 + 2a$. Prove that : $|AC| = \sqrt{3}$



Proposed by Murat Oz-Turkiye

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Sine } law \text{ on } \Delta\,\text{ABC} \Rightarrow \frac{a^2+2a}{\sin 150^\circ} = \frac{a}{\sin 10^\circ} = \frac{x}{\sin 20^\circ} \to (1)$$

$$and \text{ sine } law \text{ on } \Delta\,\text{ADC} \Rightarrow \frac{\text{m}}{\sin 150^\circ} = \frac{\text{m}-a}{\sin 10^\circ} \to (2)$$

$$\text{Sine } law \text{ on } \Delta\,\text{DBC} \Rightarrow \frac{a^2+2a}{\sin 40^\circ} = \frac{\text{m}}{\sin 20^\circ} \Rightarrow \frac{a^2+2a}{2\sin 20^\circ \cos 20^\circ} = \frac{\text{m}}{\sin 20^\circ}$$

$$\Rightarrow \text{m} \overset{(*)}{=} \frac{a^2+2a}{2\cos 20^\circ} \text{ and } (1) \Rightarrow \sin 10^\circ \overset{(**)}{=} \frac{1}{2(a+2)}$$

$$\therefore (*), (**), (2) \Rightarrow \frac{a^2+2a}{\cos 20^\circ} = 2(a+2)\left(\frac{a^2+2a}{2\cos 20^\circ} - a\right)$$

$$\Rightarrow 2\left(\frac{a+2}{2} - \cos 20^\circ\right) = 1 \Rightarrow 2\left(\frac{a+2}{2} - 1 + 2\sin^2 10^\circ\right) = 1$$

$$\overset{\text{via } (**)}{\Rightarrow} 2\left(\frac{a+2}{2} - 1 + \frac{1}{2(a+2)^2}\right) = 1 \Rightarrow \frac{(a+2)^3 - 2(a+2)^2 + 1}{2(a+2)^2} = \frac{1}{2}$$

$$\Rightarrow (a+2)^3 - 3(a+2)^2 + 1 = 0$$

$$\Rightarrow a^3 \overset{(***)}{=} 3 - 3a^2 \text{ and } also, 2\left(\frac{a+2}{2} - \cos 20^\circ\right) = 1 \Rightarrow \cos 20^\circ \overset{(****)}{=} \frac{a+1}{2}$$

$$\text{Now}, (1) \Rightarrow x^2 = \frac{a^2}{\sin^2 10^\circ}(1 - \cos^2 20^\circ) \overset{\text{via } (**),(****)}{=} 4a^2(a+2)^2\left(1 - \frac{(a+1)^2}{4}\right)$$

$$= -a^2(a^4 + 6a^3 + 9a^2 - 4a - 12) = -a^2(a(a^3 - 4) + 6a^3 + 9a^2 - 12)$$

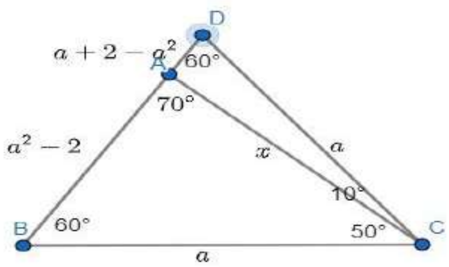
$$\overset{\text{via } (***)}{=} -a^2(a(-1 - 3a^2) + 6a^3 + 9a^2 - 12) = -a^2(3a^3 - a + 9a^2 - 12)$$

$$\overset{\text{via } (***)}{=} -a^2(9 - 9a^2 - a + 9a^2 - 12) = a^2(3 + a) = a^3 + 3a^2$$

$$\overset{\text{via } (***)}{=} 3 \Rightarrow x = \text{AC} = \sqrt{3} \text{ (QED)}$$

832. Let ABC be a triangle with $\angle\,\text{ABC} = 60^\circ$ and $\angle\,\text{ACB} = 50^\circ$;

$|\text{AB}| = a^2 - 2, |\text{BC}| = a$. Prove that : $|\text{AC}| = \sqrt{3}$.



Proposed by Murat Oz-Turkiye

Solution by Soumava Chakraborty-Kolkata-India

Sine law on Δ ABC $\Rightarrow \frac{a^2-2}{\sin 50°}=\frac{a}{\sin 70°}=\frac{x}{\sin 60°}\to(1)$ and

via Δ ADC, $\frac{a}{\sin 70°}=\frac{a+2-a^2}{\sin 10°}\left(=\frac{(a+1)(2-a)}{\sin 10°}\Rightarrow a<2\right)=\frac{x}{\sin 60°}\to(2)$

$(1),(2)\Rightarrow\frac{a^2-2}{\sin 50°}=\frac{a+2-a^2}{\sin 10°}\Rightarrow\frac{a^2-2}{a+2-a^2}=\frac{\sin 50°}{\sin 10°}\Rightarrow\frac{(a^2-2)+(a+2-a^2)}{(a^2-2)-(a+2-a^2)}$

$=\frac{\sin 50°+\sin 10°}{\sin 50°-\sin 10°}=\frac{2\sin 30°\cos 20°}{2\cos 30°\sin 20°}=\frac{1}{\sqrt{3}\tan 20°}\Rightarrow\tan 20°\overset{(*)}{=}\frac{2a^2-a-4}{\sqrt{3}a}$

$\Rightarrow\tan^2 20°=\frac{(2a^2-a-4)^2}{3a^2}\Rightarrow a^2\sec^2 20°-a^2=\frac{(2a^2-a-4)^2}{3}$

$\Rightarrow a^2\sec^2 20°=\frac{4(a^4-a^3-3a^2+2a+4)}{3}$

$\Rightarrow\frac{a^2}{\sin^2 70°}=\frac{4(a^4-a^3-3a^2+2a+4)}{3}$

$\Rightarrow\frac{x^2}{\sin^2 60°}=\frac{4(a^4-a^3-3a^2+2a+4)}{3}\Rightarrow\boxed{x^2\overset{(**)}{=}a^4-a^3-3a^2+2a+4}$

Napier's $analogy\Rightarrow\tan\frac{A-C}{2}=\frac{a-c}{a+c}\cot\frac{B}{2}\Rightarrow\tan\frac{70°-50°}{2}$

$=\frac{a-(a^2-2)}{a+(a^2-2)}\cot 30°\Rightarrow\tan 10°\overset{(***)}{=}\frac{a+2-a^2}{a^2+a-2}.\sqrt{3}$

Now, $(*)\Rightarrow\frac{2\tan 10°}{1-\tan^2 10°}=\frac{2a^2-a-4}{\sqrt{3}a}\overset{via\ (***)}{\Rightarrow}\frac{\frac{2\sqrt{3}(a+2-a^2)}{a^2+a-2}}{1-\frac{3(a+2-a^2)^2}{(a^2+a-2)^2}}=\frac{2a^2-a-4}{\sqrt{3}a}$

$\Rightarrow 6a(a+2-a^2)(a^2+a-2)$

$-(2a^2-a-4)\left((a^2+a-2)^2-3(a+2-a^2)^2\right)=0$

$\Rightarrow\boxed{(a^3-3a-1)(a^3-6a^2+8)\overset{(\bullet)}{=}0}$

If $a^3-6a^2+8=0$, then : $a^2(a-2)=4(a^2-2)>0\Rightarrow a>2$, a contradiction

$\Rightarrow a^3-6a^2+8\neq 0\therefore(\bullet)\Rightarrow a^3-3a-1=0\Rightarrow\boxed{a^3\overset{(\bullet\bullet)}{=}3a+1}$

$\therefore a^4-a^3-3a^2+2a+4\overset{via\ (\bullet\bullet)}{=}a^4-3a-1-3a^2+2a+4=a^4-a-3a^2+3$

$=a(a^3-1)-3a^2+3\overset{via\ (\bullet\bullet)}{=}a(3a)-3a^2+3=3$

$\overset{via\ (**)}{\Rightarrow}x^2=3\Rightarrow x=AC=\sqrt{3}$ (QED)

833. ***If in ΔABC holds :***

$$4\prod_{cyc}(a^2+ab+b^2)=3\left(\sum_{cyc}ab(a+b)\right)^2$$

then the triangle is an isoscele one.

Proposed by Daniel Sitaru-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$4\prod_{cyc}(a^2+ab+b^2)=3\left(\sum_{cyc}ab(a+b)\right)^2\quad(*)$$

$$We\ have:\ \ 4(a^2+ab+b^2)(c^2+ca+a^2)=$$

$$=[(a^2+ab+b^2)+(c^2+ca+a^2)]^2-[(a^2+ab+b^2)-(c^2+ca+a^2)]^2=$$

$$=[(2a^2+ab+ca+2bc)+(b-c)^2]^2-[(a+b+c)(b-c)]^2=$$

$$=(2a^2+ab+ca+2bc)^2+2(2a^2+ab+ca+2bc)(b-c)^2+(b-c)^4-(a+b+c)^2(b-c)^2$$

$$Then:\ \ 4(a^2+ab+b^2)(c^2+ca+a^2)=(2a^2+ab+ca+2bc)^2+3a^2(b-c)^2.$$

$$Also\ we\ have:\ \ 4(b^2+bc+c^2)=3(b+c)^2+(b-c)^2.$$

$$Multipying\ these\ two\ identities\ we\ get:$$

$$4LHS_{(*)}=[(2a^2+ab+ca+2bc)^2+3a^2(b-c)^2][3(b+c)^2+(b-c)^2]\geq$$

$$\overset{CBS}{\geq}\left[\sqrt{3}(2a^2+ab+ca+2bc)(b+c)+\sqrt{3}a(b-c)^2\right]^2=$$

$$=3\left(2\sum_{cyc}ab(a+b)\right)^2=4RHS_{(*)}.$$

$$Then:\ \ LHS_{(*)}\geq RHS_{(*)},\ \ with\ equality\ when:$$

$$(b-c)^2=0\ or\ \frac{2a^2+ab+ca+2bc}{\sqrt{3}(b+c)}=\sqrt{3}a\Leftrightarrow b=c\ or\ 2(a-b)(a-c)=0$$

$$\Leftrightarrow\ a=b\ or\ b=c\ or\ c=a.\ \ Then\ the\ triangle\ ABC\ is\ an\ isoscele\ one.$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\mathrm{RHS}=3\left(\sum_{cyc}ab(2s-c)\right)^2=3\left(2s(s^2+4Rr+r^2)-12Rrs\right)^2$$

$$\therefore\ \mathrm{RHS}\overset{(1)}{=}3.4s^2\left(s^2-2Rr+r^2\right)^2$$

$$\prod_{cyc}(a-b)^2\overset{(\bullet)}{=}\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4+2abc\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)$$
$$-2\sum_{cyc}a^3b^3-2abc\sum_{cyc}a^3-6a^2b^2c^2$$
$$\Rightarrow\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4+2abc\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)$$
$$\overset{(*)}{=}\prod_{cyc}(a-b)^2+2\sum_{cyc}a^3b^3+2abc\sum_{cyc}a^3+6a^2b^2c^2$$
$$\text{Now, LHS}=4\prod_{cyc}(a^2+ab+b^2)$$
$$=4\left(\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4+2abc\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)\right)$$
$$+4\left(\sum_{cyc}a^3b^3+abc\sum_{cyc}a^3+3a^2b^2c^2\right)$$
$$\overset{\text{via }(*)}{=}4\left(\prod_{cyc}(a-b)^2+3\sum_{cyc}a^3b^3+3abc\sum_{cyc}a^3+9a^2b^2c^2\right)\overset{(2)}{=}\text{LHS}$$
$$\therefore(1),(2)\Rightarrow\prod_{cyc}(a-b)^2+3\sum_{cyc}a^3b^3+3abc\sum_{cyc}a^3+9a^2b^2c^2$$
$$=3s^2(s^2-2Rr+r^2)^2\Rightarrow\prod_{cyc}(a-b)^2$$
$$=3\left(s^2(s^2-2Rr+r^2)^2-\left((s^2+4Rr+r^2)^3-3.4Rrs.2s(s^2+2Rr+r^2)\right)\right.$$
$$\left.-8Rrs^2(s^2-6Rr-3r^2)-48R^2r^2s^2\right)$$
$$\Rightarrow\boxed{\prod_{cyc}(a-b)^2\overset{(i)}{=}-3r^2\left(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\right)}$$
$$\text{Again, via }(\bullet),\prod_{cyc}(a-b)^2$$
$$=\sum_{cyc}\left(a^2b^2\left(\sum_{cyc}a^2-c^2\right)\right)+2abc\sum_{cyc}ab(2s-c)$$
$$-2\left((s^2+4Rr+r^2)^3-3.4Rrs.2s(s^2+2Rr+r^2)\right)-16Rrs^2(s^2-6Rr-3r^2)$$
$$-6a^2b^2c^2=2(s^2-4Rr-r^2)\left((s^2+4Rr+r^2)^2-16Rrs^2\right)-240R^2r^2s^2$$
$$+16Rrs^2(s^2+4Rr+r^2)-2\left((s^2+4Rr+r^2)^3-3.4Rrs.2s(s^2+2Rr+r^2)\right)$$
$$-16Rrs^2(s^2-6Rr-3r^2)$$

$$\Rightarrow \boxed{\frac{3}{4}\prod_{cyc}(a-b)^2 \overset{(ii)}{=} -3r^2\left(s^4 - s^2\left(4R^2+20Rr-2r^2\right)+r(4R+r)^3\right)}$$

$$\therefore (i),(ii) \Rightarrow \prod_{cyc}(a-b)^2 = \frac{3}{4}\prod_{cyc}(a-b)^2 \Rightarrow \prod_{cyc}(a-b)^2 = 0$$

$$\Rightarrow \boxed{(a-b)(b-c)(c-a)=0} \Rightarrow 3 \text{ possibilities arise}:$$

$$a-b=0, b-c=0, c-a=0$$

$$\Rightarrow \textit{at least one among } (a-b),(b-c),(c-a)=0 \Rightarrow \Delta\, ABC \text{ is isoscele (QED)}$$

834. ***If $0 < x < 1$ then in ΔABC holds :***

$$2^{x-1}\left(\frac{an_a - \left(\sqrt{bn_b}-\sqrt{cn_c}\right)^2}{4\sqrt{bcn_bn_c}}\right)^{\frac{x}{2}} \le \frac{(an_a)^{\frac{x}{2}}}{(bn_b)^{\frac{x}{2}}+(cn_c)^{\frac{x}{2}}}.$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{We have}:$$

$$(an_a)^2 = a^2\left(s(s-a)+\frac{s(b-c)^2}{a}\right) = s(s-a)\left[a^2-(b-c)^2\right]+s^2(b-c)^2 =$$

$$= 4s(s-a)(s-b)(s-c)+s^2(b-c)^2 = 4s^2r^2+s^2(b-c)^2 = s^2\left(4r^2+(b-c)^2\right).$$

$$\textit{Then}: \ an_a = s\sqrt{4r^2+(b-c)^2} \ \ (\textit{and analogs})$$

$$\textit{Now}, \ \left(\sqrt{an_a}+\sqrt{bn_b}\right)^2 > an_a+bn_b = s\left(\sqrt{(2r)^2+(b-c)^2}+\sqrt{(2r)^2+(c-a)^2}\right) \ge$$

$$\overset{Minkowski}{\ge} \ s\sqrt{(2r+2r)^2+[(b-c)+(c-a)]^2} = s\sqrt{16r^2+(a-b)^2} > s\sqrt{4r^2+(a-b)^2} = cn_c.$$

$$\textit{Then } \sqrt{an_a}+\sqrt{bn_b} > \sqrt{cn_c} \ (\textit{and analogs})$$

$$\Rightarrow \sqrt{an_a}, \sqrt{bn_b}, \sqrt{cn_c} \textit{ can be the sides of triangle}.$$

$$\textit{So it suffices to prove}: \ 2^{x-1}\left(\frac{a^2-(b-c)^2}{4bc}\right)^{\frac{x}{2}} \le \frac{a^x}{b^x+c^x}, \ \ \forall \Delta ABC.$$

$$\textit{We have}: \ 2^{x-1}\left(\frac{a^2-(b-c)^2}{4bc}\right)^{\frac{x}{2}} = 2^{x-1}\left(\frac{(s-b)(s-c)}{bc}\right)^{\frac{x}{2}} = 2^{x-1}\sin^x\frac{A}{2} =$$

$$\overset{Mollweide}{\cong} 2^{x-1}\left(\frac{a\cos\frac{B-C}{2}}{b+c}\right)^x \le 2^{x-1}\left(\frac{a}{b+c}\right)^x \overset{?}{\lesssim} \frac{a^x}{b^x+c^x} \Leftrightarrow b^x+c^x \le 2\left(\frac{b+c}{2}\right)^x,$$

which is true by Jensen's inequality using the concave function $t \to t^x$, $(x \in (0,1))$.

So the proof is completed.

835. $A_1A_2 \dots A_n \to$ **convex polygon** and $\mathrm{n} \ge 3$, $a_1a_2 \dots a_n \to$ **sides and** $a_1+a_2+\dots+a_n = 2\mathrm{p}$. **Prove that :**

$$\left(\frac{\hat{A}_1^3}{\hat{A}_1^2+\hat{A}_2^2}+\frac{\hat{A}_2^3}{\hat{A}_2^2+\hat{A}_3^2}+\dots+\frac{\hat{A}_n^3}{\hat{A}_n^2+\hat{A}_1^2}\right)\left(\frac{a_1^3}{a_1^2+a_2^2}+\frac{a_2^3}{a_2^2+a_3^2}+\dots+\frac{a_n^3}{a_n^2+a_1^2}\right) \ge \frac{(\mathrm{n}-2)\pi.\mathrm{p}}{2}$$

Proposed by Radu Diaconu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{\hat{A}_1^3}{\hat{A}_1^2+\hat{A}_2^2} = \frac{\hat{A}_1(\hat{A}_1^2+\hat{A}_2^2-\hat{A}_2^2)}{\hat{A}_1^2+\hat{A}_2^2} = \hat{A}_1 - \frac{\hat{A}_1.\hat{A}_2^2}{\hat{A}_1^2+\hat{A}_2^2} \overset{A-G}{\ge} \hat{A}_1 - \frac{\hat{A}_1.\hat{A}_2^2}{2\hat{A}_1.\hat{A}_2}$$

$$\Rightarrow \frac{\hat{A}_1^3}{\hat{A}_1^2+\hat{A}_2^2} \ge \hat{A}_1 - \frac{\hat{A}_2}{2} \text{ and analogs} \Rightarrow \frac{\hat{A}_1^3}{\hat{A}_1^2+\hat{A}_2^2}+\frac{\hat{A}_2^3}{\hat{A}_2^2+\hat{A}_3^2}+\dots+\frac{\hat{A}_n^3}{\hat{A}_n^2+\hat{A}_1^2}$$

$$\ge \hat{A}_1-\frac{\hat{A}_2}{2}+\hat{A}_2-\frac{\hat{A}_3}{2}+\dots+\hat{A}_n-\frac{\hat{A}_1}{2} = \sum_{i=1}^{n}\hat{A}_i - \frac{1}{2}\sum_{i=1}^{n}\hat{A}_i = \frac{1}{2}\sum_{i=1}^{n}\hat{A}_i = \frac{(n-2)\pi}{2}$$

$$\therefore \frac{\hat{A}_1^3}{\hat{A}_1^2+\hat{A}_2^2}+\frac{\hat{A}_2^3}{\hat{A}_2^2+\hat{A}_3^2}+\dots+\frac{\hat{A}_n^3}{\hat{A}_n^2+\hat{A}_1^2} \overset{(*)}{\ge} \frac{(n-2)\pi}{2}$$

$$\text{Similarly, } \frac{a_1^3}{a_1^2+a_2^2}+\frac{a_2^3}{a_2^2+a_3^2}+\dots+\frac{a_n^3}{a_n^2+a_1^2} \overset{A-G}{\ge} a_1-\frac{a_2}{2}+a_2-\frac{a_3}{2}+\dots a_n-\frac{a_1}{2}$$

$$= \frac{1}{2}\sum_{i=1}^{n} a_i = \frac{2p}{2} \therefore \frac{a_1^3}{a_1^2+a_2^2}+\frac{a_2^3}{a_2^2+a_3^2}+\dots+\frac{a_n^3}{a_n^2+a_1^2} \overset{(**)}{\ge} p \therefore (*)\bullet(**) \Rightarrow$$

$$\left(\frac{\hat{A}_1^3}{\hat{A}_1^2+\hat{A}_2^2}+\frac{\hat{A}_2^3}{\hat{A}_2^2+\hat{A}_3^2}+\dots+\frac{\hat{A}_n^3}{\hat{A}_n^2+\hat{A}_1^2}\right)\left(\frac{a_1^3}{a_1^2+a_2^2}+\frac{a_2^3}{a_2^2+a_3^2}+\dots+\frac{a_n^3}{a_n^2+a_1^2}\right)$$

$$\ge \frac{(n-2)\pi.p}{2}, \text{ with equality iff } A_1A_2\dots A_n \text{ is a regular convex polygon (QED)}$$

836.

In ΔABC the following relationship holds :

$$\frac{a^3}{12a^2-8ab+3b^2}+\frac{b^3}{12b^2-8bc+3c^2}+\frac{c^3}{12c^2-8ca+3a^2} \le \frac{3\sqrt{3}R}{7}$$

Proposed by Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{Lemma}: \ \text{If } a,b>0 \text{ then}: \ \frac{a^3}{12a^2-8ab+3b^2}\le\frac{5a+2b}{49}.$$

$$\text{Proof}: \text{We have}: \ \frac{a^3}{12a^2-8ab+3b^2}\le\frac{5a+2b}{49}$$

$$\Leftrightarrow \ (5a+2b)(12a^2-8ab+3b^2)-49a^3\ge 0$$

$$\Leftrightarrow 11a^3-16a^2b-ab^2+6b^3\ge 0 \ \Leftrightarrow \ (a-b)^2(11a+6b)\ge 0$$

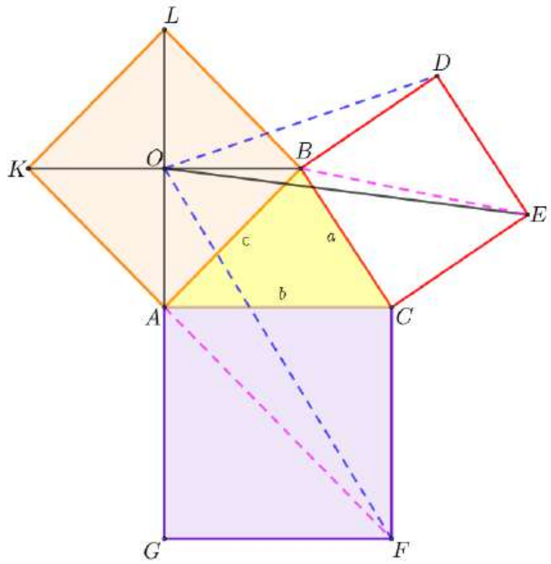
$$\text{which is true for all } a,b>0.$$

$$\text{Then we have}: \ \frac{a^3}{12a^2-8ab+3b^2}\le\frac{5a+2b}{49} \ \ (\text{and analogs})$$

$$\text{Therefore}, \ \sum_{cyc}\frac{a^3}{12a^2-8ab+3b^2}\le\sum_{cyc}\frac{5a+2b}{49}=\frac{a+b+c}{7}=\frac{2s}{7}\overset{\text{Mitrinovic}}{\le}\frac{3\sqrt{3}R}{7}.$$

$$\text{Equality holds iff } \Delta ABC \text{ is equilateral}.$$

837. On the sides of the triangle ABC the squares $BDEC, ACFG, ABLK$ are drawn outside the triangle. Let O be the centroid of the square $ABLK$. Prove that: $OG^2+OE^2=OF^2+OD^2$



Proposed by Mehmet Şahin, Alican Gullu-Ankara-Turkiye

Solution by Hikmat Mammadov-Azerbaijan

$$R\text{ −radius of circumcircle of }\Delta ABC$$

$$OA = OB = \frac{c}{\sqrt{2}}$$

$$AC = AG = b, BD = BC = a$$

$$AF = b\sqrt{2}, BE = a\sqrt{2}$$

$$OG^2 = \left(\frac{c}{\sqrt{2}}\right)^2 + b^2 - 2\cdot\frac{c}{\sqrt{2}}\cdot b\cos\left(A + \frac{\pi}{2} + \frac{\pi}{4}\right)$$

$$OG^2 = \frac{c^2}{2} + b^2 + \sqrt{2}bc\sin\left(A + \frac{\pi}{4}\right)$$

$$OG^2 = \frac{c^2}{2} + b^2 + bc(\sin A + \cos A)$$

$$OG^2 = \frac{c^2}{2} + b^2 + bc\left(\frac{a}{2R} + \frac{c^2 + b^2 - a^2}{2bc}\right)$$

$$OG^2 = c^2 + \frac{abc}{2R} + \frac{3b^2 - a^2}{2}$$

$$OE^2 = \left(\frac{c}{\sqrt{2}}\right)^2 + \left(\sqrt{2}a\right)^2 - 2ca\cos\left(B + \frac{\pi}{4} + \frac{\pi}{4}\right)$$

$$OE^2 = \frac{c^2}{2} + 2a^2 + 2ca\sin B \Rightarrow OE^2 = \frac{c^2}{2} + 2a^2 + \frac{abc}{R}$$

$$OG^2 = OE^2 = \frac{3}{2}\left(a^2 + b^2 + c^2 + \frac{abc}{R}\right)$$

Similarly,

$$OD^2 + OE^2 = \frac{3}{2}\left(a^2 + b^2 + c^2 + \frac{abc}{R}\right)$$

Therefore,

$$OG^2 + OE^2 = OF^2 + OD^2$$

838.

In any Δ ABC holds:

$$4\sqrt{3}S \le 3\sqrt[3]{abc}.\sqrt[6]{a^2b^2c^2 - (a-b)^2(b-c)^2(c-a)^2} \le a^2 + b^2 + c^2$$

Proposed by Alp Eren Koken-Turkiye

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\prod_{cyc}(a-b)^2=\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4+2abc\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)$$

$$-2\sum_{cyc}a^3b^3-2abc\sum_{cyc}a^3-6a^2b^2c^2$$

$$=\sum_{cyc}\left(a^2b^2\left(\sum_{cyc}a^2-c^2\right)\right)+2abc\sum_{cyc}ab(2s-c)$$

$$-2\left((s^2+4Rr+r^2)^3-3.4Rrs.2s(s^2+2Rr+r^2)\right)$$

$$-16Rrs^2(s^2-6Rr-3r^2)-6a^2b^2c^2$$

$$=2(s^2-4Rr-r^2)\left((s^2+4Rr+r^2)^2-16Rrs^2\right)-240R^2r^2s^2$$

$$+16Rrs^2(s^2+4Rr+r^2)-2\left((s^2+4Rr+r^2)^3-3.4Rrs.2s(s^2+2Rr+r^2)\right)$$

$$-16Rrs^2(s^2-6Rr-3r^2)$$

$$\Rightarrow (a-b)^2(b-c)^2(c-a)^2\overset{(*)}{=}-4r^2\left(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\right)$$

$$\text{Now, }4\sqrt{3}S\le 3\sqrt[3]{abc}.\sqrt[6]{a^2b^2c^2-(a-b)^2(b-c)^2(c-a)^2}\overset{\text{via }(*)}{\Leftrightarrow}4^6.3^3.S^6$$

$$\le 3^6.16R^2S^2.\left(16R^2r^2s^2+4r^2\left(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\right)\right)$$

$$\Leftrightarrow 64S^2.s^2\overset{(i)}{\le}27R^2\left(s^4-s^2(20Rr-2r^2)+r(4R+r)^3\right)\text{ and }\because 27R^2\overset{\text{Mitrinovic}}{\ge}4s^2$$

$$\therefore \text{in order to prove (i), it suffices to prove :}$$

$$s^4-s^2(20Rr-2r^2)+r(4R+r)^3\ge 16r^2s^2$$

$$\Leftrightarrow s^4-s^2(20Rr+14r^2)+r(4R+r)^3\overset{(ii)}{\ge}0$$

$$\text{and }\because \text{LHS of (ii)}\overset{\text{Gerretsen}}{\ge}s^2(16Rr-5r^2)-s^2(20Rr+14r^2)+r(4R+r)^3$$

$$\therefore \text{in order to prove (ii), it sufficess to prove :}$$

$$r(4R+r)^3\ge s^2(4Rr+19r^2)\Leftrightarrow (4R+19r)s^2\overset{(iii)}{\le}(4R+r)^3$$

$$\text{Now, LHS of (iii)}\overset{\text{Gerretsen}}{\le}(4R+19r)(4R^2+4Rr+3r^2)\overset{?}{\le}(4R+r)^3$$

$$\Leftrightarrow 12R^3-11R^2r-19Rr^2-14r^3\overset{?}{\ge}0\Leftrightarrow (R-2r)(12R^2+13Rr+7r^2)\overset{?}{\ge}0$$

$$\to \text{true }\because R\overset{\text{Euler}}{\ge}2r\Rightarrow \text{(iii)}\Rightarrow\text{(ii)}\Rightarrow\text{(i) is true}$$

$$\therefore \boxed{4\sqrt{3}S \le 3\sqrt[3]{abc}.\sqrt[6]{a^2b^2c^2-(a-b)^2(b-c)^2(c-a)^2}}$$

$$\text{Also, } 3\sqrt[3]{abc}.\sqrt[6]{a^2b^2c^2-(a-b)^2(b-c)^2(c-a)^2} \le 3\sqrt[3]{abc}.\sqrt[6]{a^2b^2c^2}$$

$$= 3\sqrt[3]{a^2b^2c^2} \overset{A-G}{\le} a^2+b^2+c^2$$

$$\therefore \boxed{3\sqrt[3]{abc}.\sqrt[6]{a^2b^2c^2-(a-b)^2(b-c)^2(c-a)^2} \le a^2+b^2+c^2}$$

$$\text{So, in any } \Delta\, ABC, 4\sqrt{3}S \le 3\sqrt[3]{abc}.\sqrt[6]{a^2b^2c^2-(a-b)^2(b-c)^2(c-a)^2}$$

$$\le a^2+b^2+c^2, \text{with equalities iff } \Delta\, ABC \text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$4\sqrt{3}S \overset{(ii)}{\le} 3\sqrt[3]{abc}\sqrt[6]{a^2b^2c^2-(a-b)^2(b-c)^2(c-a)^2} \overset{(i)}{\le} a^2+b^2+c^2$$

We have:

$$a^2+b^2+c^2 \overset{AM-GM}{\ge} 3\sqrt[3]{a^2b^2c^2} = 3\sqrt[3]{abc}\sqrt[6]{a^2b^2c^2}$$
$$\ge 3\sqrt[3]{abc}\sqrt[6]{a^2b^2c^2-(a-b)^2(b-c)^2(c-a)^2}.$$

$$\textit{Now, let: } a := x+y,\ b := y+z,\ c := z+x,\ x,y,z>0,\ s = \frac{a+b+c}{2} = x+y+z \textit{ and}$$

$$r = \sqrt[3]{xyz}.$$

$$\textit{We have: } S = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{xyz(x+y+z)} = \sqrt{sr^3} \quad (1)$$

$$\textit{Also, } a^2b^2c^2 - \prod_{cyc}(a-b)^2 = \left(abc - \prod_{cyc}(a-b)\right)\left(abc + \prod_{cyc}(a-b)\right) =$$

$$= \left(\prod_{cyc}(x+y) - \prod_{cyc}(x-y)\right)\left(\prod_{cyc}(x+y) + \prod_{cyc}(x-y)\right)$$

$$= 2\left(\sum_{cyc} x^2y + xyz\right).2\left(\sum_{cyc} xy^2 + xyz\right).$$

$$\textit{And since: } \sum_{cyc} x^2y = \sum_{cyc} \frac{1}{3}(x^2y+x^2y+z^2x) \overset{AM-GM}{\ge} \sum_{cyc} \sqrt[3]{x^5y^2z^2} = r^2s,$$

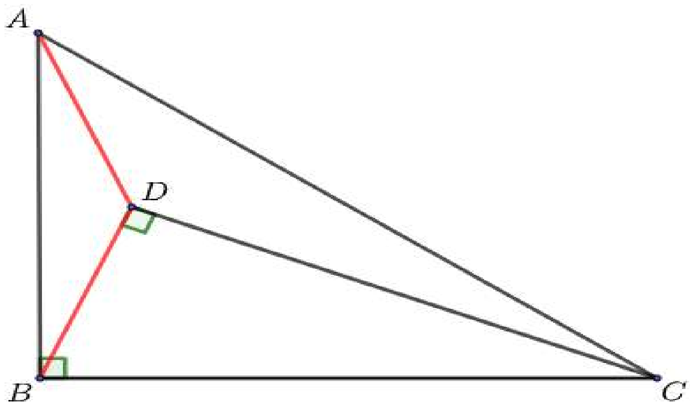
$$\textit{then we have: } a^2b^2c^2 - \prod_{cyc}(a-b)^2 \ge \left(2(sr^2+r^3)\right)^2 \quad (2)$$

$$Also,\ \ abc=\prod_{cyc}(x+y)\geq\frac{8(x+y+z)(xy+yz+zx)}{9}\overset{AM-GM}{\geq}\frac{8s}{9}.3\sqrt[3]{x^2y^2z^2}=\frac{8sr^2}{3}\ \ (3)$$

$From\ (1),(2)\ and\ (3), to\ prove\ the\ left\ side\ inequality\ it\ suffices\ to\ prove:$

$$4\sqrt{3sr^3}\leq 3\sqrt[3]{\frac{8sr^2}{3}}\sqrt[6]{\left(2r^2(s+r)\right)^2}\Leftrightarrow 16sr\leq 3(s+r)^2\ \Leftrightarrow\ (s-3r)(3s-r)\geq 0$$

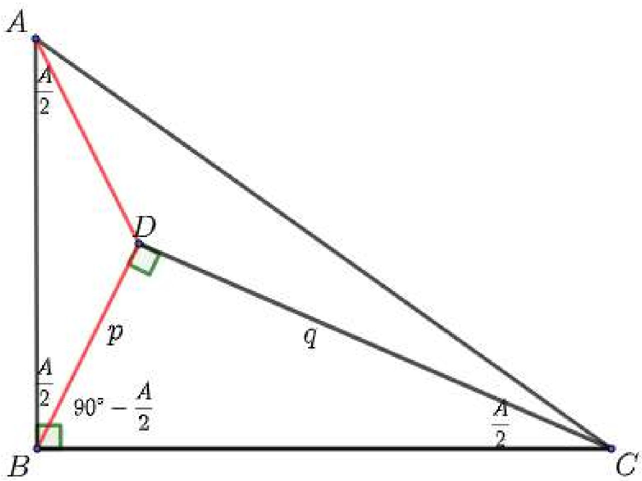
$Which\ is\ true\ by\ AM-GM\ inequality\ \therefore s\geq 3r.\ So\ the\ proof\ is\ done.$

839. 

$$\frac{CD}{AB}=?$$

Proposed by Murat Oz-Turkiye

Solution 1 by Ravi Prakash-New Delhi-India



$Let\ CD=q; BD=p; BC=a; AD=p, then:$

$$p = a\sin\frac{A}{2} \text{ and } q = a\cos\frac{A}{2}$$

$$\mu(\widehat{ADB}) = \pi - A$$

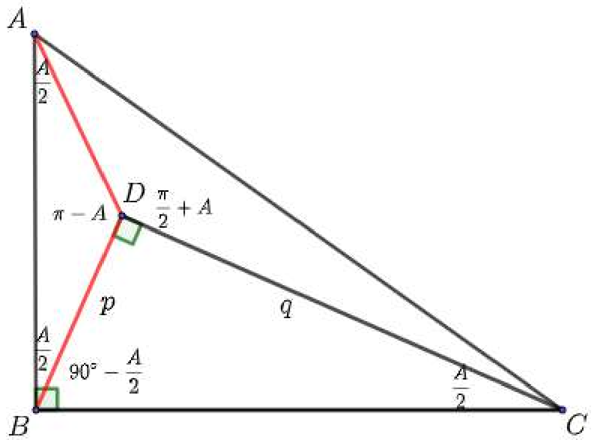
$$AB^2 = p^2 + p^2 - 2p^2\cos(\pi - A)$$

$$AB^2 = 2p^2(1+\cos A) = 4p^2\cos^2\frac{A}{2}$$

$$AB = 2p\cos\frac{A}{2}$$

$$\frac{CD}{AB} = \frac{a\cos\frac{A}{2}}{2p\cos\frac{A}{2}} = \frac{a}{2p} = \frac{1}{2\sin\frac{A}{2}}$$

Solution 2 by Tapas Das-India



$$Let: AD = BD = x.$$

$$Now, \quad [ABC] = [ADB] + [BDC] + [ADC]; \quad (1)$$

$$From\ \Delta BDC, we\ get: \sin\frac{A}{2} = \frac{BD}{BC} \Rightarrow \sin\frac{A}{2} = \frac{x}{BC} \Rightarrow BC = \frac{x}{\sin\frac{A}{2}}$$

$$[ABC] = \frac{1}{2}AB\cdot BC = \frac{1}{2}AB\cdot\frac{x}{\sin\frac{A}{2}}; \quad (2)$$

$$[ADB] + [BDC] + [ADC] = \frac{1}{2}AB\cdot x\cdot\sin\frac{A}{2} + \frac{1}{2}BD\cdot DC + \frac{1}{2}AD\cdot DC\cdot\sin\left(\frac{\pi}{2}+A\right) =$$

$$= \frac{1}{2}AB\cdot x\cdot\sin\frac{A}{2} + \frac{1}{2}\cdot x\cdot DC + \frac{1}{2}\cdot x\cdot DC\cdot\cos A =$$

$$=\frac{1}{2}x\left(AB\cdot\sin\frac{A}{2}+DC+DC\cdot\cos A\right)$$

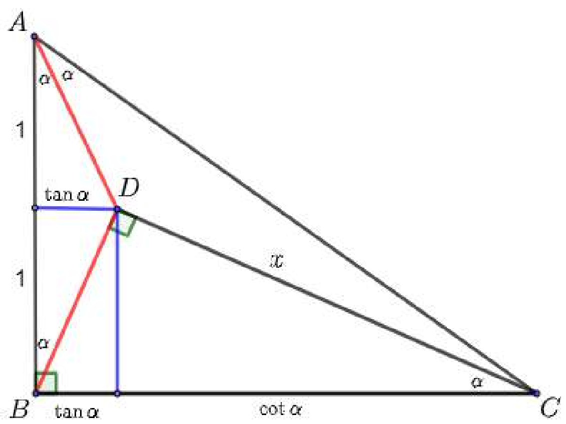
From (1) *and* (2) *we have*:

$$\frac{1}{2}\cdot AB\cdot\frac{x}{\sin\frac{A}{2}}=\frac{1}{2}x\left(AB\cdot\sin\frac{A}{2}+DC+DC\cdot\cos A\right)$$

$$\frac{AB}{\sin\frac{A}{2}}=AB\cdot\sin\frac{A}{2}+DC+DC\cdot\cos A$$

$$AB\cdot\frac{1-\sin^2\frac{A}{2}}{\sin\frac{A}{2}}=DC(1+\cos A),\ \ \frac{CD}{AB}=\frac{1-\sin^2\frac{A}{2}}{1+\cos A}=\frac{1}{2\sin\frac{A}{2}}$$

Solution 3 by Ertan Yildirim-Turkiye



$$In\ \Delta ABC:\ \tan(2\alpha)=\frac{BC}{AB}=\frac{\tan\alpha+\cot\alpha}{2}.$$

$$Let:\tan\alpha=m, then\tan(2\alpha)=\frac{2m}{1-m^2}=\frac{m+\frac{1}{m}}{2}$$

$$\frac{2m}{1-m^2}=\frac{m^2+1}{2m}\Rightarrow 4m^2=1-m^4\Rightarrow m^4+4m^2=1$$

$$m^4+4m^2+4=5\Rightarrow(m^2+2)^2=5\Rightarrow m^2=\sqrt{5}-2$$

$$\tan^2\alpha=\sqrt{5}-2$$

$$In\ \Delta DHC: x^2=1+\cot^2\alpha=1+\frac{1}{\sqrt{5}-2}=3+\sqrt{5}$$

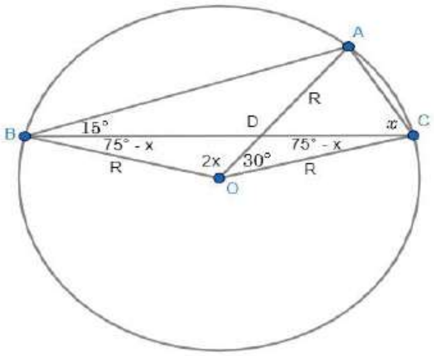
$$\phi=\frac{\sqrt{5}+1}{2}\Rightarrow\phi^2=\frac{6+2\sqrt{5}}{4}=\frac{3+\sqrt{5}}{2}$$

$$\frac{x^2}{4}=\frac{3+\sqrt{5}}{4}=\frac{\phi^2}{2}\Rightarrow\frac{x}{2}=\frac{\phi}{\sqrt{2}}=\frac{CD}{AB}$$

840. **ABC is a triangle with $\angle BAC>90^\circ$ and $\angle ABC=15^\circ$.**

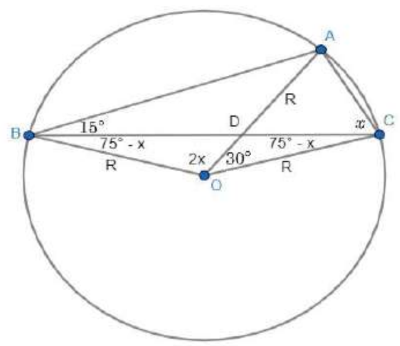
The circumcenter of Δ ABC is O and AO intersects BC at D.

If $|OD|^2+|OC||DC|=|OC|^2$, then, what is the value of $\angle BCA$?



Proposed by Alp Eren Koken-Turkiye

Solution 1 by Soumava Chakraborty-Kolkata-India



Angle *at* center is 2 times *angle at* circumference

$$\therefore\angle AOC=30^\circ\ \textit{and}\ \angle AOB=2x\Rightarrow\angle OCD=\frac{180^\circ-30^\circ-2x}{2}=75^\circ-x$$

$$\textit{and via sine law on }\Delta\,OCD,\ \frac{DC}{\sin 30^\circ}=\frac{OD}{\sin(75^\circ-x)}=\frac{R}{\sin\left(180^\circ-(75^\circ-x+30^\circ)\right)}$$

$$\therefore DC\overset{(*)}{=}\frac{R}{2\sin(75^\circ+x)}\ \textit{and}\ OD\overset{(**)}{=}\frac{R\sin(75^\circ-x)}{\sin(75^\circ+x)}$$

$$\text{Now}, |OD|^2+|OC||DC|=|OC|^2 \overset{via\ (*),(**)}{\Rightarrow} \frac{R^2\sin^2(75^\circ-x)}{\sin^2(75^\circ+x)}+\frac{R^2}{2\sin(75^\circ+x)}=R^2$$

$$\Rightarrow \frac{2\sin^2(75^\circ-x)+\sin(75^\circ+x)}{2\sin^2(75^\circ+x)}=1$$

$$\Rightarrow 1-\cos(150^\circ-2x)+\sin(75^\circ+x)=1-\cos(150^\circ+2x)$$

$$\Rightarrow \sin(75^\circ+x)=\cos(150^\circ-2x)-\cos(150^\circ+2x)=2\sin 150^\circ.\sin 2x=\sin 2x$$

$$\Rightarrow \sin(75^\circ+x)-\sin 2x=0 \Rightarrow 2\cos\frac{75^\circ+3x}{2}.\sin\frac{75^\circ-x}{2}=0$$

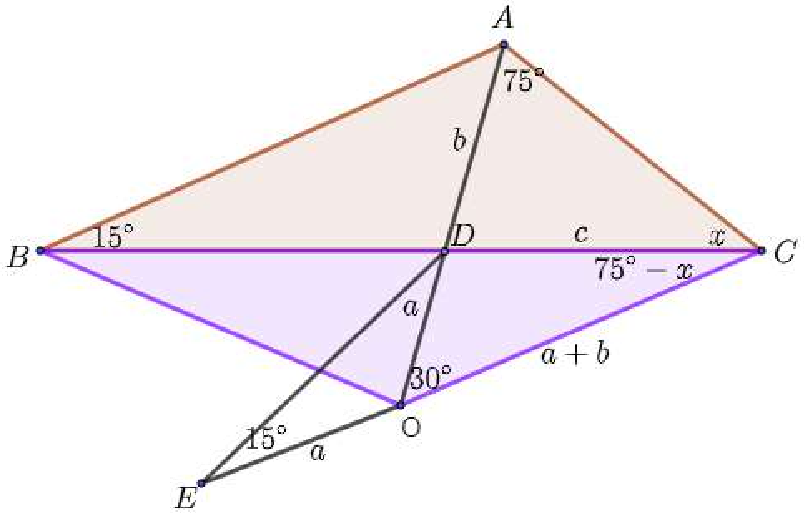
$$\Rightarrow 2 \text{ possibilities}: \sin\frac{75^\circ-x}{2}=0;\ \cos\frac{75^\circ+3x}{2}=0$$

$$\text{and} \because -7\frac{1}{2}^\circ<\frac{75^\circ-x}{2}<37\frac{1}{2}^\circ \text{ and } 37\frac{1}{2}^\circ<\frac{75^\circ+3x}{2}<172\frac{1}{2}^\circ\ (\because 0<x<90^\circ)$$

$$\therefore 2 \text{ possibilities are}: \frac{75^\circ-x}{2}=0;\ \frac{75^\circ+3x}{2}=90^\circ$$

$$\Rightarrow \frac{75^\circ+3x}{2}=90^\circ\ (\because 75^\circ-x>0) \Rightarrow x=m(\measuredangle BCA)=35^\circ\ (ans)$$

Solution 2 by Murat Oz-Turkiye



$$a^2+(a+b)c=(a+b)^2 \Rightarrow \mu(\widehat{ACB})=x=?$$

$$a^2+ac+bc=a^2+b^2+2ab$$

$$\frac{a+b}{b}=\frac{2a+b}{c} \text{ and } \frac{2a+b}{c}=\frac{\sin(90^\circ-x)}{\sin 15^\circ}$$

$$\begin{cases}\dfrac{b}{\sin x}=\dfrac{c}{\sin 75^\circ}\\ \dfrac{a}{\sin(75^\circ-x)}=\dfrac{c}{\sin 30^\circ}\end{cases} \Rightarrow \frac{a}{b}\cdot\frac{\sin x}{\sin(75^\circ-x)}=2\sin 75^\circ$$

$$\frac{a+b}{b}=\frac{a}{b}+1=\frac{2a+b}{c}\Rightarrow \frac{2\sin 75^\circ\sin(75^\circ-x)}{\sin x}+1=\frac{\cos x}{\sin 15^\circ}$$

$$\sin x\cos x=2\sin 15^\circ\cos 15^\circ\sin(75^\circ-x)+\sin 15^\circ\sin x$$

$$\sin 2x = \sin(75° - x) + 2\sin 15° \sin x$$

$$\sin 2x = \cos(15 + x) + 2\left(-\frac{1}{2}\right)[\cos(x + 15°) - \cos(x - 15°)]$$

$$\sin 2x = \cos(x - 15°) \Rightarrow x = 75° \text{ or } x = 35°$$

$$\mu(\widehat{BAC}) > 90°, \text{then } x = 35°$$

841. ***In acute ΔABC holds :***

$$\sqrt{\frac{(\pi - \hat{A})(\pi - \hat{B})}{\hat{A}\hat{B}}} + \sqrt{\frac{(\pi - \hat{B})(\pi - \hat{C})}{\hat{B}\hat{C}}} + \sqrt{\frac{(\pi - \hat{C})(\pi - \hat{A})}{\hat{C}\hat{A}}} < \frac{1}{2}\sqrt{9 + \frac{10\pi^3 R^2}{S}}$$

Proposed by Radu Diaconu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{We know that: } \sin\hat{A}\sin\hat{B}\sin\hat{C} = \frac{S}{2R^2} \text{ and } \sin x < x, \ \forall x \in \left(0, \frac{\pi}{2}\right],$$

$$\text{Then we have: } \frac{2R^2}{S} = \frac{1}{\sin\hat{A}\sin\hat{B}\sin\hat{C}} > \frac{1}{\hat{A}\hat{B}\hat{C}}.$$

$$\text{So it suffices to prove that:}$$

$$\sqrt{\frac{(\hat{B} + \hat{C})(\hat{C} + \hat{A})}{\hat{A}\hat{B}}} + \sqrt{\frac{(\hat{C} + \hat{A})(\hat{A} + \hat{B})}{\hat{B}\hat{C}}} + \sqrt{\frac{(\hat{A} + \hat{B})(\hat{B} + \hat{C})}{\hat{C}\hat{A}}} \le \frac{1}{2}\sqrt{9 + \frac{5(\hat{A} + \hat{B} + \hat{C})^3}{\hat{A}\hat{B}\hat{C}}} \quad (1)$$

$$\text{Let } p := \hat{A} + \hat{B} + \hat{C}, \ q := \hat{A}\hat{B} + \hat{B}\hat{C} + \hat{C}\hat{A}, \ r := \hat{A}\hat{B}\hat{C}.$$

$$LHS_{(1)} \overset{CBS}{\le} \sqrt{\left(\sum_{cyc}(\hat{B} + \hat{C})(\hat{C} + \hat{A})\right)\left(\sum_{cyc}\frac{1}{\hat{A}\hat{B}}\right)} = \sqrt{(p^2 + q).\frac{p}{r}} =$$

$$= \sqrt{\frac{4p^3 + 4pq}{4r}} \overset{Schur}{\le} \sqrt{\frac{4p^3 + (p^3 + 9r)}{4r}} = \frac{1}{2}\sqrt{9 + \frac{5p^3}{r}} = RHS_{(1)}.$$

842. **In ΔABC the following relationship holds:**

$$\sum_{cyc}\frac{\cos^3\frac{A}{2}}{\cos\frac{B}{2} + \sqrt{\cos\frac{B}{2}\cos\frac{C}{2}} + \sqrt[3]{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}} \ge 3\left(\frac{r}{R}\right)^2$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\cos\frac{B}{2}+\sqrt{\cos\frac{B}{2}\cos\frac{C}{2}}+\sqrt[3]{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}\overset{AM-GM}{\le}$$

$$\le\cos\frac{B}{2}+\frac{\cos\frac{B}{2}+\cos\frac{C}{2}}{2}+\frac{\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}}{3}=\frac{11}{6}\cos\frac{B}{2}+\frac{5}{6}\cos\frac{C}{2}+\frac{1}{3}\cos\frac{A}{2}$$

$$\sum_{cyc}\frac{\cos^3\frac{A}{2}}{\cos\frac{B}{2}+\sqrt{\cos\frac{B}{2}\cos\frac{C}{2}}+\sqrt[3]{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}}$$

$$\ge\sum_{cyc}\frac{\cos^3\frac{A}{2}}{\frac{11}{6}\cos\frac{B}{2}+\frac{5}{6}\cos\frac{C}{2}+\frac{1}{3}\cos\frac{A}{2}}\overset{Holder}{\ge}$$

$$\ge\frac{\left(\sum\cos\frac{A}{2}\right)^3}{3[\left(\frac{11}{6}+\frac{5}{6}+\frac{1}{3}\right)\sum\cos\frac{A}{2}}=\frac{\left(\sum\cos\frac{A}{2}\right)^2}{3\cdot3}\overset{AM-GM}{\ge}$$

$$\ge\frac{\left(3\cdot\sqrt[3]{\prod\cos\frac{A}{2}}\right)^2}{9}=\left(\prod_{cyc}\cos\frac{A}{2}\right)^{\frac{2}{3}}=\left(\frac{s}{4R}\right)^{\frac{2}{3}}\ge$$

$$\ge\left(\frac{3\sqrt{3}r}{4R}\right)^{\frac{2}{3}}=\left(\frac{3\sqrt{3}r\cdot r^2}{4R\cdot r^2}\right)^{\frac{2}{3}}\ge\left[\frac{3\sqrt{3}r^3}{4R\cdot\left(\frac{R}{2}\right)^2}\right]^{\frac{2}{3}}=3\left(\frac{r}{R}\right)^2$$

843. Let $\Delta I_aI_bI_c$ −be the excentral triangle of acute $\Delta ABC, P, Q, R$ −the midpoints of arcs BC, CA, AB and ρ −inradii of ΔPQR. Prove that:

$$4[PQR]=[I_aI_bI_c],\qquad \rho\le\frac{3\sqrt{3}}{2}R$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution by Marian Dincă-Romania

$$arc(PQ)=A+B$$

$$arc(QR)=B+C$$

$$arc(RP)=C+A$$

$$[PQR]=2R^2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A+C}{2}\right)\sin\left(\frac{B+C}{2}\right)$$

Lemma.

The triangle formed by the feet of the heights of a sharp triangle ABC denote by DEF, will have side lengths equal to $a\cos A$, $b\cos B$, $c\cos C$.

Back to the triangle ABC is formed by the feet of the heights of the triangle $I_aI_bI_c$.

According to Lemma:

$$BC=a=I_bI_c\cos(\widehat{I_aI_bI_c})=I_bI_c\cos\left(\frac{B+C}{2}\right)$$

$$a=2R\sin A=2R\sin(B+C)=4R\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B+C}{2}\right)$$

$$4R\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B+C}{2}\right)=I_bI_c\cos\left(\frac{B+C}{2}\right)\Rightarrow I_bI_c=4R\sin\left(\frac{B+C}{2}\right)$$

Similarly:

$$I_aI_c=4R\sin\left(\frac{A+C}{2}\right)\text{ and }I_aI_b=4R\sin\left(\frac{A+B}{2}\right)$$

It follows that the radius of the circle circumscribed by the triangle $I_aI_bI_c$ is equal to $2R$ and

$$[I_aI_bI_c]=2(2R)^2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A+C}{2}\right)\sin\left(\frac{B+C}{2}\right)=$$

$$=8R^2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A+C}{2}\right)\sin\left(\frac{B+C}{2}\right)$$

For the ΔABC:

$$r=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

For the ΔPQR:

$$\rho=4R\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A+C}{2}\right)\sin\left(\frac{B+C}{2}\right)\le$$

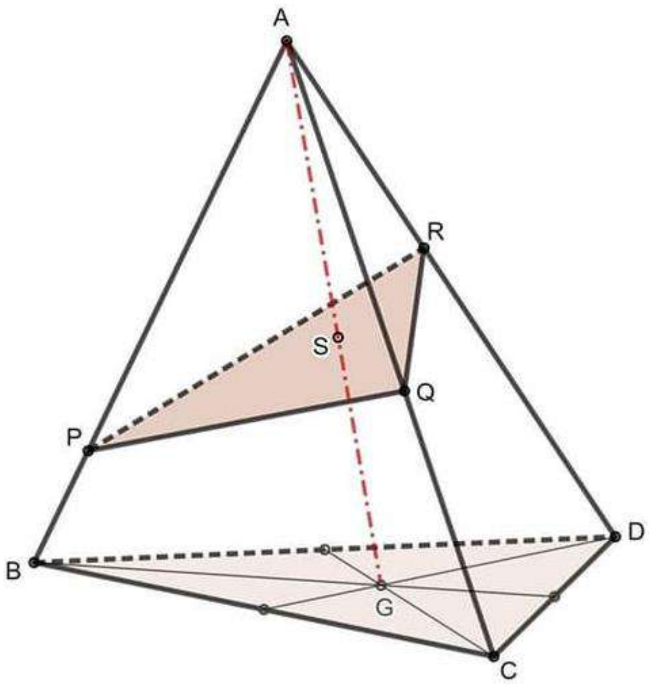
$$\le 4R\sin^3\left(\frac{\frac{A+B}{4}+\frac{B+C}{4}+\frac{C+B}{4}}{3}\right)=4R\sin^3\left(\frac{\pi}{6}\right)=4R\cdot\frac{3\sqrt{3}}{8}=\frac{3\sqrt{3}}{2}R$$

Where we use Jensen's inequality for the concave function

$$f(x) = \log(\sin x)\,;\, x \in (0,\pi)$$

844. $ABCD$ − tetrahedron, G − centroid of ΔBCD, $AP = pAB$, $AQ = qAC$, $AR = rAD$, $AS = sAG$. Prove that:

$$\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)s = 3$$



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Rajarshi Chakraborty-India

$$\vec{S} = s\left(\frac{\vec{d}+\vec{b}+\vec{c}}{3}\right)$$

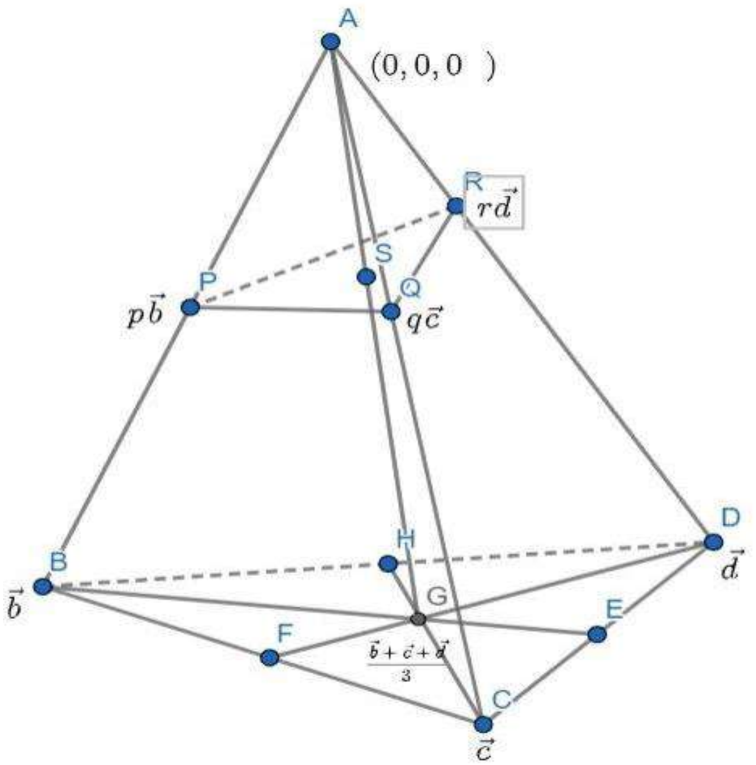
Since P, Q, R *and* S *are in the same plane, we have*:

$$\left[(p\vec{b}-r\vec{d})\times(q\vec{c}-r\vec{d})\right]\cdot(\vec{S}-r\vec{d}) = 0$$

$$\left[pq(\vec{b}\times\vec{c})-rq(\vec{d}\times\vec{c})-pr(\vec{b}\times\vec{d})\right],(\vec{S}-r\vec{d}) = 0; \qquad (1)$$

$$Now, (\vec{b}\times\vec{c})\cdot\vec{S} = s\cdot\frac{(\vec{b}\times\vec{c})\cdot\vec{d}}{3}; (\vec{d}\times\vec{c})\cdot\vec{S} = s\cdot\frac{(\vec{d}\times\vec{c})\cdot\vec{b}}{3} = -s\cdot\frac{(\vec{c}\times\vec{d})\cdot\vec{b}}{3}$$

$$(\vec{b}\times\vec{d})\cdot\vec{S} = s\cdot\frac{(\vec{b}\times\vec{d})\cdot\vec{c}}{3} = -s\cdot\frac{(\vec{d}\times\vec{b})\cdot\vec{c}}{3}$$



$$So\ (1)\ can\ be\ written\ as:$$

$$[pqs\cdot\frac{(\vec{b}\times\vec{c})\cdot\vec{d}}{3}+rqs\cdot\frac{(\vec{c}\times\vec{d})\cdot\vec{b}}{3}+prs\cdot\frac{(\vec{d}\times\vec{b})\cdot\vec{c}}{3}=pqr[(\vec{b}\times\vec{c})\cdot\vec{d}$$

$$\Rightarrow\frac{pqs}{3}+\frac{rqs}{3}+\frac{prs}{3}=pqr\Rightarrow\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)\cdot s=3$$

845. If $x,y,z>0$ then in ΔABC holds:

$$\frac{x^2a^3}{(y+z)^2}+\frac{y^2b^3}{(z+x)^2}+\frac{z^2c^3}{(x+y)^2}\geq 6rF$$

Proposed by D.M. Bătineţu-Giurgiu, Florică Anastase-Romania

Solution by Tapas Das-India

$$\frac{x^2a^3}{(y+z)^2}+\frac{y^2b^3}{(z+x)^2}+\frac{z^2c^3}{(x+y)^2}=\frac{\left(\frac{x}{y+z}a^2\right)^2}{a}+\frac{\left(\frac{y}{z+x}b^2\right)^2}{b}+\frac{\left(\frac{z}{x+y}c^2\right)^2}{c}\geq$$

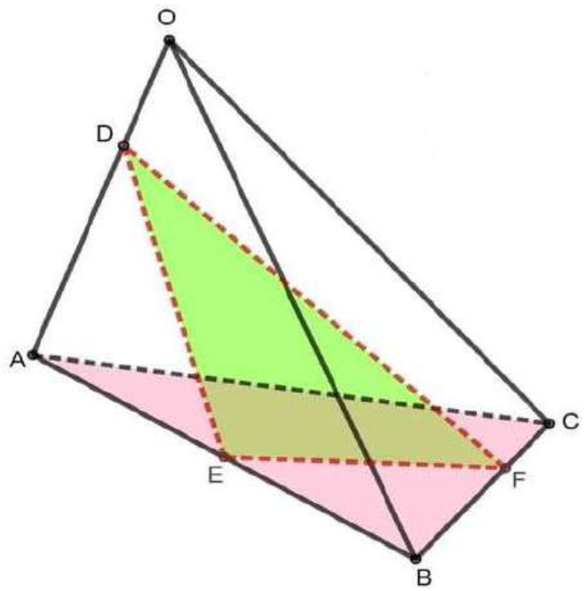
$$\geq\frac{\left(\frac{x}{y+z}a^2+\frac{y}{z+x}b^2+\frac{z}{x+y}c^2\right)^2}{a+b+c}\overset{Tsintsifas}{\geq}\frac{\left(2\sqrt{3}F\right)^2}{2s}=\frac{12F^2}{2s}=\frac{6F\cdot rs}{s}=6rF$$

846. $OABC-$ tetrahedron, $OA=6, OB=OC=12,$

$$OD=\frac{1}{3}OA, AE=\frac{1}{2}AB, BF=\frac{2}{3}BC$$

$$\measuredangle BOC=\measuredangle COA=\measuredangle AOB=\theta=60^\circ, \text{plane}(D,E,F)=(P),$$

$$(P)\cap OC=K, (P)\cap OB=L, (P)\cap AC=M. \text{Volume}(OKLM)=V=?$$



Proposed by Thanasis Gkopoulos-Farsala-Greece

Solution by proposer

Plagiogonal system 3D:

$$OA\equiv Ox; OB\equiv Oy; OC\equiv Oz$$

$$O(0,0,0), A(6,0,0), D(2,0,0), B(0,12,0), E(3,6,0), C(0,0,12), F(0,4,8)$$

$$(P):\begin{vmatrix}1 & 1 & 1 & 1\\ x & 2 & 3 & 0\\ y & 0 & 6 & 4\\ 2 & 0 & 0 & 8\end{vmatrix}=0\Rightarrow 6x-y+2z-12=0;\quad (1)$$

$$OC:\{x=0, y=0\}; (2)$$

$$(1),(2)\Rightarrow K(0,0,6)$$

$$OB:\{x=0, z=0\};\quad (3)$$

$$(1),(3)\Rightarrow L(0,-12,0)$$

$$AC:\left\{\frac{x-6}{0-6}=\frac{z-0}{12-0}\Rightarrow 2x+z=12, y=0\right\};\quad (4)$$

$$(1),(4)\Rightarrow M(6,0,24)$$

$$\text{Is } OK^2=36\Rightarrow OL^2=144, OM^2=6^2+24^2+6\cdot 24=756$$

$$KL^2=12^2+6^2+12\cdot 6=252$$

$$LM^2=6^2+12^2+24^2+6\cdot 12+12\cdot 24+6\cdot 24=1260$$

$$KM^2=6^2+18^2+6\cdot 18=468$$

$$288\cdot V^2=|\Delta|, \text{where } \Delta=\begin{vmatrix}0 & 1 & 1 & 1 & 1\\ 1 & 0 & 36 & 144 & 756\\ 1 & 36 & 0 & 252 & 468\\ 1 & 144 & 252 & 0 & 1260\\ 1 & 756 & 468 & 1260 & 0\end{vmatrix}$$

$$V=\frac{6\cdot 12\cdot 12}{6}\cdot\frac{1}{\sqrt{2}}=72\sqrt{2}$$

847. If $m,n,p,x,y,z>0$ then in ΔABC holds:

$$\frac{(m+n)(z+x)}{py}\cdot ab+\frac{(p+m)(y+z)}{mz}\cdot bc+\frac{(n+p)(x+y)}{nx}\cdot ca\geq 16\sqrt{3}\cdot F$$

Proposed by D.M.Bătinețu-Giurgiu, Florică Anastase-Romania

Solution by Tapas Das-India

$$\frac{(m+n)(z+x)}{py}\cdot ab+\frac{(p+m)(y+z)}{mz}\cdot bc+\frac{(n+p)(x+y)}{nx}\cdot ca\overset{AM-GM}{\geq}$$

$$\geq\frac{2\sqrt{mn}\cdot 2\sqrt{zx}}{py}\cdot ab+\frac{2\sqrt{pm}\cdot 2\sqrt{yz}}{mz}\cdot bc+\frac{2\sqrt{np}\cdot 2\sqrt{xy}}{nx}\cdot ca\overset{AM-GM}{\geq}$$

$$\geq 3\left[\frac{2\sqrt{mn}\cdot 2\sqrt{zx}}{py}\cdot ab\cdot\frac{2\sqrt{pm}\cdot 2\sqrt{yz}}{mz}\cdot bc\cdot\frac{2\sqrt{np}\cdot 2\sqrt{xy}}{nx}\cdot ca\right]^{\frac{1}{3}}=$$

$$=3\left[8\cdot 8\cdot\frac{mnp\cdot xyz}{mnp\cdot xyz}\cdot(abc)^2\right]^{\frac{1}{3}}\geq 4\cdot 3\left[\left(\frac{4F}{\sqrt{3}}\right)^3\right]^{\frac{1}{3}}=16\sqrt{3}\cdot F$$

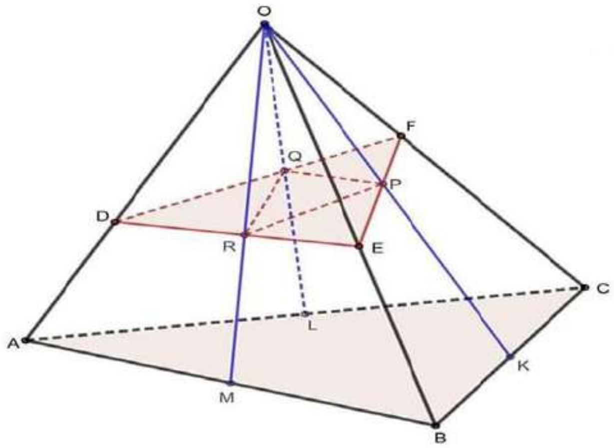
$$\because (abc)^2\geq\left(\frac{4F}{\sqrt{3}}\right)^3$$

848. $OABC-\text{tetrahedron}, KB=KC, LC=LA, MA=MB,$

$$OP=kOK, OQ=qOL, OR=rOM, \text{plane}(P,Q,R)=(\pi),$$

$$(\pi)\cap OA=D, (\pi)\cap OB=E, (\pi)\cap OC=F$$

$\text{Volume}(OABC) = V_0, \text{Volume}(ODEF) = V. \text{Find: } \frac{V_0}{V} = ?$



Proposed by Thanasis Gakopoulos-Farsala-Greece

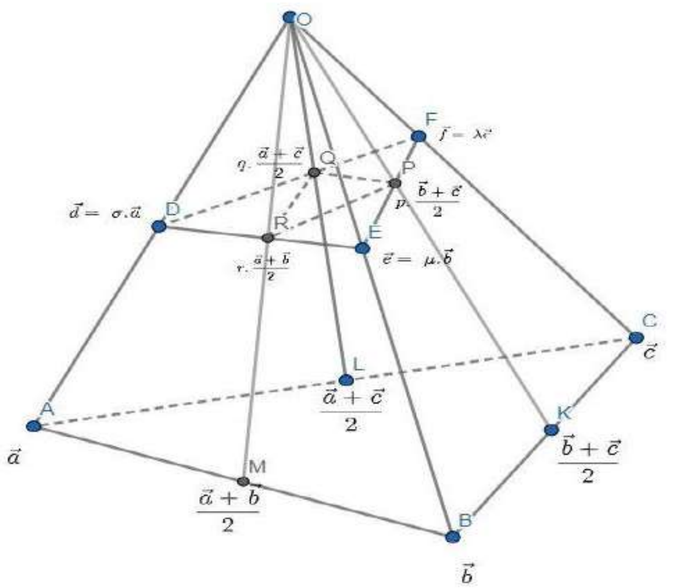
Solution by Rajarshi Chakraborty-India

$$V_0 = \frac{1}{6}\left[\vec{a} \cdot \left(\vec{b} \times \vec{c}\right)\right], \qquad V = \frac{1}{6}\left[\vec{d} \cdot \left(\vec{e} \times \vec{f}\right)\right], \vec{e} = \mu\vec{b}, \vec{d} = \sigma\vec{a}, \vec{f} = \lambda\vec{c}$$

$$V = \frac{1}{6} \cdot \lambda\mu\sigma\left[\vec{a} \cdot \left(\vec{b} \times \vec{c}\right)\right] \Rightarrow \frac{V_0}{V} = \frac{1}{\lambda\mu\sigma}; \quad (l)$$

$$\textit{Since } D, R \textit{ and } E \textit{ are collinear } \left(\sigma\vec{a} - \mu\vec{b}\right) \times \left(r\frac{\vec{a} + \vec{b}}{2} - \mu\vec{b}\right) = 0$$

$$\sigma\frac{r}{2}\left(\vec{a} \times \vec{b}\right) + \mu\frac{r}{2}\left(\vec{a} \times \vec{b}\right) = \sigma\mu\left(\vec{a} \times \vec{b}\right), \qquad \frac{\sigma + \mu}{2} \cdot r = \sigma\mu \Rightarrow \frac{1}{2}\left(\frac{1}{\mu} + \frac{1}{\sigma}\right) = \frac{1}{r}; \quad (1)$$



$$Similarly, \frac{1}{2}\left(\frac{1}{\mu}+\frac{1}{\lambda}\right)=\frac{1}{p}; \quad (2)$$

$$\frac{1}{2}\left(\frac{1}{\lambda}+\frac{1}{\sigma}\right)=\frac{1}{q}; \quad (3)$$

From $(1),(2)$ *and* (3):

$$\frac{1}{\mu}+\frac{1}{\sigma}+\frac{1}{\lambda}=\frac{1}{r}+\frac{1}{p}+\frac{1}{q}; \quad (4)$$

From (1) *and* (4):

$$\frac{1}{\lambda}=-\frac{1}{r}+\frac{1}{p}+\frac{1}{q}; \quad (m)$$

$$\frac{1}{\sigma}=\frac{1}{r}-\frac{1}{p}+\frac{1}{q}; \quad (n)$$

$$\frac{1}{\mu}=\frac{1}{r}-\frac{1}{p}+\frac{1}{q}; \quad (o)$$

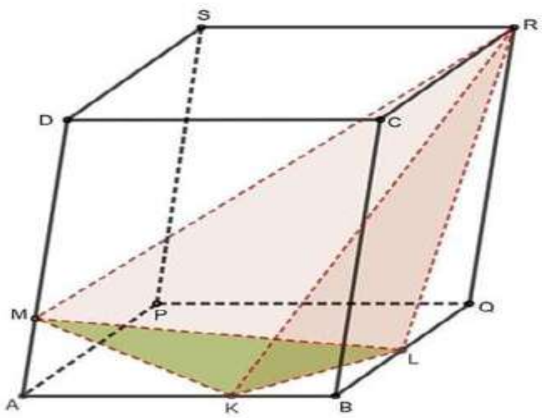
$$Thus, \frac{V_0}{V}=\frac{1}{\lambda\mu\sigma}=\left(\frac{1}{r}-\frac{1}{p}+\frac{1}{q}\right)\left(\frac{1}{r}-\frac{1}{p}+\frac{1}{q}\right)\left(-\frac{1}{r}+\frac{1}{p}+\frac{1}{q}\right)$$

849. $ABCDPQRS$ **− parallelipiped,**

$$AP=8, AB=12, AD=24, AM=\frac{1}{3}AD$$

$$BL=\frac{1}{2}BQ, AK=\frac{2}{3}AB, \measuredangle PAD=\theta_1=60^\circ, \measuredangle BAD=\theta_2=45^\circ,$$

$$\measuredangle BAP=\theta_3=30^\circ. \text{ Volume(RKLM)} = V = ?$$



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

$$\boldsymbol{Plagiogonal\ 3D\ system: AB \equiv Ax, AP \equiv Ay, AD \equiv Az}$$

$$\boldsymbol{A(0,0,0), B(12,0,0), K(8,0,0), D(0,0,24), M(0,0,8), Q(12,8,0), L(12,4,0), R(12,8,24)}$$

$$\boldsymbol{Transformation: A(0,0,0) \to A(0,0,0)}$$

$$\boldsymbol{L(12,4,0) \to L\left(12+2\sqrt{3},2,0\right)}$$

$$\boldsymbol{K(8,0,0) \to K(8,0,0)}$$

$$\boldsymbol{M(0,0,8) \to M\left(4\sqrt{2},8-4\sqrt{6},8\sqrt{\sqrt{6}-2}\right)}$$

$$\boldsymbol{R(12,8,24) \to R\left(12+4\sqrt{3}+12\sqrt{2},28-12\sqrt{6},24\sqrt{\sqrt{6}-2}\right)}$$

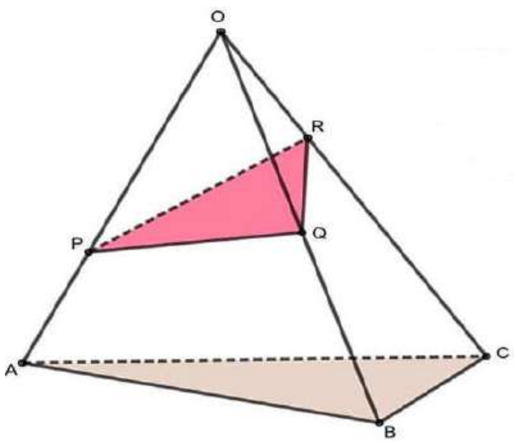
$$\boldsymbol{V = \frac{1}{6}|\Delta|, where\ \Delta = \begin{vmatrix} 8 & 0 & 0 & 1 \\ 12+2\sqrt{3} & 2 & 0 & 1 \\ 4\sqrt{2} & 8-4\sqrt{6} & 8\sqrt{\sqrt{6}-2} & 1 \\ 12+4\sqrt{3}+12\sqrt{2} & 28-12\sqrt{6} & 24\sqrt{\sqrt{6}-2} & 1 \end{vmatrix}}$$

$$\boldsymbol{V = \frac{160\sqrt{\sqrt{6}-2}}{3}}$$

850. $\boldsymbol{OABC}$ **–tetrahedron,** $\boldsymbol{\measuredangle BOC = \measuredangle COA = \measuredangle AOB = \theta = 60^\circ}$

$$\boldsymbol{OA = 4a, OB = 4b, OC = 4c, OP = p \cdot OA, OQ = q \cdot OB, OR = r \cdot OC}$$

Find $\boldsymbol{Area_{(PQR)} = S = ?}$



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

$$\boldsymbol{Plagiogonal\ system}: \boldsymbol{OA \equiv Ox; OB \equiv Oy; OC \equiv Oz}$$

$$\boldsymbol{O(0,0,0), A(4a,0,0), B(0,4b,0), C(0,0,4c)}$$

$$\boldsymbol{P(4pa,0,0), Q(0,4bq,0), R(0,0,4rc)}$$

$$\overrightarrow{\boldsymbol{PQ}} = (-4pa, 4qb, 0), \overrightarrow{\boldsymbol{PR}} = (-4pa, 0, 4rc)$$

$$\left|\overrightarrow{\boldsymbol{PQ}}\right|^2 = 16p^2a^2 + 16q^2b^2 - 16pqab$$

$$\left|\overrightarrow{\boldsymbol{PR}}\right|^2 = 16p^2a^2 + 16r^2c^2 - 16prac$$

$$\boldsymbol{Is}\ 4S^2 = \left|\overrightarrow{\boldsymbol{PQ}}\right|^2 \cdot \left|\overrightarrow{\boldsymbol{PR}}\right|^2 - \left(\overrightarrow{\boldsymbol{PQ}} \cdot \overrightarrow{\boldsymbol{RC}}\right)$$

$$S = 4\sqrt{3(a^2b^2p^2q^2 + b^2c^2q^2r^2 + c^2a^2r^2p^2) - 2(ab^2cp^2qr + ab^2cpq^2r + abc^2pqr^2)}$$

$$S = 4abcpqr\sqrt{3\left(\frac{1}{a^2p^2} + \frac{1}{b^2q^2} + \frac{1}{c^2r^2}\right) - 2\left(\frac{1}{abpq} + \frac{1}{bcqr} + \frac{1}{carp}\right)}$$

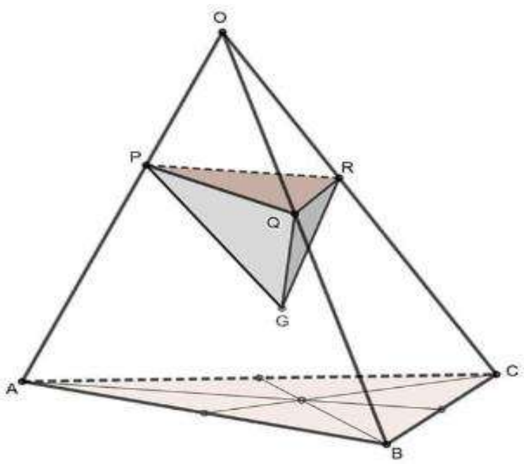
$$S = \frac{OP \cdot OQ \cdot OR}{4}\sqrt{3\left(\frac{1}{OP^2} + \frac{1}{OQ^2} + \frac{1}{OR^2}\right) - 2\left(\frac{1}{OP \cdot OQ} + \frac{1}{OQ \cdot OR} + \frac{1}{OR \cdot OP}\right)}$$

851. $OABC$ **–tetrahedron,** G **–centroid of** $OABC$,

$$\sphericalangle BOC = \sphericalangle COA = \sphericalangle AOB = \theta = 60^\circ$$

$$OA = 4a, OB = 4b, OC = 4c, OP = pOA, OQ = qOB, OR = rOC$$

Find: $Volume_{(GPQR)} = V = ?$



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

$$Plagiogonal\ 3D\ system: OA \equiv Ox; OB \equiv Oy; OC \equiv Oz$$

$$O(0,0,0), A(4a,0,0), B(0,4b,0), C(0,0,4c), G(a,b,c)$$

$$P(4pa,0,0), Q(0,4qb,0), R(0,0,4rc)$$

$$Transformation: A_1 = 4pa; A_2 = 0; A_3 = 0; B_1 = 4qb \cdot \frac{1}{2}; B_2 = 4qb \cdot \frac{\sqrt{3}}{2}$$

$$C_1 = 4rc \cdot \frac{1}{2}; C_2 = 4rc \cdot \frac{\sqrt{3}}{6}; C_3 = 4rc \cdot \frac{\sqrt{6}}{3}$$

$$D_1 = a + \frac{b}{2} + \frac{c}{3}; D_2 = b \cdot \frac{\sqrt{3}}{2} + C \cdot \frac{\sqrt{3}}{6}; D_3 = \frac{\sqrt{6}}{3}$$

$$V = \frac{1}{6}|\Delta|, where\ \Delta = \begin{vmatrix} 1 & A_1 & A_2 & A_3 \\ 1 & B_1 & B_2 & B_3 \\ 1 & C_1 & C_2 & C_3 \\ 1 & D_1 & D_2 & D_3 \end{vmatrix}. Thus,$$

$$V = \frac{4\sqrt{2}}{3} abcpqr \left|\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 4\right|$$

$$Note: Is\ V = \frac{1}{3}[PQR] \cdot h; h = d\big(O,(PQR)\big)$$

$$h = \sqrt{2} \cdot \frac{\left|\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 4\right|}{\sqrt{3\left(\frac{1}{a^2p^2} + \frac{1}{b^2q^2} + \frac{1}{c^2r^2}\right) - 2\left(\frac{1}{abpq} + \frac{1}{bcqr} + \frac{1}{carp}\right)}}$$

$$V = \frac{\sqrt{2}}{48} OP \cdot QO \cdot OR \cdot \left|\frac{OA}{OP} + \frac{OB}{OQ} + \frac{OC}{OR} - 4\right|$$

852. In ΔABC the following relationship holds:

$$\sqrt{162Rr(s^2 + 9r^2)} \leq 4s^2 \leq 17R^2 + 18Rr + 4r^2$$

Proposed by Radu Diaconu-Romania

Solution by Tapas Das-India

$$s^2 \geq \frac{27Rr}{2} \Rightarrow 162Rr(s^2 + 9r^2) = 12 \cdot \frac{27Rr}{2}(s^2 + 9r^2) \leq 12s^2(s^2 + 9r^2)$$

$$\sqrt{162Rr(s^2 + 9r^2)} \leq s\sqrt{12(s^2 + 9r^2)}$$

$$We\ need\ to\ show:$$

$$s\sqrt{12(s^2 + 9r^2)} \leq 9s^2 \Leftrightarrow \sqrt{12(s^2 + 9r^2)} \leq 4s \Leftrightarrow 12(s^2 + 9r^2) \leq 16s^2$$

$$\Leftrightarrow 4s^2 \geq 108r^2 \text{ which is true since } s^2 \geq 27r^2 (Mitrinovic)$$

$$Now, 4s^2 \leq 4(4R^2 + 4Rr + 3r^2); (Gerretsen)$$

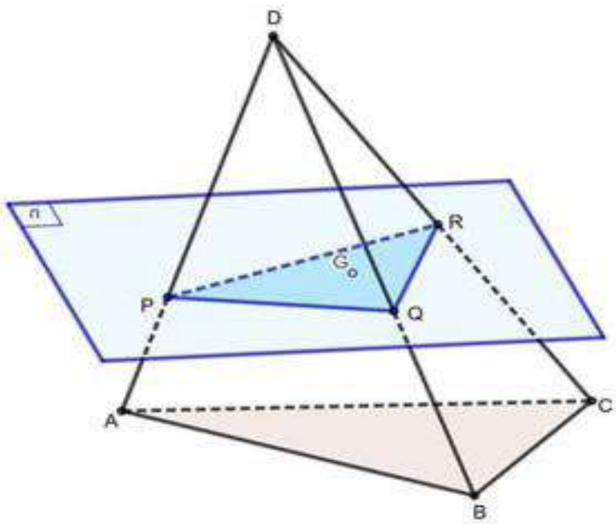
$$4(4R^2 + 4Rr + 3r^2) \leq 17R^2 + 18Rr + 4r^2 \Leftrightarrow$$

$$(R + 4r)(R - 2r) \geq 0 \text{ which is true from } R \geq 2(Euler).$$

853. $DABC$ **−tetrahedron,** G **−centroid of** $DABC$**,**

plane$(P, Q, R) = (\pi), G \in (\pi)$

Prove: $\frac{DA}{DP} + \frac{DB}{DQ} + \frac{DC}{DR} = 4$ **or** $\frac{PA}{PD} + \frac{QB}{QD} + \frac{RC}{RD} = 1.$



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

$$Plagiogonal\ 3D\ system: DA \equiv Dx; DB \equiv Dy; DC \equiv Dz$$

$$D(0,0,0), G\left(\frac{DA}{4}, \frac{DB}{4}, -\frac{DC}{4}\right)$$

$$(\pi): \frac{x}{DP} + \frac{y}{DQ} + \frac{z}{DR} = 1; G \in (\pi) \Rightarrow \frac{DA}{DP} + \frac{DB}{DQ} + \frac{DC}{DR} = 4$$

$$\frac{DP + PA}{DP} + \frac{DQ + QB}{DQ} + \frac{DR + RC}{DR} = 4 \Rightarrow \frac{PA}{PD} + \frac{QB}{QD} + \frac{RC}{RD} = 1.$$

854. **In** ΔABC **the following relationship holds:**

$$a^2 m_a + b^2 m_b + c^2 m_c \geq 54Rr^2$$

Proposed by Marin Chirciu-Romania

Solution 1 by Marian Ursărescu-Romania

$$\sum_{cyc} a^2 m_a = \sum_{cyc} \frac{a^2}{\frac{1}{m_a}} \overset{Bergstrom}{\geq} \frac{(a+b+c)^2}{\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}}$$

$$We\ must\ show\ that:$$

$$\frac{4s^2}{\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}} \geq 54Rr^2 \Leftrightarrow \frac{2s^2}{\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}} \geq 27Rr^2;\ \ (1)$$

$$But\ 2s^2 \geq 27Rr\ (Cosnita - Turtoiu); (2)$$

$$From\ (1)\ and\ (2), we\ must\ show:$$

$$\frac{1}{\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}} \geq \frac{1}{r} \Leftrightarrow \frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c} \leq \frac{1}{r}\ true\ because$$

$$m_a \geq h_a\ and\ \frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c} \leq \frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}.$$

Solution 2 by Tapas Das-India

$$\sum_{cyc} a^2 m_a \geq \sum_{cyc} a^2 h_a = \sum_{cyc} a^2 \cdot \frac{2F}{a} = 2F\sum_{cyc} a = 4Fs$$

$$Now, we\ need\ to\ prove:$$

$$4Fs \geq 54Rr^2 \Leftrightarrow 4rs^2 \geq 27Rr \Leftrightarrow 2(16Rr - 5r^2) \geq 27Rr(Gerretsen)$$

$$32Rr - 10r^2 \geq 27Rr \Leftrightarrow 5Rr - 10r^2 \geq 0 \Leftrightarrow 5r(R-2r) \geq 0\ true\ from$$

$$R \geq 2r\ (Euler).$$

855. In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{\sqrt{s_a}}{\sqrt{h_b}+\sqrt{h_c}}(w_a^2 + m_b m_c) + \sum_{cyc} h_a^2 \geq \frac{108r^3}{R}$$

Proposed by Florică Anastase, Mihai Ghenoiu-Romania

Solution by Tapas Das-India

$$Because: s_a \geq h_a, w_a \geq h_a, m_a \geq h_a, we\ have:$$

$$\sum_{cyc} \frac{\sqrt{s_a}}{\sqrt{h_b}+\sqrt{h_c}}(w_a^2 + m_b m_c) \geq \sum_{cyc} \frac{\sqrt{h_a}}{\sqrt{h_b}+\sqrt{h_c}}(h_a^2 + h_b h_c) \geq$$

$$\overset{AM-GM}{\geq}\sum_{cyc}\frac{\sqrt{h_a}}{\sqrt{h_b}+\sqrt{h_c}}\cdot 2\sqrt{h_a^2h_bh_c}=2\sum_{cyc}\frac{h_a\sqrt{h_ah_bh_c}}{\sqrt{h_b}+\sqrt{h_c}}=$$

$$=2\sqrt{h_ah_bh_c}\sum_{cyc}\frac{\left(\sqrt{h_a}\right)^2}{\sqrt{h_b}+\sqrt{h_c}}\geq 2\sqrt{h_ah_bh_c}\cdot\frac{\left(\sqrt{h_a}+\sqrt{h_b}+\sqrt{h_c}\right)^2}{2\left(\sqrt{h_a}+\sqrt{h_b}+\sqrt{h_c}\right)}=$$

$$=\sqrt{h_ah_bh_c}\left(\sqrt{h_a}+\sqrt{h_b}+\sqrt{h_c}\right)\overset{AM-GM}{\geq}3\sqrt{h_ah_bh_c}\cdot\sqrt[6]{h_ah_bh_c}=$$

$$=3\sqrt[3]{(h_ah_bh_c)^2}\geq 3\sqrt[3]{(27r^3)^2}=27r^2=\frac{27r^3}{r}\geq\frac{27r^3}{\frac{R}{2}}=\frac{54r^3}{R}$$

$$\sum_{cyc}h_a^2=h_a^2+h_b^2+h_c^2\geq\frac{(h_a+h_b+h_c)^2}{3}\geq\frac{(9r)^2}{3}=27r^2=$$

$$=\frac{27r^3}{r}\geq\frac{27r^3}{\frac{R}{2}}=\frac{54r^3}{R}$$

$$\mathit{Therefore},$$

$$\sum_{cyc}\frac{\sqrt{s_a}}{\sqrt{h_b}+\sqrt{h_c}}(w_a^2+m_bm_c)+\sum_{cyc}h_a^2\geq\frac{108r^3}{R}$$

856. Find $\lambda>0$ so that the double inequality holds in any ΔABC:

$$R\geq\sum_{cyc}\frac{b+c}{\lambda\cdot\cos\frac{A}{2}}\geq 2r$$

Proposed by Alex Szoros-Romania

Solution by Tapas Das-India

$$R\geq\sum_{cyc}\frac{b+c}{\lambda\cdot\cos\frac{A}{2}}\geq 2r;\ \ (1)$$

$$\mathit{Since\ it\ is\ true\ for\ any\ }\Delta ABC,\mathit{for\ an\ equilateral\ triangle},\mathit{we\ have}:$$

$$a=b=c\ \mathit{and}\ A=B=C=\frac{\pi}{3},R=2r$$

$$\mathit{Now},\mathit{from}\ (1),\sum_{cyc}\frac{b+c}{\lambda\cos\frac{A}{2}}=2r\Leftrightarrow\sum_{cyc}\frac{2a}{\lambda\cos\frac{A}{2}}=2r\Leftrightarrow$$

$$\frac{6a}{\lambda\cos\frac{A}{2}}=2r\Leftrightarrow\lambda=\frac{3a}{r\cos\frac{A}{2}}=\frac{3a}{\lambda\cos 30^\circ}=12$$

We know that: $s^2\ge 27r^2$ *and for an equilateral triangle*: $s^2=27r^2$

$$s=3\sqrt{3}r$$

So, it is suffices to prove that the result is true for $\lambda=12$.

$$\text{We need to prove}: R\ge\sum_{cyc}\frac{b+c}{12\cos\frac{A}{2}}\ge 2$$

$$\sum_{cyc}\frac{b+c}{12\cos\frac{A}{2}}=\sum_{cyc}\frac{2bc}{12w_a}=\frac{1}{6}\sum_{cyc}\frac{bc}{w_a}\overset{AM-GM}{\ge}\frac{1}{6}\cdot 3\sqrt[3]{\frac{(abc)^2}{w_aw_bw_c}}\ge$$

$$\ge\frac{1}{2}\cdot\sqrt[3]{\frac{4RF\cdot 4RF}{rs^2}}=\frac{1}{2}\cdot\sqrt[3]{\frac{16R^2F^2}{rs^2}}\ge\frac{1}{2}\sqrt[3]{\frac{16(2r)^2r^2s^2}{rs^2}}=2r;\quad(2)$$

On the other hand, we have:

$$\sum_{cyc}\frac{b+c}{12\cos\frac{A}{2}}=\sum_{cyc}\frac{2bc}{12w_a}=\frac{1}{6}\sum_{cyc}\frac{bc}{w_a}\le\frac{1}{6}\sum_{cyc}\frac{bc}{h_a}=$$

$$=\frac{abc}{6}\sum_{cyc}\frac{1}{2F}=\frac{abc}{6}\cdot\frac{3}{2F}=\frac{12Rrs}{12rs}=R;\quad(3)$$

From $(1),(2)$ *and* (3), *we have* $\lambda=12$.

857. In ΔABC the following relationship holds:

$$\sqrt{\frac{a^{4n}+b^{4n}+c^{4n}}{a^{2n}+b^{2n}+c^{2n}}}\cdot\sqrt{\frac{A^{4n}+B^{4n}+C^{4n}}{A^{2n}+B^{2n}+C^{2n}}}\cdot\sqrt{\frac{r_a^{4n}+r_b^{4n}+r_c^{4n}}{r_a^{2n}+r_b^{2n}+r_c^{2n}}}\ge\frac{[2\pi s(4R+r)]^n}{3^{3n}}$$

Proposed by Radu Diaconu-Romania

Solution by Tapas Das-India

$$a^{4n}+b^{4n}+c^{4n}=(a^{2n})^2+(b^{2n})^2+(c^{2n})^2\ge\frac{(a^{2n}+b^{2n}+c^{2n})^2}{3}$$

$$\frac{a^{4n}+b^{4n}+c^{4n}}{a^{2n}+b^{2n}+c^{2n}}\ge\frac{a^{2n}+b^{2n}+c^{2n}}{3}\ge\frac{1}{3}\cdot\frac{(a+b+c)^{2n}}{3^{2n-1}}=\frac{(a+b+c)^{2n}}{3^{2n}}$$

$$\sqrt{\frac{a^{4n}+b^{4n}+c^{4n}}{a^{2n}+b^{2n}+c^{2n}}}\ge\sqrt{\frac{(a+b+c)^{2n}}{3^{2n}}}=\frac{(a+b+c)^n}{3^n}=\left(\frac{2s}{3}\right)^n$$

Similarly,

$$\sqrt{\frac{A^{4n}+b^{4n}+C^{4n}}{A^{2n}+B^{2n}+C^{2n}}}\ge\frac{(A+B+C)^n}{3^n}=\frac{\pi^n}{3^n}$$

$$\sqrt{\frac{r_a^{4n}+r_b^{4n}+r_c^{4n}}{r_a^{2n}+r_b^{2n}+r_c^{2n}}}\ge\frac{(r_a+r_b+r_c)^n}{3^n}=\frac{(4R+r)^n}{3^n}$$

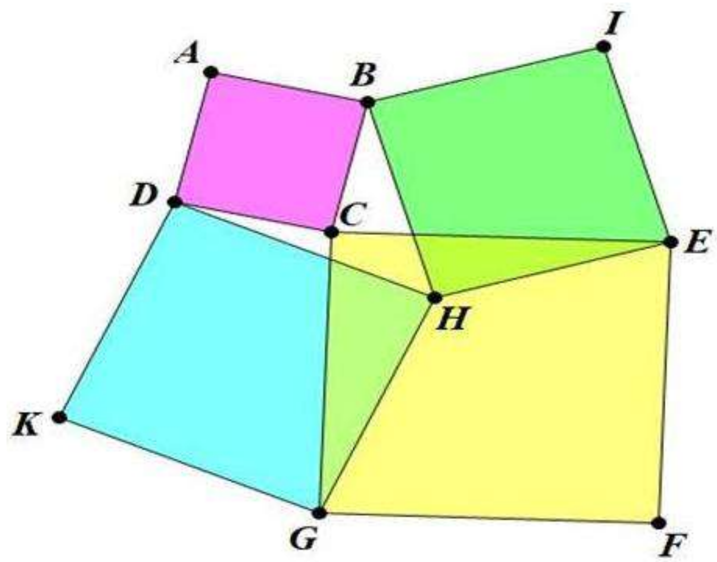
Therefore,

$$\sqrt{\frac{a^{4n}+b^{4n}+c^{4n}}{a^{2n}+b^{2n}+c^{2n}}}\cdot\sqrt{\frac{A^{4n}+b^{4n}+C^{4n}}{A^{2n}+B^{2n}+C^{2n}}}\cdot\sqrt{\frac{r_a^{4n}+r_b^{4n}+r_c^{4n}}{r_a^{2n}+r_b^{2n}+r_c^{2n}}}\ge$$

$$\ge\left(\frac{3s}{2}\right)^n\cdot\left(\frac{\pi}{3}\right)^n\cdot\left(\frac{4R+r}{3}\right)^n=\frac{[2\pi s(4R+r)]^n}{3^{3n}}$$

858. **Given four squares. Prove that:**

$$[ABCD]+[CEFG]=[BHEI]+[DHGK]$$



Proposed by Binh Luc-Vietnam

Solution 1 by Jose Ferreira Queiroz-Olinda-Brazil

$$Let\ GF=a;GH=c;HE=b;AD=d$$

$$In\ \Delta GEH:2a^2=b^2+c^2-2bc\cdot\cos\theta\ ,where\ \theta=\mu(\widehat{GHE});\ \ (1)$$

$$In\ \Delta HBD:2d^2=b^2+c^2-2bc\cdot\cos(\pi-\theta)$$

$$2d^2=b^2+c^2+2bc\cos\theta;\ \ (2)$$

$$From\ (1)\ and\ (2): 2a^2+2d^2=2b^2+2c^2$$

$$a^2+d^2=b^2+c^2$$

$$Therefore,\qquad [ABCD]+[CEFG]=[BHEI]+[DHGK]$$

Solution 2 by Hikmat Mammadov-Azerbaijan

$$BE^2=a^2+b^2-2ab\cos\left(\pi-\frac{\pi}{2}-\theta\right)$$

$$BE^2=a^2+b^2-2ab\sin\theta=\left(\sqrt{2}c\right)^2;\ \ (i)$$

$$Replace\ \theta\ with-\theta, we\ get: BD^2=a^2+b^2+2ab\sin\theta=\left(\sqrt{2}d\right)^2;\ \ (ii)$$

$$From\ (i)\ and\ (ii), it\ follows:$$

$$2(a^2+b^2)=2c^2+2d^2\Rightarrow a^2+b^2=c^2+d^2$$

It can also be proved that the blue square and the green square always touch each other as shown.

$$\sin\left(\alpha+\frac{\pi}{4}\right)=\frac{a\cos\theta}{\sqrt{a^2+b^2-2ab\sin\theta}}$$

$$\sin\left(\beta+\frac{\pi}{4}\right)=\frac{a\sin\theta}{\sqrt{a^2+b^2-2ab\sin\theta}}$$

$$\left(\sqrt{2}a\right)^2=c^2+d^2+2cd\sin(\alpha+\beta);(to\ prove)$$

$$2a^2=c^2+d^2-2cd\cos\left(\alpha+\frac{\pi}{4}+\beta+\frac{\pi}{4}\right)$$

$$b^2-a^2\sin^2\theta=\sqrt{(a^2\sin^2\theta+b^2)^2-4a^2b^2\sin^2\theta}$$

$$b^4+a^4\sin^4\theta=\sqrt{(a^2\sin^2\theta+b^2)^2-2a^2b^2\sin^2\theta};(true)$$

859.

$$A_1A_2\dots A_n-convex\ polygon,\ \frac{1}{\mu(A_1)}+\frac{1}{\mu(A_2)}+\cdots+\frac{1}{\mu(A_n)}=\frac{n^2}{(n-3)\pi}, n\ge 4.$$

$$Prove\ that:\sum_{cyc}\sqrt[3]{\frac{\mu^3(A_1)+\mu^3(A_2)}{2}}\le(n-1)\pi$$

Proposed by Radu Diaconu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let\ x,y>0.\ By\ AM-GM\ inequality, we\ have:$$

$$(x+y)\sqrt[3]{\frac{x^3+y^3}{2}}=2\sqrt[3]{\left(\frac{x+y}{2}\right)^4(x^2-xy+y^2)}\le$$

$$\le\frac{2}{3}\left[\left(\frac{x+y}{2}\right)^2+\left(\frac{x+y}{2}\right)^2+(x^2-xy+y^2)\right]=x^2+y^2.$$

$$Then:\ \sqrt[3]{\frac{x^3+y^3}{2}}\le\frac{x^2+y^2}{x+y}=x+y-\frac{2xy}{x+y},\quad \forall x,y>0.$$

$$Therefore,\ \sum_{cyc}\sqrt[3]{\frac{\mu^3(A_1)+\mu^3(A_2)}{2}}\le\sum_{cyc}\left(\mu(A_1)+\mu(A_2)-\frac{2\mu(A_1)\mu(A_2)}{\mu(A_1)+\mu(A_2)}\right)=$$

$$=2\sum_{cyc}\mu(A_1)-2\sum_{cyc}\frac{1}{\frac{1}{\mu(A_1)}+\frac{1}{\mu(A_2)}}\overset{CBS}{\le}2(n-2)\pi-2.\frac{n^2}{\sum_{cyc}\left(\frac{1}{\mu(A_1)}+\frac{1}{\mu(A_2)}\right)}=$$

$$=2(n-2)\pi-2.\frac{n^2}{2.\frac{n^2}{(n-3)\pi}}=(n-1)\pi,\ \ as\ desired.$$

860. $A_1A_2\ldots A_n$ **−convexe polygon** $n\ge 3, a_1,a_2,\ldots,a_n$ **−sides,**

$a_1+a_2+\cdots+a_n=2p$. **Prove that:**

$$\left(\sum_{cyc}\frac{A_1A_2}{10A_1+A_2}\right)\left(\sum_{cyc}\frac{a_1a_2}{10a_1+a_2}\right)\le\frac{2p(n-2)\pi}{121}$$

Proposed by Radu Diaconu-Romania

Solution 1 by Tapas Das-India

$$121A_1A_2-(10A_1+A_2)(10A_2+A_1)=$$

$$=121A_1A_2-(100A_1A_2+10A_1^2+10A_2^2+A_1A_2)<0$$

$$Since:$$

$$100A_1A_2+10A_1^2+10A_2^2+A_1A_2\ge 100A_1A_2+A_1A_2+20A_1A_2\ge 121A_1A_2$$

$$\frac{A_1A_2}{10A_1+A_2}<\frac{10A_2+A_1}{121}\ (and\ analogs)$$

$$\sum_{cyc}\frac{A_1A_2}{1-A_1+A_2}\le\sum_{cyc}\frac{10A_2+A_1}{121}=\frac{11(A_1+A_2+\cdots+A_n)}{121}=$$

$$=\frac{1}{11}(A_1+A_2+\cdots+A_n)=\frac{1}{11}\left[2(n-2)\cdot\frac{\pi}{2}\right]=\frac{(n-2)\pi}{11}$$

$$Similarly,\quad \sum_{cyc}\frac{a_1a_2}{10a_1+a_2}\le\sum_{cyc}\frac{10a_2+a_1}{121}=\frac{11(a_1+a_2+\cdots+a_n)}{121}=$$

$$=\frac{1}{11}(a_1+a_2+\cdots+a_n)=\frac{1}{11}\cdot 2p$$

$$Therefore,\quad \left(\sum_{cyc}\frac{A_1A_2}{10A_1+A_2}\right)\left(\sum_{cyc}\frac{a_1a_2}{10a_1+a_2}\right)\le\frac{2p(n-2)\pi}{121}$$

Solution 2 by Marian Dincă-Romania

$$\frac{a_1a_2}{10a_1+a_2}=\frac{1}{\frac{10}{a_2}+\frac{1}{a_1}}\le\frac{1}{(10+1)\cdot\frac{1}{\frac{10a_2+a_1}{10+1}}}=\frac{10a_2+a_1}{11^2}$$

$$use\ weighted\ Jensen's inequality\ for\ convex\ function\ f(x)=\frac{1}{x};x>0$$

$$\frac{10}{a_2}+\frac{1}{a_1}=10f(a_2)+f(a_1)\ge(10+1)f\left(\frac{10a_2+a_1}{10+1}\right)$$

$$we\ obtain:$$

$$\sum_{cyc}\frac{a_1a_2}{10a_1+a_2}\le\sum_{cyc}\frac{10a_2+a_1}{11^2}=\frac{1}{11^2}\cdot 11\sum_{k=1}^{n}a_k=\frac{1}{11}\sum_{k=1}^{n}a_k$$

$$Similarly,$$

$$\sum_{cyc}\frac{A_1A_2}{10A_1+A_2}\le\sum_{cyc}\frac{10A_2+A_1}{11^2}=\frac{1}{11^2}\cdot 11\sum_{k=1}^{n}A_k=\frac{1}{11}\sum_{k=1}^{n}A_k=\frac{\pi(n-2)}{11}$$

861. $A_1A_2\dots A_p$ **−convex polygon,** $p\in\mathbb{N},p\ge 3,a_1,a_2,\dots,a_p$ **−sides,**

$a_1+a_2+\cdots+a_p=2s,m,n,k>0,m+n=k^4$**. Prove that:**

$$\left(\sum_{cyc}\sqrt[4]{na_1^4+ma_2^4}\right)\left(\sum_{cyc}\sqrt[4]{nA_1^4+mA_2^4}\right)\geq 2k^2s(p-2)\pi$$

Proposed by Radu Diaconu-Romania

Solution by Tapas Das-India

$$(m+n)^{\frac{3}{4}}\cdot\sqrt[4]{na_1^4+ma_2^4}\overset{Holder}{\geq} na_1+ma_2$$

$$\sqrt[4]{na_1^4+ma_2^4}\geq\frac{na_1+ma_2}{(m+n)^{\frac{3}{4}}};\ \ (m+n=k^4)$$

$$\sqrt[4]{na_1^4+ma_2^4}\geq\frac{na_1+ma_2}{k^3}$$

$$\sum_{cyc}\sqrt[4]{na_1^4+ma_2^4}\geq\frac{1}{k^3}(m+n)(a_1+a_2+\cdots+a_p)=\frac{1}{k^3}\cdot k^4\cdot 2s=k\cdot 2s;\ \ (1)$$

$$\sum_{cyc}\sqrt[4]{nA_1^4+mA_2^4}\geq\frac{1}{k^3}(m+n)(A_1+A_2+\cdots+A_p)=\frac{1}{k^3}\cdot k^4(p-2)\pi$$

$$=k(p-2)\pi;\ \ (2)$$

From (1) *and* (2), *it follows*:

$$\left(\sum_{cyc}\sqrt[4]{na_1^4+ma_2^4}\right)\left(\sum_{cyc}\sqrt[4]{nA_1^4+mA_2^4}\right)\geq k\cdot 2s\cdot k(p-2)\pi=2k^2s(p-2)\pi$$

862. *In ΔABC holds :*

$$\frac{2}{\pi^2}<\frac{\widehat{A}+\pi}{\widehat{A}^3+\pi^3}+\frac{\widehat{B}+\pi}{\widehat{B}^3+\pi^3}+\frac{\widehat{C}+\pi}{\widehat{C}^3+\pi^3}\leq\frac{27}{7\pi^2}$$

Proposed by Radu Diaconu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let\ f(x)=\frac{x+\pi}{x^3+\pi^3}=\frac{1}{x^2-\pi x+\pi^2},\ \ x\in(0,\pi).$$

$$We\ have:\ f'(x)=\frac{\pi-2x}{(x^2-\pi x+\pi^2)^2}\ \ and\ \ f'(x)=\frac{-6x(\pi-x)}{(x^2-\pi x+\pi^2)^3}\leq 0,$$

$$So\ f\ is\ concave\ on\ (0,\pi)\ and\ by\ Jensen's\ inequality, we\ have:$$

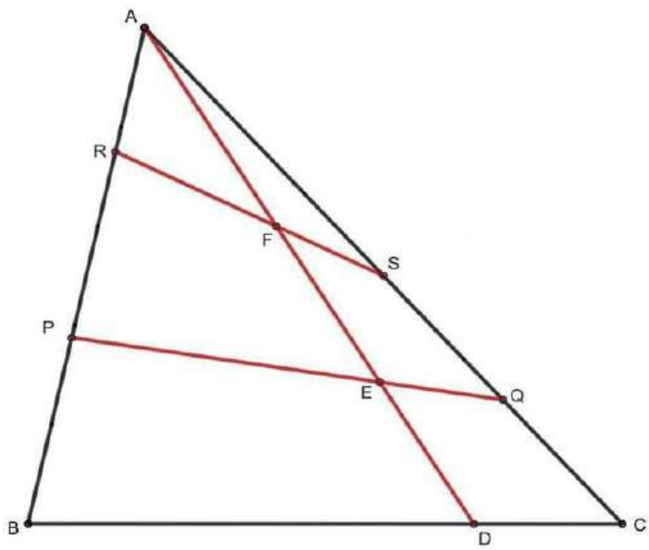
$$\sum_{cyc}\frac{\hat{A}+\pi}{\hat{A}^3+\pi^3}=f(\hat{A})+f(\hat{B})+f(\hat{C})\le 3f\left(\frac{\hat{A}+\hat{B}+\hat{C}}{3}\right)=3f\left(\frac{\pi}{3}\right)=\frac{27}{7\pi^2}.$$

$$Now, we\ have: \sum_{cyc}\frac{\hat{A}+\pi}{\hat{A}^3+\pi^3}=\sum_{cyc}\left(\frac{1}{\pi^2}+\frac{\hat{A}(\pi^2-\hat{A}^2)}{\pi^2(\hat{A}^3+\pi^3)}\right)\overset{\pi>\hat{A}}{>}\sum_{cyc}\frac{1}{\pi^2}=\frac{3}{\pi^2}>\frac{2}{\pi^2}.$$

$$Therefore,\ \frac{2}{\pi^2}<\frac{\hat{A}+\pi}{\hat{A}^3+\pi^3}+\frac{\hat{A}+\pi}{\hat{A}^3+\pi^3}+\frac{\hat{A}+\pi}{\hat{A}^3+\pi^3}\le\frac{27}{7\pi^2}.$$

863. $AB=mAR, AC=nAS, PR=PB, QS=QC$

$\frac{RF}{FS}=x,\frac{PE}{EQ}=y, y=x+1,\frac{BD}{DC}=z, m,n,z-$ **integers. Find:** $(m,n,z)=?$



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

$$\frac{AP}{AB}=\frac{AR+\frac{BR}{2}}{AB}=\frac{AR}{AB}+\frac{AB-AR}{2AB}=\frac{AR+AB}{2AB}=\frac{1+m}{m}$$

$$Similarly,\ \frac{AQ}{AC}=\frac{1+n}{2n}$$

$$\frac{RF}{FS}=\frac{BD}{DC}\cdot\frac{AR}{AB}\cdot\frac{AC}{AS}\Rightarrow x=\frac{zn}{m};\quad (Gakopoulos\ Lemma);\quad (1)$$

$$\frac{PE}{EQ}=\frac{BD}{DC}\cdot\frac{AP}{AB}\cdot\frac{AC}{AQ}\Rightarrow y=z\cdot\frac{1+m}{2m}\cdot\frac{2n}{2+n}\Rightarrow y=\frac{n}{m}\cdot\frac{m+1}{n+1}\cdot z$$

$$\frac{x}{y}=\frac{n+1}{m+1}\Rightarrow\frac{x}{x+1}=\frac{n+1}{m+1}\Rightarrow x=\frac{n+1}{m-n}, m>n;\quad (2)$$

$$From\ (1)\ and\ (2): \frac{n+1}{m-n}=z\cdot\frac{n}{m}\Rightarrow z=\frac{m}{n}\cdot\frac{n+1}{m-n}$$

$$Is\ m,n,z\ integer, m>1, n>1, then:$$

$$(m,n,z)=(4,2,3),(8,2,2),(9,3,2)$$

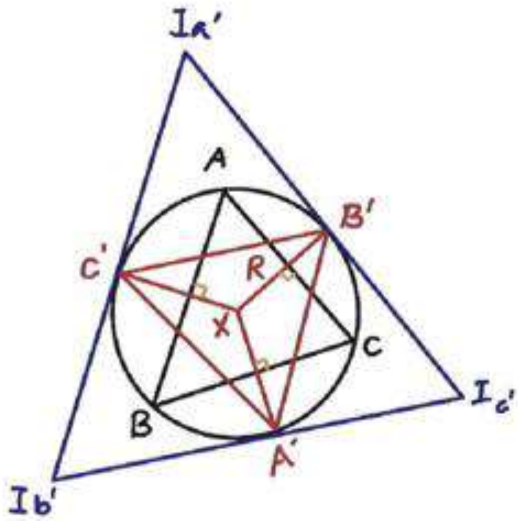
$$If\ m=n, then\ x=z, y=z,(y=x+1).$$

864. If $X\in Int(\Delta ABC), XA'=XB'=XC'=R, R$ –circumradii, $XA'\perp BC$, $XB'\perp CA, XC'\perp AB, a',b',c',I_{a'},I_{b'},I_{c'}$ –sides and excenters in $\Delta A'B'C'$, then:

$$[I_{a'}I_{b'}I_{c'}]=2R^2\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution by Hikmat Mammadov-Azerbaijan



$$a'=BC=2R\sin\frac{\pi-A}{2}=2R\sin A', A'=\frac{\pi}{2}-\frac{A}{2}\Rightarrow\sin A'=\cos\frac{A}{2}$$

$$Similarly, B'=\frac{\pi}{2}-\frac{B}{2}\Rightarrow\sin B'=\cos\frac{B}{2}$$

$$C' = \frac{\pi}{2} - \frac{C}{2} \Rightarrow \sin C' = \cos\frac{C}{2}$$

$$[I_{a'}I_{b'}I_{c'}] = 8R^2\cos\frac{A'}{2}\cos\frac{B'}{2}\cos\frac{C'}{2}, \qquad [I_{a'}I_{b'}I_{c'}] = 2R^2(\sin A' + \sin B' + \sin C')$$

$$Therefore, [I_{a'}I_{b'}I_{c'}] = 2R^2\left(\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}\right)$$

865.

In acute $\Delta ABC, N -$ first Nagel's point, $G -$ first Gergonne's point,

$I -$ incenter, $\Omega -$ centroid. Prove that : $NG \ge I\Omega$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\ I\Omega^2 = \frac{s^2 - 16Rr + 5r^2}{9}\ and\ NG^2 = \frac{16Rs^2(R+r)}{(4R+r)^2} - 16Rr,$$

See : Gergonne's point and outstanding distances – www.ssmrmh.ro.

$$Then:\ NG^2 - I\Omega^2 = \left(\frac{16Rs^2(R+r)}{(4R+r)^2} - 16Rr\right) - \frac{s^2 - 16Rr + 5r^2}{9} =$$

$$= \frac{(128R^2 + 136Rr - r^2)s^2}{9(4R+r)^2} - \frac{128Rr + 5r^2}{9} \ge$$

$$\overset{Gerretsen}{\ge} \frac{(128R^2 + 136Rr - r^2)(16Rr - 5r)}{9(4R+r)^2} - \frac{128Rr + 5r^2}{9} =$$

$$= \frac{48Rr^2(R-2r)}{(4R+r)^2} \overset{Euler}{\ge} 0.\ Hence,\ NG \ge I\Omega.$$

Equality holds iff ΔABC is equilateral.

866. *In $\Delta ABC, P \in (BC), Q \in (CA), R \in (AB),$*

$$3(AR + AQ) = 3(BP + BR) = 3(CQ + CP) = AB + BC + CA.$$

$$Prove\ that:\ PR + RQ + QP \ge 3\sqrt{3}r$$

Proposed by Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let\ 2s := AB + BC + CA.\ We\ have:\ AR + AQ = BP + BR = CQ + CP = \frac{2s}{3}.$$

$$By\ the\ Law\ of\ Cosines\ in\ \Delta BPR, we\ have:$$

$$PR = \sqrt{BP^2 + BR^2 - 2BP.BR.\cos B} = \sqrt{(BP + BR)^2 - 2BP.BR.(1 + \cos B)} =$$

$$= \sqrt{\left(\frac{2s}{3}\right)^2 - 2BP.BR.2\cos^2\frac{B}{2}} \overset{AM-GM}{\underset{\smile}{\geq}} \sqrt{\left(\frac{2s}{3}\right)^2 - (BP + BR)^2.\cos^2\frac{B}{2}} =$$

$$= \frac{2s}{3}.\sqrt{1 - \cos^2\frac{B}{2}} = \frac{2s}{3}.\sin\frac{B}{2} \Rightarrow PR \geq \frac{2s}{3}.\sin\frac{B}{2}\ (and\ analogs)$$

$$Then:\ PR + RQ + QP \geq \frac{2s}{3}.\sum_{cyc}\sin\frac{A}{2} \overset{AM-GM}{\underset{\smile}{\geq}} 2s.\sqrt[3]{\prod_{cyc}\sin\frac{A}{2}} = 2s\sqrt[3]{\frac{r}{4R}} =$$

$$= \sqrt[3]{\frac{r.s.2s^2}{R}} \overset{Mitrinovic\ \&\ Cosnita-Turtoiu}{\underset{\smile}{\geq}} \sqrt[3]{\frac{r.3\sqrt{3}r.27Rr}{R}} = 3\sqrt{3}r,\ as\ desired.$$

$$Equality\ holds\ iff\ P, Q, R\ are\ the\ midpoints\ of\ the\ sides\ of\ equilateral\ \Delta ABC.$$

867. In any acute Δ ABC holds:

$$\frac{54\sqrt{3}}{13\pi} < \left(\sum_{cyc}\frac{1}{\hat{A}^2 + \pi\hat{A} + \pi^2}\right)\left(\sum_{cyc}\sqrt{\hat{A}^2 + \pi\hat{A} + \pi^2}\right) < \frac{12}{\pi}$$

Proposed by Radu Diaconu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\left(\sum_{cyc}\frac{1}{\hat{A}^2 + \pi\hat{A} + \pi^2}\right)\left(\sum_{cyc}\sqrt{\hat{A}^2 + \pi\hat{A} + \pi^2}\right)$$

$$= \left(\sum_{cyc}\frac{1}{\hat{A}^2 + \pi\hat{A} + \pi^2}\right)\left(\sum_{cyc}\sqrt{\hat{A}^2 + \pi\hat{A} + \pi^2}\right)\left(\sum_{cyc}\sqrt{\hat{A}^2 + \pi\hat{A} + \pi^2}\right).\frac{1}{\left(\sum_{cyc}\sqrt{\hat{A}^2 + \pi\hat{A} + \pi^2}\right)}$$

$$\overset{Holder}{\geq} \frac{\left(1^{\frac{1}{3}} + 1^{\frac{1}{3}} + 1^{\frac{1}{3}}\right)^3}{\left(\sum_{cyc}\sqrt{\hat{A}^2 + \pi\hat{A} + \pi^2}\right)} \overset{?}{>} \frac{54\sqrt{3}}{13\pi} \Leftrightarrow \sum_{cyc}\sqrt{\hat{A}^2 + \pi\hat{A} + \pi^2} \underset{(*)}{\overset{?}{\underset{\smile}{<}}} \frac{13\pi}{2\sqrt{3}}$$

$$\because A,B,C<\frac{\pi}{2}\therefore\sum_{cyc}\sqrt{\widehat{A}^2+\pi\widehat{A}+\pi^2}<\sum_{cyc}\sqrt{\widehat{A}\left(\frac{\pi}{2}\right)+\pi\widehat{A}+\pi^2}$$

$$=\sqrt{\pi}.\sum_{cyc}\sqrt{\pi+\frac{3A}{2}}\overset{\text{Jensen}}{\le}3\sqrt{\pi}.\sqrt{\pi+\frac{3}{2}.\frac{\pi}{3}}$$

$$\left(\because f(x)=\sqrt{\pi+\frac{3x}{2}}\ \forall x\in\left(0,\frac{\pi}{2}\right)\text{ is concave as } f''(x)=\frac{-9}{16\left(\pi+\frac{3x}{2}\right)^{\frac{3}{2}}}<0\right)$$

$$\overset{?}{<}\frac{13\pi}{2\sqrt{3}}\Leftrightarrow 9.\frac{3}{2}\overset{?}{<}\frac{169}{12}\Leftrightarrow 162\overset{?}{<}169\to\text{true}\therefore\sum_{cyc}\sqrt{\widehat{A}^2+\pi\widehat{A}+\pi^2}<\frac{13\pi}{2\sqrt{3}}$$

$$\Rightarrow(*)\text{ is true}\Rightarrow\left(\sum_{cyc}\frac{1}{\widehat{A}^2+\pi\widehat{A}+\pi^2}\right)\left(\sum_{cyc}\sqrt{\widehat{A}^2+\pi\widehat{A}+\pi^2}\right)>\frac{54\sqrt{3}}{13\pi}$$

$$Also,\because 0<A,B,C<\frac{\pi}{2}\therefore\left(\sum_{cyc}\frac{1}{\widehat{A}^2+\pi\widehat{A}+\pi^2}\right)\left(\sum_{cyc}\sqrt{\widehat{A}^2+\pi\widehat{A}+\pi^2}\right)$$

$$<\left(\sum_{cyc}\frac{1}{\pi^2}\right)\left(\sum_{cyc}\sqrt{\frac{\pi^2}{4}+\frac{\pi^2}{2}+\pi^2}\right)=\frac{3\pi}{\pi^2}.\frac{3\sqrt{7}}{2}\overset{?}{<}\frac{12}{\pi}\Leftrightarrow 3\sqrt{7}\overset{?}{<}8\Leftrightarrow 63\overset{?}{<}64$$

$$\to\text{true}\Rightarrow\left(\sum_{cyc}\frac{1}{\widehat{A}^2+\pi\widehat{A}+\pi^2}\right)\left(\sum_{cyc}\sqrt{\widehat{A}^2+\pi\widehat{A}+\pi^2}\right)<\frac{12}{\pi}\therefore\text{in any acute }\Delta\,ABC,$$

$$\frac{54\sqrt{3}}{13\pi}<\left(\sum_{cyc}\frac{1}{\widehat{A}^2+\pi\widehat{A}+\pi^2}\right)\left(\sum_{cyc}\sqrt{\widehat{A}^2+\pi\widehat{A}+\pi^2}\right)<\frac{12}{\pi}\ \text{(QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let\ f(x)=\frac{1}{x^2+\pi x+\pi^2},\quad x\in\left(0,\frac{\pi}{2}\right).$$

$$We\ have:\ f'(x)=-\frac{2x+\pi}{(x^2+\pi x+\pi^2)^2}\quad and\quad f''(x)=\frac{6x(x+\pi)}{(x^2+\pi x+\pi^2)^3}\ge 0$$

$$So\ f\ is\ convex\ on\ \left(0,\frac{\pi}{2}\right)\ and\ by\ Jensen's\ inequality, we\ have:$$

$$\sum_{cyc}\frac{1}{\widehat{A}^2+\pi\widehat{A}+\pi^2}=f(\widehat{A})+f(\widehat{B})+f(\widehat{C})\ge 3f\left(\frac{\widehat{A}+\widehat{B}+\widehat{C}}{3}\right)=3f\left(\frac{\pi}{3}\right)=\frac{27}{13\pi^2}.$$

And,

$$\sum_{cyc}\sqrt{\hat{A}^2+\pi\hat{A}+\pi^2}=\sum_{cyc}\sqrt{\frac{3(\hat{A}+\pi)^2+(\hat{A}-\pi)^2}{4}}>\sum_{cyc}\frac{\sqrt{3}(\hat{A}+\pi)}{2}=\frac{\sqrt{3}(\pi+3\pi)}{2}=2\sqrt{3}\pi.$$

$$Then:\left(\sum_{cyc}\frac{1}{\hat{A}^2+\pi\hat{A}+\pi^2}\right)\left(\sum_{cyc}\sqrt{\hat{A}^2+\pi\hat{A}+\pi^2}\right)>\frac{27}{13\pi^2}.2\sqrt{3}\pi=\frac{54\sqrt{3}}{13\pi}.$$

$$Now, we\ have:\sum_{cyc}\sqrt{\hat{A}^2+\pi\hat{A}+\pi^2}<\sum_{cyc}\sqrt{(\hat{A}+\pi)^2}=\sum_{cyc}(\hat{A}+\pi)=\pi+3\pi=4\pi.$$

$$And,\ \sum_{cyc}\frac{1}{\hat{A}^2+\pi\hat{A}+\pi^2}<\sum_{cyc}\frac{1}{\pi^2}=\frac{3}{\pi^2}.$$

$$Then:\left(\sum_{cyc}\frac{1}{\hat{A}^2+\pi\hat{A}+\pi^2}\right)\left(\sum_{cyc}\sqrt{\hat{A}^2+\pi\hat{A}+\pi^2}\right)<\frac{3}{\pi^2}.4\pi=\frac{12}{\pi}.$$

868. *In ΔABC the following relationship holds:*

$$\prod_{cyc}(2+a^4)\geq\left(12F+\frac{\sqrt{3}}{2}.\sum_{cyc}(a-b)^2\right)^2$$

Proposed by D.M.Bătinețu-Giurgiu,Florică Anastase-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\ (2+a^4)(2+b^4)=(a^4+1)(1+b^4)+(a^4+b^4)+3\geq$$

$$\overset{CBS}{\geq}(a^2+b^2)^2+\frac{(a^2+b^2)^2}{2}+3=3\left[\frac{(a^2+b^2)^2}{2}+1\right].$$

$$Then:\prod_{cyc}(2+a^4)\geq 3\left[\frac{(a^2+b^2)^2}{2}+1\right](2+c^4)\overset{CBS}{\geq}3(a^2+b^2+c^2)^2=$$

$$=\left(\sqrt{3}\sum_{cyc}ab+\frac{\sqrt{3}}{2}.\sum_{cyc}(a-b)^2\right)^2\overset{Gordon}{\geq}\left(\sqrt{3}.4\sqrt{3}F+\frac{\sqrt{3}}{2}.\sum_{cyc}(a-b)^2\right)^2=$$

$$=\left(12F+\frac{\sqrt{3}}{2}\cdot\sum_{cyc}(a-b)^2\right)^2.\ So\ the\ proof\ is\ completed.$$

$$Equality\ holds\ iff\ a=b=c=1.$$

869. In any acute $\Delta\, ABC$,

$$\frac{\pi}{2}\le\frac{\hat{A}^{n+1}}{\hat{B}^n+\hat{C}^n}+\frac{\hat{B}^{n+1}}{\hat{C}^n+\hat{A}^n}+\frac{\hat{C}^{n+1}}{\hat{A}^n+\hat{B}^n}<\frac{\pi^{n+1}}{2^{n+2}}\cdot\left(\frac{p^2+r^2+4Rr}{2rp}\right)^n\ \forall\, n\in\mathbb{N}-\{0\}$$

Proposed by Radu Diaconu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{\hat{A}^{n+1}}{\hat{B}^n+\hat{C}^n}+\frac{\hat{B}^{n+1}}{\hat{C}^n+\hat{A}^n}+\frac{\hat{C}^{n+1}}{\hat{A}^n+\hat{B}^n}\overset{\text{repeated Chebyshev}}{\le}\sum_{cyc}\frac{\hat{A}.\hat{A}^n}{\frac{1}{2^{n-1}}(\hat{B}+\hat{C})^n}$$

$$=2^{n-1}.\sum_{cyc}\hat{A}.\left(\frac{\hat{A}}{\pi-\hat{A}}\right)^n\overset{\hat{A}<\frac{\pi}{2}\text{ and analogs}}{<}2^{n-1}.\sum_{cyc}\hat{A}.\left(\frac{\frac{\pi}{2}}{\pi-\frac{\pi}{2}}\right)^n=2^{n-1}.\sum_{cyc}\hat{A}$$

$$\Rightarrow\frac{\hat{A}^{n+1}}{\hat{B}^n+\hat{C}^n}+\frac{\hat{B}^{n+1}}{\hat{C}^n+\hat{A}^n}+\frac{\hat{C}^{n+1}}{\hat{A}^n+\hat{B}^n}\overset{(*)}{<}\pi.\frac{2^n}{2}$$

$$\text{Again, }\frac{\pi^{n+1}}{2^{n+2}}\cdot\left(\frac{p^2+r^2+4Rr}{2rp}\right)^n\overset{\text{Gordon}}{\ge}\frac{\pi^{n+1}}{2^{n+2}}\cdot\left(2\sqrt{3}\right)^n=\frac{\pi^{n+1}}{2^{n+2}}.2^n.\left(\sqrt{3}\right)^n\overset{?}{>}\pi.\frac{2^n}{2}$$

$$\Leftrightarrow\left(\frac{\pi\sqrt{3}}{2}\right)^n\overset{?}{>}2\to\text{true}\because\frac{\pi\sqrt{3}}{2}\approx 2.720699>2\text{ and }n\ge 1$$

$$\therefore\forall\, n\in\mathbb{N}-\{0\},\frac{\pi^{n+1}}{2^{n+2}}\cdot\left(\frac{p^2+r^2+4Rr}{2rp}\right)^n>\pi.\frac{2^n}{2}$$

$$\overset{\text{via }(*)}{>}\frac{\hat{A}^{n+1}}{\hat{B}^n+\hat{C}^n}+\frac{\hat{B}^{n+1}}{\hat{C}^n+\hat{A}^n}+\frac{\hat{C}^{n+1}}{\hat{A}^n+\hat{B}^n}$$

$$\Rightarrow\boxed{\frac{\hat{A}^{n+1}}{\hat{B}^n+\hat{C}^n}+\frac{\hat{B}^{n+1}}{\hat{C}^n+\hat{A}^n}+\frac{\hat{C}^{n+1}}{\hat{A}^n+\hat{B}^n}<\frac{\pi^{n+1}}{2^{n+2}}\cdot\left(\frac{p^2+r^2+4Rr}{2rp}\right)^n\ \forall\, n\in\mathbb{N}-\{0\}}$$

$$\text{Now, WLOG we may assume }\hat{A}\ge\hat{B}\ge\hat{C}\text{ and then : }\frac{\hat{A}^n}{\hat{B}^n+\hat{C}^n}\overset{?}{\ge}\frac{\hat{B}^n}{\hat{C}^n+\hat{A}^n}$$

$$\Leftrightarrow\hat{A}^{2n}-\hat{B}^{2n}+\hat{C}^n(\hat{A}^n-\hat{B}^n)\overset{?}{\ge}0\Leftrightarrow(\hat{A}^n-\hat{B}^n)(\hat{A}^n+\hat{B}^n+\hat{C}^n)\overset{?}{\ge}0$$

$$\to\text{true via assumption}\therefore\frac{\hat{A}^n}{\hat{B}^n+\hat{C}^n}\ge\frac{\hat{B}^n}{\hat{C}^n+\hat{A}^n}\text{ and similarly, }\frac{\hat{B}^n}{\hat{C}^n+\hat{A}^n}\ge\frac{\hat{C}^n}{\hat{A}^n+\hat{B}^n}$$

$$\Rightarrow \frac{\hat{A}^n}{\hat{B}^n+\hat{C}^n} \ge \frac{\hat{B}^n}{\hat{C}^n+\hat{A}^n} \ge \frac{\hat{C}^n}{\hat{A}^n+\hat{B}^n} \therefore \frac{\hat{A}^{n+1}}{\hat{B}^n+\hat{C}^n}+\frac{\hat{B}^{n+1}}{\hat{C}^n+\hat{A}^n}+\frac{\hat{C}^{n+1}}{\hat{A}^n+\hat{B}^n}$$

$$= \hat{A}.\frac{\hat{A}^n}{\hat{B}^n+\hat{C}^n}+\hat{B}.\frac{\hat{B}^n}{\hat{C}^n+\hat{A}^n}+\hat{C}.\frac{\hat{C}^n}{\hat{A}^n+\hat{B}^n}$$

$$\overset{\text{Chebyshev}}{\ge} \frac{1}{3}\left(\sum_{\text{cyc}}\hat{A}\right)\left(\sum_{\text{cyc}}\frac{\hat{A}^n}{\hat{B}^n+\hat{C}^n}\right) \overset{\text{Nesbitt}}{\ge} \frac{\pi}{3}.\frac{3}{2}=\frac{\pi}{2}$$

$$\therefore \boxed{\frac{\pi}{2} \le \frac{\hat{A}^{n+1}}{\hat{B}^n+\hat{C}^n}+\frac{\hat{B}^{n+1}}{\hat{C}^n+\hat{A}^n}+\frac{\hat{C}^{n+1}}{\hat{A}^n+\hat{B}^n}\ \forall\, n\in\mathbb{N}-\{0\}} \text{ and } \therefore \text{ in any acute } \Delta\, ABC,$$

$$\frac{\pi}{2} \le \frac{\hat{A}^{n+1}}{\hat{B}^n+\hat{C}^n}+\frac{\hat{B}^{n+1}}{\hat{C}^n+\hat{A}^n}+\frac{\hat{C}^{n+1}}{\hat{A}^n+\hat{B}^n} < \frac{\pi^{n+1}}{2^{n+2}}.\left(\frac{p^2+r^2+4Rr}{2rp}\right)^n$$

$\forall\, n\in\mathbb{N}-\{0\}$, **with equality iff $\Delta\, ABC$ is equilateral (QED)**

Solution 2 by Tapas Das-India

$$\text{In } \Delta ABC,\ \sin A < A.$$

$$\text{In acute triangle } A<\frac{\pi}{2}, B<\frac{\pi}{2}, C<\frac{\pi}{2}.$$

$$\frac{A^{n+1}}{B^n+C^n}+\frac{B^{n+1}}{C^n+A^n}+\frac{C^{n+1}}{A^n+B^n} \le \frac{\left(\frac{\pi}{2}\right)^{n+1}}{B^n+C^n}+\frac{\left(\frac{\pi}{2}\right)^{n+1}}{C^n+A^n}+\frac{\left(\frac{\pi}{2}\right)^{n+1}}{A^n+B^n}<$$

$$<\frac{\pi^{n+1}}{2^{n+1}}\cdot\frac{1}{4}\left[\left(\frac{1}{B^n}+\frac{1}{C^n}\right)+\left(\frac{1}{C^n}+\frac{1}{A^n}\right)+\left(\frac{1}{A^n}+\frac{1}{B^n}\right)\right]=\frac{\pi^{n+1}}{2^{n+1}}\cdot\frac{1}{4}\left(\frac{1}{A^n}+\frac{1}{B^n}+\frac{1}{C^n}\right)<$$

$$<\frac{\pi^{n+1}}{2^{n+1}}\cdot\left(\frac{1}{\sin^n A}+\frac{1}{\sin^n B}+\frac{1}{\sin^n C}\right)=\frac{\pi^{n+1}}{2^{n+2}}\cdot\frac{\sum(\sin A\sin B)^n}{(\sum \sin A)^n}<$$

$$<\frac{\pi^{n+1}}{2^{n+2}}\cdot\left(\frac{s^2+r^2+4Rr}{4R^2}\cdot\frac{2R^2}{Rs}\right)^n=\frac{\pi^{n+1}}{2^{n+2}}\cdot\left(\frac{s^2+r^2+4Rr}{2rs}\right)^n$$

Using Chebyshev's and Nesbitt's inequalities:

$$\frac{A^{n+1}}{B^n+C^n}+\frac{B^{n+1}}{C^n+A^n}+\frac{C^{n+1}}{A^n+B^n}=A\cdot\frac{A^n}{B^n+C^n}+B\cdot\frac{B^n}{C^n+A^n}+C\cdot\frac{C^n}{A^n+B^n}\ge$$

$$\ge\frac{1}{3}\left(\sum_{cyc}A\right)\left(\sum_{cyc}\frac{A^n}{B^n+C^n}\right)=\frac{\pi}{3}\cdot\frac{3}{2}=\frac{\pi}{2}$$

870. *In ΔABC the following relationship holds*:

$$1+\frac{1}{s^2}\left(\sum_{cyc}n_a n_b\right)^{\frac{1}{2}}\left(\sum_{cyc}m_a m_b\right)^{\frac{1}{2}}\le\frac{R}{r}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$(*):\ 1+\frac{1}{s^2}\left(\sum_{cyc} n_a n_b\right)^{\frac{1}{2}}\left(\sum_{cyc} m_a m_b\right)^{\frac{1}{2}} \le \frac{R}{r}$$

$$We\ have:\ {n_a}^2 = s(s-a)+\frac{s(b-c)^2}{a} = s^2-\frac{s\left[a^2-(b-c)^2\right]}{a} =$$

$$= s^2-\frac{s.4(s-b)(s-c)}{a} = s^2-\frac{4s.sr^2}{a(s-a)} = s^2-2r_a h_a\ \ (i)$$

$$Also,\ 2r_a(n_a+h_a) \overset{AM-GM}{\leq} {r_a}^2+{n_a}^2+2r_a h_a \overset{(i)}{=} {r_a}^2+s^2 = s^2\left(\tan^2\frac{A}{2}+1\right) =$$

$$= \frac{s^2}{\cos^2\frac{A}{2}} = \frac{s.bc}{s-a} = \frac{r_a}{r}.2Rh_a.\ \Rightarrow\ n_a+h_a \le \frac{Rh_a}{r}\ \ or\ \ n_a \le \left(\frac{R}{r}-1\right)h_a\ \ (and\ analogs)$$

$$Then:\ \sum_{cyc} n_a n_b \le \left(\frac{R}{r}-1\right)^2.\sum_{cyc} h_a h_b = \left(\frac{R}{r}-1\right)^2.\frac{2s^2r}{R}\ \ (1)$$

$$Now,\ \sum_{cyc} m_a m_b \overset{Panaitopol}{\leq} \sum_{cyc}\frac{Rh_a}{2r}.\frac{Rh_b}{2r} = \left(\frac{R}{2r}\right)^2.\sum_{cyc} h_a h_b = \left(\frac{R}{2r}\right)^2.\frac{2s^2r}{R} = \frac{Rs^2}{2r}\ \ (2)$$

$$Therefore,\ LHS_{(*)} \overset{(1)\,\&\,(2)}{\leq} 1+\frac{1}{s^2}.\sqrt{\left(\frac{R}{r}-1\right)^2.\frac{2s^2r}{R}.\frac{Rs^2}{2r}} = 1+\frac{1}{s^2}.\left(\frac{R}{r}-1\right)s^2 = \frac{R}{r}.$$

So the proof is complete. Equality holds iff ΔABC is equilateral.

871. If $M \in Int(\Delta ABC), d_a = d(M,BC), d_b = d(M,CA), d_c = d(M,AB)$ and g_a, g_b, g_c –Gergonne's cevians, then holds:

$$\frac{g_a}{d_a}+\frac{g_b}{d_b}+\frac{g_c}{d_c} \ge 9$$

Proposed by D.M. Bătinețu-Giurgiu, Florică Anastase-Romania

Solution by Tapas Das-India

$$Let\ MD = d_a, MF = d_b, ME = d_c,\qquad [ABC] = [AMB]+[BMC]+[CMA]$$

$$[ABC]=\frac{1}{2}a\cdot d_a+\frac{1}{2}b\cdot d_b+\frac{1}{2}c\cdot d_c$$

$$F=\frac{1}{2}(a\cdot d_a+b\cdot d_b+c\cdot d_c)$$

$$2F=a\cdot d_a+b\cdot d_b+c\cdot d_c;\quad (1)$$

$$Now,\qquad g_a\ge h_a, g_b\ge h_b, g_c\ge h_c, thus$$

$$\frac{g_a}{d_a}+\frac{g_b}{d_b}+\frac{g_c}{d_c}\ge\frac{h_a}{d_a}+\frac{h_b}{d_b}+\frac{h_c}{d_c}=\frac{2F}{ad_a}+\frac{2F}{bd_b}+\frac{2F}{cd_c}=$$

$$=2F\left(\frac{1}{ad_a}+\frac{1}{bd_b}+\frac{1}{cd_c}\right)\ge 2F\cdot\frac{(1+1+1)^2}{ad_a+bd_b+cd_c}=2F\cdot\frac{9}{2F}=9$$

872. In $a$ny Δ ABC holds:

$$\left(2(a^3+b^3+c^3)+\frac{1}{9}\right)\left(\sum_{cyc}\frac{1}{\left(1+\sqrt{\hat{A}}\right)^2}\right)\ge\frac{9r(a^2+b^2+c^2)}{(\pi+3)R}$$

Proposed by Radu Diaconu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$1\overset{Euler}{\ge}\frac{8r^3}{R^3}\Rightarrow 2(a^3+b^3+c^3)+\frac{1}{9}\ge\sum_{cyc}a^3+\sum_{cyc}a^3+\frac{8r^3}{9R^3}$$

$$\overset{A-G}{\ge}3.\sqrt[3]{\left(\sum_{cyc}a^3\right)^2.\frac{8r^3}{9R^3}}\overset{?}{\ge}\frac{2r\left(\sum_{cyc}a^2\right)}{R}\Leftrightarrow 27\left(\sum_{cyc}a^3\right)^2\overset{?}{\ge}9\left(\sum_{cyc}a^2\right)^3$$

$$\Leftrightarrow\sqrt{3}.\left(\sum_{cyc}a^3\right)^{\frac{1}{3}}\overset{?}{\ge}\sqrt[3]{3}.\left(\sum_{cyc}a^2\right)^{\frac{1}{2}}\Leftrightarrow\left(\frac{\sum_{cyc}a^3}{3}\right)^{\frac{1}{3}}\overset{?}{\ge}\left(\frac{\sum_{cyc}a^2}{3}\right)^{\frac{1}{2}}$$

$$\to \text{true via Power} - \text{Mean ineq}a\text{lity} \therefore 2(a^3+b^3+c^3)+\frac{1}{9}\overset{(*)}{\ge}\frac{2r(a^2+b^2+c^2)}{R}$$

$$\text{Again,}\sum_{cyc}\frac{1}{\left(1+\sqrt{\hat{A}}\right)^2}=\sum_{cyc}\frac{1^3}{\left(1+\sqrt{\hat{A}}\right)^2}\overset{Radon}{\ge}\frac{27}{\left(3+\sum_{cyc}\sqrt{\hat{A}}\right)^2}$$

$$\overset{CBS}{\ge}\frac{27}{\left(3+\sqrt{3}.\sqrt{\sum_{cyc}\hat{A}}\right)^2}=\frac{27}{3\left(\sqrt{3}+\sqrt{\pi}\right)^2}\overset{CBS}{\ge}\frac{9}{\left(\sqrt{2}.\sqrt{3+\pi}\right)^2}$$

$$\therefore \sum_{cyc} \frac{1}{\left(1+\sqrt{\hat{A}}\right)^2} \overset{(**)}{\geq} \frac{9}{2(\pi+3)}$$

$$\therefore (*)\bullet(**) \Rightarrow \left(2(a^3+b^3+c^3)+\frac{1}{9}\right)\left(\sum_{cyc} \frac{1}{\left(1+\sqrt{\hat{A}}\right)^2}\right) \geq \frac{9r(a^2+b^2+c^2)}{(\pi+3)R} \text{ (QED)}$$

Solution 2 by Tapas Das-India

$$\frac{a^3+b^3+c^3}{3} \geq \left(\frac{a+b+c}{3}\right)^3 \Rightarrow a^3+b^3+c^3 \geq \frac{(a+b+c)^3}{9}$$

$$2(a^3+b^3+c^3)+\frac{1}{9}. = \frac{2}{9}(a+b+c)^3+\frac{1}{9}$$

$$= \frac{1}{9}(a+b+c)^3+\frac{1}{9}(a+b+c)^3+\frac{1}{9} \overset{AM-GM}{\geq}$$

$$\geq 3\sqrt[3]{\frac{1}{9}(a+b+c)^3 \cdot \frac{1}{9}(a+b+c)^3 \cdot \frac{1}{9}} = \frac{3}{9}(a+b+c)^2 = \frac{1}{3}(a+b+c)^2; \quad (1)$$

$$\textit{Now, we need to show:}$$

$$\frac{1}{6}(a+b+c)^2 \geq \frac{r}{R}(a^2+b^2+c^2), \qquad R(a+b+c)^2 \geq 6r(a^2+b^2+c^2)$$

$$R \cdot 4s^2 \geq 6r(a^2+b^2+c^2), \qquad 4R \cdot s^2 \geq 6r(s^2-r^2-4Rr)$$

$$2Rs^2 \geq 6r(s^2-r^2-4Rr)$$

$$\textit{By Gerretsen's inequality: } 2R(16Rr-5r^2) \geq 6r(4R^2+4Rr+3r^2-r^2-4Rr)$$

$$16R^2-12R^2r-5Rr^2-6r^3 \geq 0$$

$$(R-2r)(4R+3r) \geq 0 \textit{ true from } R \geq 2r \textit{ (Euler)}.$$

$$\textit{So,} \qquad 2(a^3+b^3+c^3)+\frac{1}{9} \geq \frac{r}{R}(a^2+b^2+c^2); \quad (2)$$

$$2(a^3+b^3+c^3)+\frac{1}{9} \geq \frac{1}{3}(a+b+c)^2 \geq \frac{1}{6}(a+b+c)^2 \geq \frac{r}{R}(a^2+b^2+c^2)$$

$$\left(1+\sqrt{A}\right)^2 \leq (1+1)(1+A)$$

$$\left(1+\sqrt{A}\right)^2 \leq 2(1+A) \ \textit{(and analogs)}$$

$$\sum_{cyc} \frac{1}{\left(1+\sqrt{A}\right)^2} \geq \sum_{cyc} \frac{1}{2(1+A)} = \frac{1}{2}\sum_{cyc} \frac{1}{1+A}$$

$$\textit{Let } f(x) = \frac{1}{1+x}, \textit{ then } f'(x) = -\frac{1}{(1+x)^2} \textit{ and } f''(x) = \frac{2}{(1+x)^3} > 0.$$

$$\textit{Using Jensen's inequality, we get:}$$

$$f(A)+f(B)+f(C) \geq 3f\left(\frac{A+B+C}{3}\right)$$

$$\frac{1}{1+A}+\frac{1}{1+B}+\frac{1}{1+C} \geq 3f\left(\frac{\pi}{3}\right)$$

$$\sum_{cyc}\frac{1}{1+A}\geq 3\cdot\frac{1}{1+\frac{\pi}{3}}$$

$$\textit{Now,}\quad \sum_{cyc}\frac{1}{\left(1+\sqrt{A}\right)^2}\geq\sum_{cyc}\frac{1}{2(1+A)}\geq\frac{9}{2(\pi+3)};\ (3)$$

Hence,

$$\left(2\sum_{cyc}a^2+\frac{1}{9}\right)\left(\sum_{cyc}\frac{1}{\left(1+\sqrt{A}\right)^2}\right)\geq\frac{1}{3}\left(\sum_{cyc}a\right)^2\cdot\frac{9}{2(\pi+3)}=\frac{1}{6}\left(\sum_{cyc}a\right)^2\cdot\frac{9}{\pi+3}$$

$$\geq\frac{r}{R}\left(\sum_{cyc}a^2\right)\cdot\frac{9}{\pi+3}=\frac{9r}{(\pi+3)R}\sum_{cyc}a^2$$

873. If $x,y,z\in\left[0,\frac{\pi}{2}\right)$, then in Δ ABC,

$$\frac{a^2.e^x}{2+\sin y+\sin z}+\frac{b^2.e^y}{2+\sin z+\sin x}+\frac{a^2.e^z}{2+\sin x+\sin y}\geq 2\sqrt{3}.F$$

Proposed by D.M.Bătinețu-Giurgiu, Florică Anastase-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$e^x\geq 1+x\geq 1+\sin x\ \forall\, x\in\left[0,\frac{\pi}{2}\right)\ \textit{and analogs}$$

$$\Rightarrow \text{LHS}\geq a^2.\frac{\alpha}{\beta+\gamma}+b^2.\frac{\beta}{\gamma+\alpha}+c^2.\frac{\gamma}{\alpha+\beta}$$

$$(\alpha=1+\sin x,\beta=1+\sin y,\gamma=1+\sin z\ \textit{and}\ \alpha,\beta,\gamma\geq 1>0)$$

$$\overset{\text{Oppenheim}}{\geq}\ 4F.\sqrt{\frac{\alpha}{\beta+\gamma}.\frac{\beta}{\gamma+\alpha}+\frac{\beta}{\gamma+\alpha}.\frac{\gamma}{\alpha+\beta}+\frac{\gamma}{\alpha+\beta}.\frac{\alpha}{\beta+\gamma}}\overset{?}{\geq}2\sqrt{3}.F$$

$$\Leftrightarrow 4\sum_{cyc}\left(\frac{\alpha}{\beta+\gamma}.\frac{\beta}{\gamma+\alpha}\right)\overset{?}{\geq}3\Leftrightarrow\frac{4\sum_{cyc}\alpha\beta(\alpha+\beta)}{(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)}\overset{?}{\geq}3$$

$$\Leftrightarrow 4\sum_{cyc}\alpha^2\beta+4\sum_{cyc}\alpha\beta^2\overset{?}{\geq}3\left(\sum_{cyc}\alpha^2\beta+\sum_{cyc}\alpha\beta^2+2\alpha\beta\gamma\right)$$

$$\Leftrightarrow\sum_{cyc}\alpha^2\beta+\sum_{cyc}\alpha\beta^2\overset{?}{\geq}6\alpha\beta\gamma\Leftrightarrow\sum_{cyc}\alpha(\beta-\gamma)^2\overset{?}{\geq}0\to\text{true}$$

$$\therefore\ \text{if } x,y,z\in\left[0,\frac{\pi}{2}\right),\text{then in }\Delta\text{ ABC},$$

$$\frac{a^2.e^x}{2+\sin y+\sin z}+\frac{b^2.e^y}{2+\sin z+\sin x}+\frac{a^2.e^z}{2+\sin x+\sin y}\geq 2\sqrt{3}.F,$$

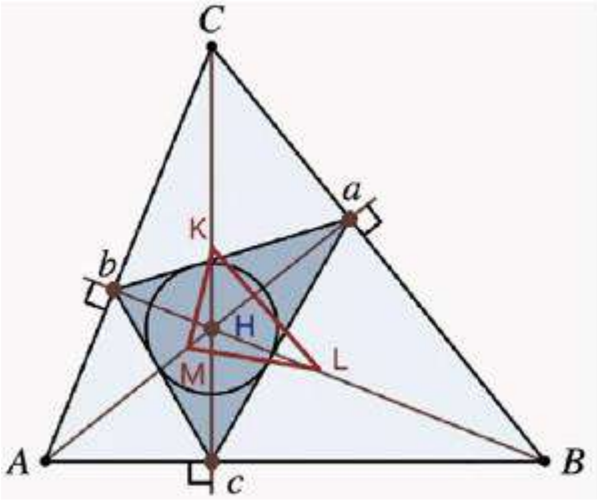
with equality iff $x=y=z=0$ (QED)

874. Let $\Delta I_a I_b I_c$ –be the excentral triangle of ΔABC and M, L, K the midpoints of AI_a, BI_b, CI_c. Prove that:

$$[ABC] = 4[MLK],\ \ AB + BC + CA \leq 2(ML + LK + KM)$$

Proposed by Mehmet Şahin, Alican Gullu-Turkiye

Solution by Hikmat Mammadov-Azerbaijan



$$M - midpoint\ of\ Aa;\ L - midpoint\ of\ Bb,\ K - midpoint\ of\ Cc$$

$$AH = \frac{Ac}{\sin B} = \frac{bc\cos A}{\sin B} = \frac{2R\sin B\cos A}{\sin B} = 2R\cos A$$

$$Similarly,\qquad BH = 2R\cos B\ \ and\ CH = 2R\cos C$$

$$HM = AH - \frac{AH + Ha}{2} = \frac{1}{2}(AH - Ha) = R(\cos A - \cos B\cos C)$$

$$HL = BH - \frac{BH - Hb}{2} = \frac{1}{2}(BH - Hb) = R(\cos B - \cos C\cos A)$$

$$HK = CH - \frac{CH - Hc}{2} = \frac{1}{2}(CH - Hc) = R(\cos C - \cos A\cos B)$$

$$2\Delta_{abc} = HM \cdot HL\sin C + HL \cdot HK\sin A + HK \cdot HM\sin B =$$

$$= 4R^2\left(\sum_{cyc}\cos A\cos B\cos^2 C\right) = 4R^2\prod_{cyc}\cos A \cdot \sum_{cyc}\sin 2A =$$

$$= 4R^2\prod_{cyc}\sin A\cos A = \frac{R^2}{2}\prod_{cyc}\sin 2A$$

$$2\Delta_{MLK} = HM \cdot HL\sin C + HL \cdot HK\sin A + HK \cdot HM\sin B$$

$$\frac{2}{R^2}\Delta_{MLK}=\sum_{cyc}(\cos A-\cos B\cos C)(\cos B-\cos C\cos A)\sin C=$$

$$=\prod_{cyc}\cos A\cdot\sum_{cyc}\tan A-\frac{1}{2}\sum_{cyc}\sin 2A-\frac{1}{4}\sum_{cyc}(\cos 2A+\cos 2B)\sin 2C+$$

$$+\frac{1}{2}\prod_{cyc}\cos A\cdot\sum_{cyc}\sin 2A=$$

$$=\prod_{cyc}\cos A\cdot\sum_{cyc}\tan A-\frac{1}{2}\sum_{cyc}\sin 2A-\frac{1}{4}\sum_{cyc}\sin 2(A+B)+\frac{1}{2}\prod_{cyc}\cos A\cdot\sum_{cyc}\sin 2A=$$

$$=\prod_{cyc}\cos A\cdot\prod_{cyc}\tan A-\frac{1}{2}\sum_{cyc}\sin 2A+\frac{1}{4}\sum_{cyc}\sin 2A+\frac{1}{4}\prod_{cyc}\cos A\cdot\sum_{cyc}\sin 2A=$$

$$=\sin A\sin B\sin C-\sin A\sin B\sin C+2\cos A\cos B\cos C\sin A\sin B\sin C=$$

$$=\frac{1}{4}\sin 2A\sin 2B\sin 2C,\qquad \Delta MLK=\frac{R^2}{8}\sin 2A\sin 2B\sin 2C$$

$$Therefore,\qquad \Delta_{abc}=4\Delta_{MLK}.$$

$$Note:\ \frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

$$\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$$

$$\tan A+\tan B+\tan C=\tan A\tan B\tan C$$

$$Ca=b\cos C=2R\cos B\cos C$$

$$Cb=a\cos A\cos C$$

$$(ab)^2=(2R\cos B\cos C)^2+(2R\cos A\cos C)^2+2\cdot 2R\cos B\cos C\cdot 2R\cos A\cos C\cdot\cos C$$

$$\frac{(ab)^2}{4R^2}=\left(\sum_{cyc}\cos^2 A+2\prod_{cyc}\cos A-\cos^2 C\right)\cos^2 C$$

$$=\left(3-\sum_{cyc}\sin^2 A+2\prod_{cyc}\cos A-\cos^2 C\right)\cos^2 C=\sin^2 C\cos^2 C$$

$$ab=R\sin 2C$$

$$ab+bc+ca=R\sum_{cyc}\sin 2A=4R\prod_{cyc}\sin A$$

$$KL^2=R^2(\cos B-\cos C\cos A)^2+R^2(\cos C-\cos A\cos B)^2+$$

$$+2R^2(\cos B-\cos C\cos A)(\cos C-\cos A\cos B)\cos A$$

$$\frac{KL^2}{R^2}=(\cos B-\cos C\cos A)^2+(\cos C-\cos A\cos B)^2+$$

$$+2(\cos B-\cos C\cos A)(\cos C-\cos A\cos B)\cos A=$$

$$=\sin^2 A\,(\cos^2 B+\cos^2 C-2\cos A\cos B\cos C)=$$

$$=\sin^2 A\,(3-\sin^2 A-\sin^2 B-\sin^2 C-2\cos A\cos B\cos C-\cos^2 A)=$$

$$=\sin^2 A\,(1-\cos^2 A)$$

$$\frac{KL^2}{R^2}=\sin^4 A\Rightarrow KL=R\sin^2 A$$

$$KL+LM+KM=R(\sin^2 A+\sin^2 B+\sin^2 C)$$

$$2(KL+LM+KM)=2R(\sin^2 A+\sin^2 B+\sin^2 C)\ge$$

$$\ge 2R\cdot 3(\sin A\sin B\sin C)^{\frac{2}{3}}>6R\sin A\sin B\sin C>4R\sin A\sin B\sin C$$

$$\textit{Therefore,}\qquad 2(KL+LM+KM)>ab+bc+ca$$

875. If $m,n>0$ then in ΔABC holds:

$$\frac{a^3}{ma+n\sqrt{bc}}+\frac{b^3}{mb+n\sqrt{ca}}+\frac{c^3}{mc+n\sqrt{ab}}\ge\frac{4\sqrt{3}}{m+n}\cdot F$$

Proposed by D.M. Bătinețu-Giurgiu, Florică Anastase-Romania

Solution 1 by Tapas Das-India

$$\frac{a^3}{ma+n\sqrt{bc}}+\frac{b^3}{mb+n\sqrt{ca}}+\frac{c^3}{mc+n\sqrt{ab}}\overset{Holder}{\ge}$$

$$\ge\frac{(a+b+c)^3}{3\left[m(a+b+c)+n\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)\right]}\overset{AM-GM}{\ge}$$

$$\ge\frac{(a+b+c)^3}{3\left[m(a+b+c)+n\left(\frac{a+b}{2}+\frac{b+c}{2}+\frac{c+a}{2}\right)\right]}=$$

$$=\frac{(a+b+c)^3}{3(m+n)(a+b+c)}=\frac{(a+b+c)^2}{3(m+n)}\overset{AM-GM}{\ge}$$

$$\ge\frac{3^2[(abc)^2]^{\frac{1}{3}}}{3(m+n)}\ge\frac{3}{m+n}\left[\left(\frac{4F}{\sqrt{3}}\right)^3\right]^{\frac{1}{3}}=\frac{3}{m+n}\cdot\frac{4F}{\sqrt{3}}=\frac{4\sqrt{3}}{m+n}\cdot F$$

Solution 2 by Adrian Popa-Romania

$$\frac{a^3}{ma+n\sqrt{bc}}+\frac{b^3}{mb+n\sqrt{ca}}+\frac{c^3}{mc+n\sqrt{ab}}\overset{Holder}{\geq}$$

$$\geq\frac{(a+b+c)^3}{3\left[m(a+b+c)+n\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)\right]}\overset{AM-GM}{\geq}$$

$$\geq\frac{(a+b+c)^3}{3\left[m(a+b+c)+n\left(\frac{a+b}{2}+\frac{b+c}{2}+\frac{c+a}{2}\right)\right]}=$$

$$=\frac{(a+b+c)^3}{3(m+n)(a+b+c)}=\frac{(a+b+c)^2}{3(m+n)}\overset{(?)}{\geq}\frac{4\sqrt{3}}{m+n}\cdot F\Leftrightarrow(a+b+c)^2\geq 4\cdot 3\sqrt{3}F$$

$$(a+b+c)^2=a^2+b^2+c^2+2ab+2bc+2ca>$$

$$>ab+bc+ca+2ab+2bc+2ca=3(ab+bc+ca)\geq 3\cdot 4\sqrt{3}F\ true.$$

$$\because ab+bc+ca=abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\overset{Bergstrom}{\geq}abc\cdot\frac{(1+1+1)^2}{a+b+c}=$$

$$=\frac{9abc}{2s}=\frac{9\cdot 4RF}{2s}=4F\cdot 9\cdot\frac{R}{2s}\overset{Mitrinovic}{\geq}4F\cdot 9\cdot\frac{R}{2\cdot\frac{3\sqrt{3}}{2}R}=4F\cdot\frac{3}{\sqrt{3}}=4\sqrt{3}F$$

876. *In ΔABC the following relationship holds:*

$$\left(\sum_{cyc}\frac{7\hat{A}^3+3\hat{A}^2\hat{B}}{\hat{A}+\hat{B}}\right)\left(\sum_{cyc}\frac{\hat{A}^2-3\hat{A}+4+\hat{B}}{\hat{A}\hat{B}}\right)>\frac{810}{\pi}\left(\frac{r}{R}\right)^2$$

Proposed by Radu Diaconu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\sum_{cyc}\frac{\hat{A}^2-3\hat{A}+4+\hat{B}}{\hat{A}\hat{B}}=\sum_{cyc}\frac{\left(\hat{A}-2\right)^2+\hat{A}+\hat{B}}{\hat{A}\hat{B}}\geq\sum_{cyc}\frac{\hat{A}+\hat{B}}{\hat{A}\hat{B}}=$$

$$=\sum_{cyc}\left(\frac{1}{\hat{A}}+\frac{1}{\hat{B}}\right)=2\sum_{cyc}\frac{1}{\hat{A}}\overset{CBS}{\geq}2.\frac{3^2}{\hat{A}+\hat{B}+\hat{C}}=\frac{18}{\pi}.$$

$$Also,we\ have:\sum_{cyc}\frac{7\hat{A}^3+3\hat{A}^2\hat{B}}{\hat{A}+\hat{B}}>7\sum_{cyc}\frac{\hat{A}^3}{\hat{A}+\hat{B}}\overset{Hölder}{\geq}\frac{7\left(\hat{A}+\hat{B}+\hat{C}\right)^3}{3.2\left(\hat{A}+\hat{B}+\hat{C}\right)}=\frac{7\pi^2}{6}>\frac{45}{4}.$$

$$Then:\left(\sum_{cyc}\frac{7\hat{A}^3+3\hat{A}^2\hat{B}}{\hat{A}+\hat{B}}\right)\left(\sum_{cyc}\frac{\hat{A}^2-3\hat{A}+4+\hat{B}}{\hat{A}\hat{B}}\right)>\frac{45}{4}\cdot\frac{18}{\pi}\overset{Euler}{\geq}\frac{810}{4\pi}\cdot\left(\frac{2r}{R}\right)^2=\frac{810}{\pi}\left(\frac{r}{R}\right)^2.$$

Solution 2 by Tapas Das-India

$$\sum_{cyc}\frac{7A^3+3A^2B}{A+B}=\sum_{cyc}\frac{A^2(7A+3B)}{A+B}\overset{Chebyshev}{\geq}\frac{1}{3}\sum_{cyc}\frac{A^2}{A+B}\cdot\sum_{cyc}(7A+3B)\geq$$

$$\overset{Chebyshev}{\geq}\frac{1}{3}\cdot\frac{1}{3}(A+B+C)\cdot\sum_{cyc}\frac{A}{A+B}\cdot\sum_{cyc}(7A+3B)\overset{Nesbitt}{\geq}\frac{1}{9}\cdot\pi\cdot\frac{3}{2}\cdot 10\sum_{cyc}A=$$

$$=\frac{5\pi^2}{3}>\frac{45}{4}$$

$$\sum_{cyc}\frac{A^2-3A+4+B}{AB}=\frac{1}{ABC}[(A^2C+B^2A+C^2B)-2(AB+BC+CA)+4(A+B+C)]$$

$$=\frac{1}{ABC}[C(A^2+4)+B(C^2+4)+A(B^2+4)-2(AB+BC+CA)]\overset{AM-GM}{\geq}$$

$$\geq\frac{4(AC+BC+CA)-2(AB+BC+CA)}{ABC}=\frac{2(AB+BC+CA)}{ABC}=$$

$$=2\left(\frac{1}{A}+\frac{1}{B}+\frac{1}{C}\right)\geq\frac{2(1+1+1)^2}{A+B+C}=\frac{18}{\pi}$$

$$\sum_{cyc}\frac{7A^3+3A^2B}{A+B}\cdot\sum_{cyc}\frac{A^2-3A+4+B}{AB}\geq\frac{45}{4}\cdot\frac{18}{\pi}$$

We need to show:

$$\frac{45}{4}\cdot\frac{18}{\pi}\geq\frac{810}{\pi}\left(\frac{r}{R}\right)^2\Leftrightarrow R\geq 2r\ (Euler).$$

877. In ΔABC the following relationship holds:

$$\frac{1}{R^2}\sum_{cyc}(w_a^2+w_bw_c)\leq\frac{27}{2}$$

Proposed by Ertan Yildirim-Izmir-Turkiye

Solution by Daniel Sitaru-Romania

$$w_a=\frac{2}{b+c}\sqrt{bc}\sqrt{s(s-a)}\overset{AM-GM}{\leq}\frac{2}{b+c}\cdot\frac{b+c}{2}\cdot\sqrt{s(s-a)}=\sqrt{s(s-a)}$$

$$\frac{1}{R^2}\sum_{cyc}(w_a^2+w_bw_c)=\frac{1}{R^2}\left(\sum_{cyc}w_a^2+\sum_{cyc}w_bw_c\right)\leq$$

$$\leq\frac{1}{R^2}\left(\sum_{cyc}\left(\sqrt{s(s-a)}\right)^2+\sum_{cyc}\sqrt{s(s-b)}\cdot\sqrt{s(s-c)}\right)=$$

$$=\frac{1}{R^2}\left(\sum_{cyc}s(s-a)+s\sum_{cyc}\sqrt{(s-b)(s-c)}\right)\overset{AM-GM}{\underbrace{\leq}}$$

$$\leq\frac{1}{R^2}\left(s\sum_{cyc}(s-a)+s\sum_{cyc}\frac{s-b+s-c}{2}\right)=$$

$$=\frac{1}{R^2}\left(s(3s-a-b-c)+s\sum_{cyc}\frac{2s-b-c}{2}\right)=$$

$$=\frac{1}{R^2}\left(s(3s-2s)+s\sum_{cyc}\frac{a+b+c-b-c}{2}\right)=$$

$$=\frac{1}{R^2}\left(s^2+s\sum_{cyc}\frac{a}{2}\right)=\frac{1}{R^2}(s^2+s^2)=\frac{2s^2}{R^2}\overset{MITRINOVIC}{\underbrace{\leq}}\frac{2}{R^2}\cdot\left(\frac{3\sqrt{3}}{2}\cdot R\right)^2=\frac{27}{2}$$

Equality holds for $a=b=c$.

878. *In ΔABC the following relationship holds :*

$$\frac{a^2}{(b+c)^2}+\frac{b^2}{(c+a)^2}+\frac{c^2}{(a+b)^2}+\frac{R^2}{r^2}\geq 4+\frac{1}{16}\left(\frac{(b+c)^2}{a^2}+\frac{(c+a)^2}{b^2}+\frac{(a+b)^2}{c^2}\right).$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\ \sum_{cyc}\frac{(b+c)^2}{a^2}=\sum_{cyc}\left(\frac{2s}{a}-1\right)^2=4s^2\sum_{cyc}\frac{1}{a^2}-4s\sum_{cyc}\frac{1}{a}+3\leq$$

$$\overset{Steinig\ \&\ CBS}{\underbrace{\leq}}\ 4s^2.\frac{1}{4r^2}-4s.\frac{3^2}{a+b+c}+3\overset{Gerretsen}{\underbrace{\leq}}\frac{4R^2+4Rr+3r^2}{r^2}-15.$$

$$Then:\ \frac{(b+c)^2}{a^2}+\frac{(c+a)^2}{b^2}+\frac{(a+b)^2}{c^2}\leq 4\left(\frac{R^2}{r^2}+\frac{R}{r}-3\right).$$

$$Also, we\ have:\ \sum_{cyc}\frac{a^2}{(b+c)^2}\overset{CBS}{\underbrace{\geq}}\frac{1}{3}\left(\sum_{cyc}\frac{a}{b+c}\right)^2\overset{Nesbitt}{\underbrace{\geq}}\frac{1}{3}\left(\frac{3}{2}\right)^2=\frac{3}{4}.$$

So it suffices to prove that :

$$\frac{3}{4}+\frac{R^2}{r^2}\geq 4+\frac{1}{4}\left(\frac{R^2}{r^2}+\frac{R}{r}-3\right)\Leftrightarrow\left(\frac{R}{2r}-1\right)\left(\frac{3R}{2r}+\frac{5}{2}\right)\geq 0,$$

$$\textit{which is true and the proof is done. Equality holds iff } \Delta ABC \textit{ is equilateral.}$$

Solution 2 by Tapas Das-India

$$\sum_{cyc}\frac{(b+c)^2}{a^2}\le\sum_{cyc}\frac{2(b^2+c^2)}{a^2}=2\sum_{cyc}\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)=$$

$$=2\sum_{cyc}\left[\left(\frac{a}{b}+\frac{b}{a}\right)^2-2\frac{a}{b}\cdot\frac{b}{a}\right]\overset{Bandila}{\le}2\left[3\left(\frac{R}{r}\right)^2-6\right]=6\frac{R^2}{r^2}-12$$

$$4+\frac{1}{16}\sum_{cyc}\frac{(b+c)^2}{a^2}\le4+\frac{1}{16}\left(6\frac{R^2}{r^2}-12\right)=\frac{13}{4}+\frac{3}{8}\cdot\frac{R^2}{r^2}$$

$$Now,\sum_{cyc}\frac{a^2}{(b+c)^2}\ge\frac{1}{3}\left(\sum_{cyc}\frac{a}{b+c}\right)^2\ge\frac{1}{3}\left(\frac{3}{2}\right)^2=\frac{3}{4}$$

$$We\ need\ to\ show:$$

$$\sum_{cyc}\frac{a^2}{(b+c)^2}\ge\frac{13}{4}+\frac{3}{8}\cdot\frac{R^2}{r^2}\Leftrightarrow\frac{3}{4}+\frac{R^2}{r^2}\ge\frac{13}{4}+\frac{3}{8}\cdot\frac{R^2}{r^2}\Leftrightarrow$$

$$\frac{5}{8}\cdot\frac{R^2}{r^2}\ge\frac{10}{4}\Leftrightarrow\frac{R^2}{r^2}\ge4\Leftrightarrow R^2\ge4r^2\Leftrightarrow R\ge2r\ (Euler)$$

$$Therefore,$$

$$\sum_{cyc}\frac{a^2}{(b+c)^2}\ge9+\frac{1}{16}\sum_{cyc}\frac{(b+c)^2}{a^2}$$

879. In any $\Delta\ ABC$ the following relationship holds:

$$a\sqrt[3]{\frac{a+2b}{2s}}+b\sqrt[3]{\frac{b+2c}{2s}}+c\sqrt[3]{\frac{c+2a}{2s}}\le3\sqrt{3}R$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Tapas Das-India

$$Let\ f(x)=x^{\frac{1}{3}},x>0,then\ f'(x)=\frac{1}{3}x^{-\frac{2}{3}}\ and\ f''(x)=\frac{1}{3}\cdot\left(-\frac{2}{3}\right)x^{-\frac{5}{3}}<0$$

$$f-is\ concave\ function,use\ Jensen'sinequality:$$

$$af(a+2b)+bf(b+2c)+cf(c+2a)$$
$$\leq (a+b+c)f\left(\frac{a(a+2b)+b(b+2c)+c(c+2a)}{a+b+c}\right)$$
$$a\cdot\sqrt{a+2b}+b\cdot\sqrt[3]{b+2c}+c\cdot\sqrt[3]{c+2a}\leq 2sf\left(\frac{(a+b+c)^2}{a+b+c}\right)=$$
$$=2sf(a+b+c)=2s\cdot\sqrt[3]{2s}$$
$$a\cdot\sqrt{\frac{a+2b}{2s}}+b\cdot\sqrt[3]{\frac{b+2c}{2s}}+c\cdot\sqrt[3]{\frac{c+2a}{2s}}\leq 2s\overset{Mitrinovic}{\leq}2\cdot\frac{3\sqrt{3}}{2}R$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sqrt[3]{\frac{a+2b}{2s}}+b\sqrt[3]{\frac{b+2c}{2s}}+c\sqrt[3]{\frac{c+2a}{2s}}=\sum_{cyc}a\sqrt[3]{\frac{a+2b}{2s}.1.1}$$
$$\overset{A-G}{\leq}\sum_{cyc}a\left(\frac{\frac{a+2b}{2s}+2}{3}\right)=\frac{\sum_{cyc}a^2+2\sum_{cyc}ab}{6s}+\frac{2}{3}.2s=\frac{\left(\sum_{cyc}a\right)^2}{6s}+\frac{4s}{3}=\frac{4s^2}{6s}+\frac{4s}{3}$$
$$=2s\overset{Mitrinovic}{\leq}3\sqrt{3}R$$
$$\therefore a\sqrt[3]{\frac{a+2b}{2s}}+b\sqrt[3]{\frac{b+2c}{2s}}+c\sqrt[3]{\frac{c+2a}{2s}}\leq 3\sqrt{3}R,$$

with equality iff $\Delta\,ABC$ *is equilateral* (QED)

880. **In ΔABC the following relationship holds:**

$$\sqrt[4]{\csc\frac{A}{2}}+\sqrt[4]{\csc\frac{B}{2}}+\sqrt[4]{\csc\frac{C}{2}}\leq 3\cdot\sqrt[4]{\frac{R}{r}}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum_{cyc}\csc\frac{A}{2}=\sum_{cyc}\sqrt{\frac{bc}{(s-b)(s-c)}}\overset{CBS}{\leq}\sqrt{\left(\sum_{cyc}ab\right)\left(\sum_{cyc}\frac{1}{(s-a)(s-b)}\right)}=$$
$$=\sqrt{(s^2+r^2+4Rr)\cdot\frac{s-a+s-b+s-c}{(s-a)(s-b)(s-c)}}=$$
$$=\sqrt{(s^2+r^2+4Rr)\cdot\frac{s}{sr^2}}\overset{Gerretsen}{\leq}\sqrt{(4R^2+4Rr+3r^2+r^2+4Rr)\cdot\frac{s}{sr^2}}\leq$$

$$\overset{Euler}{\leq}\sqrt{\left(4R^2+4R\cdot\frac{R}{2}+3\cdot\frac{R^2}{4}+4R\cdot\frac{R}{2}\right)\cdot\frac{1}{r^2}}=\sqrt{\frac{9R^2}{r^2}}=\frac{3R}{r}$$

$$Now,\ \sqrt[4]{\csc\frac{A}{2}}+\sqrt[4]{\csc\frac{B}{2}}+\sqrt[4]{\csc\frac{C}{2}}\overset{CBS}{\leq}3\cdot\sqrt[4]{\frac{\csc\frac{A}{2}+\csc\frac{B}{2}+\csc\frac{C}{2}}{3}}\leq 3\cdot\sqrt[4]{\frac{3R}{3r}}=3\cdot\sqrt[4]{\frac{R}{r}}$$

881. If $x,y,z>0$ then in ΔABC holds:

$$\frac{x^2a^3}{(y+z)^2}+\frac{y^2b^3}{(z+x)^2}+\frac{z^2c^3}{(x+y)^2}\geq 18\sqrt{3}r^3$$

Proposed by D.M. Bătineţu-Giurgiu, Florică Anastase-Romania

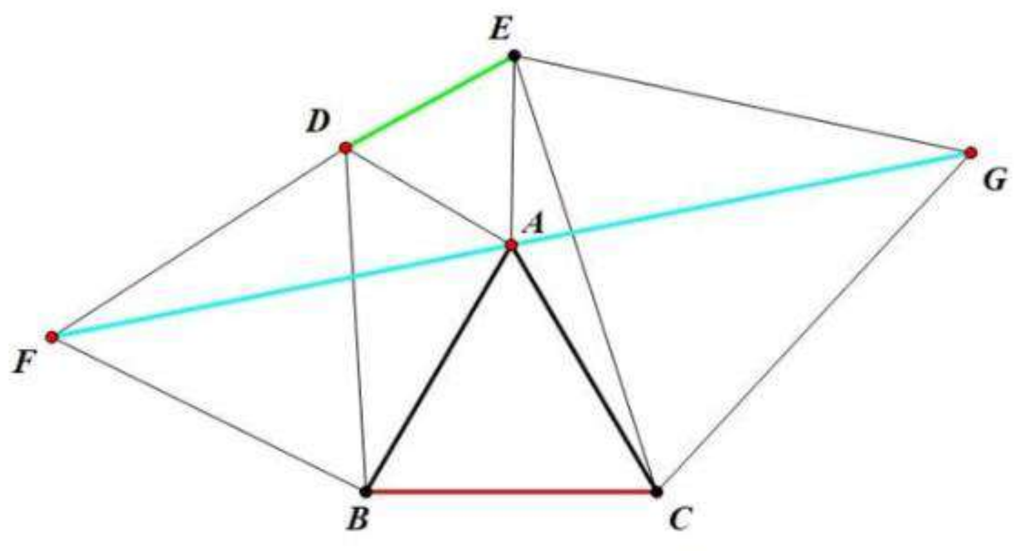
Solution by Tapas Das-India

$$\frac{x^2a^3}{(y+z)^2}+\frac{y^2b^3}{(z+x)^2}+\frac{z^2c^3}{(x+y)^2}=\frac{\left(\frac{x}{y+z}a^2\right)^2}{a}+\frac{\left(\frac{y}{z+x}b^2\right)^2}{b}+\frac{\left(\frac{z}{(x+y)}c^2\right)^2}{c}\geq$$

$$\geq\frac{\left(\frac{x}{y+z}a^2+\frac{y}{z+x}b^2+\frac{z}{x+y}c^2\right)^2}{a+b+c}\overset{Tsintsifas}{\geq}\frac{\left(2\sqrt{3}F\right)^2}{2s}=\frac{12F^2}{2s}=\frac{6F^2}{s}=$$

$$=\frac{6r^2s^2}{s}=6r^2s\overset{Mitrinovic}{\geq}6r^2\cdot 3\sqrt{3}r=18\sqrt{3}r^3$$

882. Given four equilateral triangles. Prove that: $2(BC+DE)\geq FG$



Proposed by Luc Binh-Vietnam

Solution 1 by Rajarshi Chakraborty-India

$$We\ use\ Ptolemy's\ inequality\ for\ quadrilaterals.$$

$$In\ quadrilateral\ AEGC, AE\cdot GC + AC\cdot GE \ge AG\cdot EC;\ \ (i)$$

$$Now\ \Delta EGC\ is\ equilateral, so\ EC = GC = EG;\ \ (ii)$$

$$From\ (i)\ and\ (ii), we\ get\ AE + AC \ge AG;\ \ (a)$$

$$In\ quadrilateral\ DFBA, DF\cdot AB + FB\cdot DA \ge DB\cdot FA;\ \ (iii)$$

$$Now\ \Delta DFB\ is\ quadrilateral, so\ DF = FB = DB;\ \ (iv)$$

$$From\ (iii)\ and\ (iv), we\ get\ AB + AD \ge FA;\ \ (b)$$

$$(a)+(b) \Rightarrow AE + AC + AB + AD \ge AG + FA = FG;\ \ (c)$$

$$So, AE = AD = ED\ and\ AB = BC = CA.$$

$$Thus, from\ (c)\ we\ get\ 2(BC + ED) \ge FG.$$

Solution 2 by Hikmat Mammadov-Azerbaijan

$$Let\ BC = a, DE = b\ and\ BD = c.$$

$$c^2 = a^2 + b^2 - 2ab\cos\left(\pi - \theta - \frac{\pi}{3}\right)$$

$$c^2 = a^2 + b^2 + ab\left(\cos\theta - \sqrt{3}\sin\theta\right)$$

$$\frac{\sin(\measuredangle ABD)}{AD} = \frac{\sin\left(\theta + \frac{\pi}{3}\right)}{BD} \Rightarrow \sin(\measuredangle ABD) = \frac{b}{c}\sin\left(\theta + \frac{\pi}{3}\right)$$

$$AF^2 = a^2 + c^2 - 2ac\cos\left(\measuredangle ABD + \frac{\pi}{3}\right) = a^2 + c^2 - ac\left(\cos(\measuredangle ABD) - \sqrt{3}\sin(\measuredangle ABD)\right)$$

$$= a^2 + c^2 - ac\cdot\cos(\measuredangle ABD) + \sqrt{3}ab\cdot\sin\left(\theta + \frac{\pi}{3}\right) =$$

$$= a^2 + c^2 - ac\cdot\frac{a^2 + c^2 - b^2}{ac} + \sqrt{3}ab\cdot\sin\left(\theta + \frac{\pi}{3}\right) =$$

$$= \frac{a^2 + b^2 + c^2}{2} + \sqrt{3}ab\cdot\sin\left(\theta + \frac{\pi}{3}\right) =$$

$$= \frac{a^2 + b^2 + a^2 + b^2 + ab\left(\cos\theta - \sqrt{3}\sin\theta\right)}{2} + \frac{\sqrt{3}}{2}ab\left(\sin\theta + \sqrt{3}\cos\theta\right)$$

$$AF^2 = a^2 + b^2 + 2ab\cos\theta \Rightarrow AF = \sqrt{a^2 + b^2 + 2ab\cos\theta}$$

$$Replace\ (\theta)\ with\ (-\theta)\ we\ get$$

$$AG = \sqrt{a^2 + b^2 + 2ab\cos\theta}, i.e. AF = AG$$

It can also be shown that F, A, G *are collinear*

$$FG = AF + AG = 2\sqrt{a^2 + b^2 + 2ab\cos\theta} \leq 2\sqrt{a^2 + b^2 + 2ab} = 2(a + b)$$

$$i.e.\, 2(BC + DE) \geq FG$$

883. In $a$ny Δ ABC holds

$$\frac{r^2}{R^2} \leq \frac{2(2a^2 - (b - c)^2)(2b^2 - (c - a)^2)(2c^2 - (a - b)^2)}{(a + b)^2(b + c)^2(c + a)^2}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$a^2 - (b - c)^2 = 4rr_a \text{ and analogs}$$
$$\Rightarrow (2a^2 - (b - c)^2)(2b^2 - (c - a)^2)(2c^2 - (a - b)^2)$$
$$= (a^2 + 4rr_a)(b^2 + 4rr_b)(c^2 + 4rr_c)$$
$$= a^2b^2c^2 + 64r^3.r_ar_br_c + 4r\sum_{cyc} b^2c^2r_a + 16r^2\sum_{cyc} a^2r_br_c$$
$$= 16R^2r^2s^2 + 64r^4s^2 + 4r^2s\sum_{cyc}\frac{b^2c^2}{s - a} + 16r^2s\sum_{cyc} a^2(s - a)$$
$$\overset{\text{Bergstrom}}{\geq} 16R^2r^2s^2 + 64r^4s^2 + 4r^2s.\frac{\left(\sum_{cyc} ab\right)^2}{\sum_{cyc}(s - a)} + 16r^2s\left(s\sum_{cyc} a^2 - \sum_{cyc} a^3\right)$$
$$= 16R^2r^2s^2 + 64r^4s^2 + 4r^2(s^2 + 4Rr + r^2)^2$$
$$+16r^2s^2\left(2(s^2 - 4Rr - r^2) - 2(s^2 - 6Rr - 3r^2)\right)$$
$$\overset{?}{\geq} \frac{(a + b)^2(b + c)^2(c + a)^2r^2}{2R^2} = \frac{2r^2s^2(s^2 + 2Rr + r^2)^2}{R^2}$$
$$\Leftrightarrow R^2\left(8R^2s^2 + 32r^2s^2 + 2(s^2 + 4Rr + r^2)^2 + 16s^2(2Rr + 2r^2)\right)$$
$$\overset{?}{\geq} s^2(s^2 + 2Rr + r^2)^2$$
$$\Leftrightarrow s^6 - (2R^2 - 4Rr - 2r^2)s^4 - (8R^4 + 48R^3r + 64R^2r^2 - 4Rr^3 - r^4)s^2$$
$$-2R^2r^2(4R + r)^2 \underbrace{\overset{?}{\leq}}_{(*)} 0$$

$$\text{Now, LHS of } (*) \overset{\text{Gerretsen}}{\leq} (2R^2 + 8Rr + 5r^2)s^4$$
$$-(8R^4 + 48R^3r + 64R^2r^2 - 4Rr^3 - r^4)s^2 - 2R^2r^2(4R + r)^2 \overset{\text{Gerretsen}}{\leq}$$
$$\left((2R^2 + 8Rr + 5r^2)(4R^2 + 4Rr + 3r^2) - (8R^4 + 48R^3r + 64R^2r^2 - 4Rr^3 - r^4)\right)s^2$$
$$-2R^2r^2(4R + r)^2 \overset{?}{\leq} 0 \Leftrightarrow (4R^3 + 3R^2r - 24Rr^2 - 8r^3)s^2 + R^2r(4R + r)^2 \underbrace{\overset{?}{\geq}}_{(**)} 0$$

$\boxed{\textit{Case } 1}$ $4R^3 + 3R^2r - 24Rr^2 - 8r^3 \geq 0$ *and then* : LHS of $(**) \geq R^2r(4R + r)^2$
$> 0 \Rightarrow (**)$ is true (strict inequ$a$lity)

$$\boxed{Case\ 2}\ 4R^3+3R^2r-24Rr^2-8r^3<0 \text{ and then :}$$

$$\text{LHS of }(**)-\left(-(4R^3+3R^2r-24Rr^2-8r^3)\right)s^2+R^2r(4R+r)^2$$

$$\overset{Gerretsen}{\geq}-\left(-(4R^3+3R^2r-24Rr^2-8r^3)(4R^2+4Rr+3r^2)\right)+R^2r(4R+r)^2\overset{?}{\geq}0$$

$$\Leftrightarrow 8t^5+22t^4-32t^3-59t^2-52t-12\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(8t^4+38t^3+44t^2+29t+6)\overset{?}{\geq}0\to \text{true}\because t\overset{Euler}{\geq}2$$

$$\Rightarrow (**)\text{ is true and combining cases }1,2,(**)\Rightarrow(*)\text{ is true for all triangles}$$

$$\therefore (2a^2-(b-c)^2)(2b^2-(c-a)^2)(2c^2-(a-b)^2)\geq\frac{(a+b)^2(b+c)^2(c+a)^2r^2}{2R^2}$$

$$\Rightarrow \text{in any }\Delta\ ABC,\frac{r^2}{R^2}\leq\frac{2(2a^2-(b-c)^2)(2b^2-(c-a)^2)(2c^2-(a-b)^2)}{(a+b)^2(b+c)^2(c+a)^2},$$

equality iff $\Delta\ ABC$ is equilateral (QED)

884. If $m,n\geq 0,x,y,z>0$ and $\Delta ABC,\Delta A_1B_1C_1$ has areas F,F_1, then:

$$\frac{x+y}{z}\cdot a^mb_1^n+\frac{y+z}{x}\cdot b^mc_1^n+\frac{z+x}{y}\cdot c^ma_1^n\geq 2^{1+m+n}\left(\sqrt[4]{3}\right)^{4-m-n}\left(\sqrt{F}\right)^m\left(\sqrt{F_1}\right)^n$$

Proposed by D.M. Bătineţu-Giurgiu, Florică Anastase-Romania

Solution by Tapas Das-India

$$\frac{x+y}{z}\cdot a^mb_1^n+\frac{y+z}{x}\cdot b^mc_1^n+\frac{z+x}{y}\cdot c^ma_1^n\overset{AM-GM}{\geq}$$

$$\geq\frac{2\sqrt{xy}}{z}\cdot a^mb_1^n+\frac{2\sqrt{yz}}{x}\cdot b^mc_1^n+\frac{2\sqrt{zx}}{y}\cdot c^ma_1^n\overset{AM-GM}{\geq}$$

$$\geq 2\cdot 3\sqrt[3]{\frac{\sqrt{xy}}{z}\cdot a^mb_1^n\cdot\frac{\sqrt{yz}}{x}\cdot b^mc_1^n\cdot\frac{\sqrt{zx}}{y}\cdot c^ma_1^n}=$$

$$=6\sqrt[3]{(abc)^m}\cdot\sqrt[3]{(a_1b_1c_1)^n}\geq 6\left(\frac{4F}{\sqrt{3}}\right)^{\frac{3}{2}\cdot\frac{m}{3}}\cdot\left(\frac{4F_1}{\sqrt{3}}\right)^{\frac{3}{2}\cdot\frac{n}{3}}=$$

$$=6\left(\frac{4F}{\sqrt{3}}\right)^{\frac{m}{2}}\cdot\left(\frac{4F_1}{\sqrt{3}}\right)^{\frac{n}{2}}=2^{1+m+n}\cdot\frac{\left(\sqrt[4]{4}\right)^4}{\left(\sqrt[4]{3}\right)^m\left(\sqrt[4]{3}\right)^n}\cdot\left(\sqrt{F}\right)^m\left(\sqrt{F_1}\right)^n=$$

$$=2^{1+m+n}\left(\sqrt[4]{3}\right)^{4-m-n}\left(\sqrt{F}\right)^m\left(\sqrt{F_1}\right)^n$$

885.

New Euler's inequality refinements in $\Delta ABC,\omega-$ Brocard's angle

$$R\ge 2r.max\left\{\frac{m_a}{h_a},\frac{m_b}{h_b},\frac{m_c}{h_c}\right\}\ge\frac{r}{\sin\omega}\ge r.max\left\{\frac{a}{b}+\frac{b}{a},\frac{b}{c}+\frac{c}{b},\frac{c}{a}+\frac{a}{c}\right\}\ge 2r.$$

$$R\ge 2r.max\left\{\frac{m_a}{h_a},\frac{m_b}{h_b},\frac{m_c}{h_c}\right\}\ge\frac{r}{\sin\omega}\ge r.max\left\{\frac{m_a}{m_b}+\frac{m_b}{m_a},\frac{m_b}{m_c}+\frac{m_c}{m_b},\frac{m_c}{m_a}+\frac{m_a}{m_c}\right\}\ge 2r$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$By\ Panaitopol's\ inequality, we\ have:\ \frac{m_a}{h_a}\le\frac{R}{2r}\ \ (and\ analogs)$$

$$Then:\ \ R\ge 2r.max\left\{\frac{m_a}{h_a},\frac{m_b}{h_b},\frac{m_c}{h_c}\right\}\ \ (1)$$

$$WLOG, we\ may\ assume\ that:\ \ a\ge b\ge c.\ We\ have:$$

$$\left(\frac{m_a}{h_a}\right)^2-\left(\frac{m_b}{h_b}\right)^2=\frac{a^2(2b^2+2c^2-a^2)-b^2(2c^2+2a^2-b^2)}{16F^2}=$$

$$=\frac{(a^2-b^2)(2c^2-a^2-b^2)}{16F^2}\le 0.$$

$$Similarly, we\ have:\ \left(\frac{m_b}{h_b}\right)^2-\left(\frac{m_c}{h_c}\right)^2=\frac{(b^2-c^2)(2a^2-b^2-c^2)}{16F^2}\ge 0.$$

$$Then:\ \ max\left\{\frac{m_a}{h_a},\frac{m_b}{h_b},\frac{m_c}{h_c}\right\}=\frac{m_b}{h_b}.$$

$$Now\ we\ have:\ \frac{2m_b}{h_b}\overset{?}{\ge}\frac{1}{\sin\omega}\Leftrightarrow\frac{b\sqrt{2c^2+2a^2-b^2}}{2F}\ge\frac{\sqrt{a^2b^2+b^2c^2+c^2a^2}}{2F}$$

$$\overset{squaring}{\Leftrightarrow}\ \ a^2b^2+b^2c^2-c^2a^2-b^4\ge 0\Leftrightarrow(a^2-b^2)(b^2-c^2)\ge 0,\ \ which\ is\ true.$$

$$Then:\ 2r.max\left\{\frac{m_a}{h_a},\frac{m_b}{h_b},\frac{m_c}{h_c}\right\}=r.\frac{2m_b}{h_b}\ge\frac{r}{\sin\omega}\ \ (2)$$

$$Now, we\ have:\ \frac{1}{\sin\omega}\overset{?}{\ge}\frac{b}{c}+\frac{c}{b}\Leftrightarrow\frac{\sqrt{a^2b^2+b^2c^2+c^2a^2}}{2F}\ge\frac{b^2+c^2}{bc}$$

$$\Leftrightarrow 2bc\sqrt{a^2b^2+b^2c^2+c^2a^2}\ge(b^2+c^2)\sqrt{2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)}$$

$$\overset{squaring}{\Leftrightarrow} \quad 4b^2c^2(a^2b^2+b^2c^2+c^2a^2) \geq (2b^2c^2+b^4+c^4)[2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)]$$

$$\Leftrightarrow \ 0 \geq -a^4(b^2+c^2)^2+2(b^4+c^4)(a^2b^2+c^2a^2)-(b^4+c^4)^2 = -[a^2(b^2+c^2)-(b^4+c^4)]^2,$$

$$\text{which is true} \ \Rightarrow \ \frac{1}{\sin\omega} \geq \frac{b}{c}+\frac{c}{b} \ (\text{and analogs}) \ (i)$$

$$\text{Then}: \ \frac{r}{\sin\omega} \geq r.max\left\{\frac{a}{b}+\frac{b}{a}, \frac{b}{c}+\frac{c}{b}, \frac{c}{a}+\frac{a}{c}\right\} \ (3)$$

m_a, m_b, m_c – *can be the sides of a triangle. Applying* (i) *in* $\Delta m_a m_b m_c$, *we have* :

$$\frac{m_b}{m_c}+\frac{m_c}{m_b} \leq \frac{1}{\sin\omega_m} = \frac{\sqrt{{m_a}^2{m_b}^2+{m_b}^2{m_c}^2+{m_c}^2{m_a}^2}}{2F_m} = \frac{\sqrt{\frac{9}{16}(a^2b^2+b^2c^2+c^2a^2)}}{\frac{3F}{2}} = \frac{1}{\sin\omega}.$$

$$\text{Then}: \ \frac{r}{\sin\omega} \geq r.max\left\{\frac{m_a}{m_b}+\frac{m_b}{m_a}, \frac{m_b}{m_c}+\frac{m_c}{m_b}, \frac{m_c}{m_a}+\frac{m_a}{m_c}\right\} \ (4)$$

Therefore,

$$R \overset{(1)}{\gtreqqless} 2r.max\left\{\frac{m_a}{h_a}, \frac{m_b}{h_b}, \frac{m_c}{h_c}\right\} \overset{(2)}{\gtreqqless} \frac{r}{\sin\omega} \overset{(3)}{\gtreqqless} r.max\left\{\frac{a}{b}+\frac{b}{a}, \frac{b}{c}+\frac{c}{b}, \frac{c}{a}+\frac{a}{c}\right\} \overset{AM-GM}{\gtreqqless} 2r.$$

$$R \overset{(1)}{\gtreqqless} 2r.max\left\{\frac{m_a}{h_a}, \frac{m_b}{h_b}, \frac{m_c}{h_c}\right\} \overset{(2)}{\gtreqqless} \frac{r}{\sin\omega} \overset{(4)}{\gtreqqless} r.max\left\{\frac{m_a}{m_b}+\frac{m_b}{m_a}, \frac{m_b}{m_c}+\frac{m_c}{m_b}, \frac{m_c}{m_a}+\frac{m_a}{m_c}\right\} \overset{AM-GM}{\gtreqqless} 2r.$$

886. ***In ΔABC the following relationship holds :***

$$4 < \sum_{cyc} \frac{\pi+\mu(A)}{\pi-\mu(B)} \leq 6max\left(\frac{\mu(B)}{\mu(A)}, \frac{\mu(C)}{\mu(B)}, \frac{\mu(A)}{\mu(C)}\right)$$

Proposed by Radu Diaconu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have :

$$\sum_{cyc} \frac{\pi+\mu(A)}{\pi-\mu(B)} = \sum_{cyc}\left(1+\frac{\mu(A)+\mu(B)}{\pi-\mu(B)}\right) = 3+\sum_{cyc}\frac{\mu(A)+\mu(B)}{\mu(C)+\mu(A)} \geq$$

$$\overset{AM-GM}{\geq} 3+3\sqrt[3]{\prod_{cyc}\frac{\mu(A)+\mu(B)}{\mu(C)+\mu(A)}}=3+3=6>4.$$

$$Now, since:\ 3max\left(\frac{\mu(B)}{\mu(A)},\frac{\mu(C)}{\mu(B)},\frac{\mu(A)}{\mu(C)}\right)\geq\frac{\mu(B)}{\mu(A)}+\frac{\mu(C)}{\mu(B)}+\frac{\mu(A)}{\mu(C)},$$

$$so\ it\ suffices\ to\ prove:\ \sum_{cyc}\frac{\pi+\mu(A)}{\pi-\mu(B)}\leq 2\sum_{cyc}\frac{\mu(B)}{\mu(A)}\Leftrightarrow 3+\sum_{cyc}\frac{\mu(A)+\mu(B)}{\mu(C)+\mu(A)}\leq 2\sum_{cyc}\frac{\mu(B)}{\mu(A)}$$

$$\Leftrightarrow 6\leq\sum_{cyc}\left(\frac{2\mu(B)}{\mu(A)}+1-\frac{\mu(A)+\mu(B)}{\mu(C)+\mu(A)}\right)=\sum_{cyc}\frac{\mu(B)[\mu(C)+\mu(A)]+\mu(C)[\mu(A)+\mu(B)]}{\mu(A)[\mu(C)+\mu(A)]},$$

$$which\ is\ true\ because:\ RHS\overset{AM-GM}{\geq}\sum_{cyc}\frac{2\sqrt{\mu(B)\mu(C)[\mu(C)+\mu(A)][\mu(A)+\mu(B)]}}{\mu(A)[\mu(C)+\mu(A)]}=$$

$$=2\sum_{cyc}\sqrt{\frac{\mu(B)\mu(C)[\mu(A)+\mu(B)]}{\mu^2(A)[\mu(C)+\mu(A)]}}\overset{AM-GM}{\geq}2.3\sqrt[6]{\prod_{cyc}\frac{\mu(B)\mu(C)[\mu(A)+\mu(B)]}{\mu^2(A)[\mu(C)+\mu(A)]}}=6=LHS.$$

887. *In $\Delta ABC, I-incenter, \omega-Brocard's\ angle, n_a-Nagel's\ cevian, g_a-Gergonne\ cevian.$ Prove that:*

$$\frac{1}{\sin\omega}\geq\omega_1+\omega_2,\quad where:$$

$$\omega_1=\frac{\sum_{cyc}\left(AI+\frac{m_aw_a}{h_a}\right)}{-3r+\sum_{cyc}(m_a+h_a)},\quad \omega_2=\sqrt{\frac{5R-r+\sum_{cyc}\frac{n_ag_a}{h_a}}{2\sum_{cyc}h_a}}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\ AI=\frac{r}{\sin\frac{A}{2}}=\frac{r.4R\cos\frac{A}{2}}{a}=\frac{bc\cos\frac{A}{2}}{s}=\frac{r.(b+c)w_a}{ah_a}.$$

$$Also,\ \frac{w_a}{h_a}=\frac{2\sqrt{bc.s(s-a)}}{b+c}.\frac{a}{2F}=\frac{\sqrt{2abcs}.\sqrt{a(-a+b+c)}}{2F(b+c)}\overset{AM-GM}{\leq}\frac{\sqrt{8Rs^2r}.(b+c)}{4sr(b+c)}=\sqrt{\frac{R}{2r}}.$$

$$Then:\sum_{cyc}\left(AI+\frac{m_aw_a}{h_a}\right)=\sum_{cyc}\frac{w_a}{h_a}\left(\frac{r(b+c)}{a}+m_a\right)\le\sum_{cyc}\sqrt{\frac{R}{2r}}.(-r+h_a+m_a)=$$

$$=\sqrt{\frac{R}{2r}}\left(-3r+\sum_{cyc}(m_a+h_a)\right)\ \Rightarrow\ \omega_1\le\sqrt{\frac{R}{2r}}\ (1)$$

$$Now,\ \sum_{cyc}\frac{n_ag_a}{h_a}\overset{AM-GM}{\leq}\sum_{cyc}\frac{{n_a}^2+{g_a}^2}{2h_a}$$

$$=\sum_{cyc}\frac{\left(s(s-a)+\frac{s(b-c)^2}{a}\right)+\left(s(s-a)-\frac{(s-a)(b-c)^2}{a}\right)}{2h_a}=$$

$$=\sum_{cyc}\frac{a\left[2s(s-a)+(b-c)^2\right]}{4F}=\frac{1}{2r}\sum_{cyc}a(s-a)+\frac{1}{4F}\sum_{cyc}a(b-c)^2=$$

$$=\frac{2r(4R+r)}{2r}+\frac{2s\sum_{cyc}bc-9abc}{4F}=4R+r+\frac{1}{2r}.\sum_{cyc}2Rh_a-9R=-5R+r+\frac{R}{r}\sum_{cyc}h_a.$$

$$Then:\ \ 5R-r+\sum_{cyc}\frac{n_ag_a}{h_a}\le\frac{R}{r}\sum_{cyc}h_a\ \Rightarrow\ \omega_2\le\sqrt{\frac{R}{2r}}\ (2)$$

$$Therefore,$$

$$\frac{1}{\sin\omega}=\frac{\sqrt{a^2b^2+b^2c^2+c^2a^2}}{2F}\ge\frac{\sqrt{abc(a+b+c)}}{2sr}=2\sqrt{\frac{R}{2r}}\overset{(1)\,\&\,(2)}{\geq}\omega_1+\omega_2$$

888. In $\Delta ABC, I_a, I_b, I_c$ −excenters. Prove that:

$$72Rr\le\sum_{cyc}(I_bI_c)^2\le 36R^2$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$BC=a=I_bI_c\cos\left(90^\circ-\frac{A}{2}\right)=I_bI_c\sin\frac{A}{2},\qquad a=2R\sin A=I_bI_c\sin\frac{A}{2}$$

$$2R\cdot 2\sin\frac{A}{2}\cos\frac{A}{2}=I_bI_c\sin\frac{A}{2},\qquad I_bI_c=4R\cos\frac{A}{2}$$

$$Similarly: I_aI_c = 4R\sin\frac{B}{2} \text{ and } I_aI_b = 4R\sin\frac{C}{2}$$

$$\sum_{cyc}(I_bI_c)^2 = 16R^2\sum_{cyc}\cos^2\frac{A}{2} = 16R^2\left(2+\frac{r}{2R}\right) \overset{Euler}{\leq} 16R^2\left(2+\frac{R}{2R}\right) = 36R^2; (1)$$

$$\sum_{cyc}(I_bI_c)^2 = 16R^2\sum_{cyc}\cos^2\frac{A}{2} = \frac{16R^2(4R+r)}{2R} = 8R(4R+r)$$

$$We\ need\ to\ prove:$$

$$8R(4R+r) \geq 72Rr \Leftrightarrow 4R+r \geq 9r \Leftrightarrow R \geq 2r\ (Euler)$$

$$Hence, \sum_{cyc}(I_bI_c)^2 \geq 72Rr;\ \ (2)$$

$$From\ (1)\ and\ (2), it\ follows:\ 72Rr \leq \sum_{cyc}(I_bI_c)^2 \leq 36R^2$$

889. In any Δ ABC holds:

$$\frac{a^2+b^2}{b^2+c^2}+\frac{b^2+c^2}{c^2+a^2}+\frac{c^2+a^2}{a^2+b^2}+\frac{R}{2r} \geq 1+\frac{b^2+c^2}{a^2+b^2}+\frac{c^2+a^2}{b^2+c^2}+\frac{a^2+b^2}{c^2+a^2}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a^2+b^2}{b^2+c^2}+\frac{b^2+c^2}{c^2+a^2}+\frac{c^2+a^2}{a^2+b^2}+\frac{R}{2r} \geq 1+\frac{b^2+c^2}{a^2+b^2}+\frac{c^2+a^2}{b^2+c^2}+\frac{a^2+b^2}{c^2+a^2}$$

$$\Leftrightarrow \frac{a^2+b^2+c^2-c^2}{b^2+c^2}+\frac{b^2+c^2+a^2-a^2}{c^2+a^2}+\frac{c^2+a^2+b^2-b^2}{a^2+b^2}+\frac{R}{2r}$$

$$\geq 1+\frac{b^2+c^2+a^2-a^2}{a^2+b^2}+\frac{c^2+a^2+b^2-b^2}{b^2+c^2}+\frac{a^2+b^2+c^2-c^2}{c^2+a^2}$$

$$\Leftrightarrow \left(\sum_{cyc}a^2\right)\left(\sum_{cyc}\frac{1}{b^2+c^2}\right)-\frac{c^2}{b^2+c^2}-\frac{a^2}{c^2+a^2}-\frac{b^2}{a^2+b^2}+\frac{R-2r}{2r}$$

$$\geq \left(\sum_{cyc}a^2\right)\left(\sum_{cyc}\frac{1}{b^2+c^2}\right)-\frac{a^2}{a^2+b^2}-\frac{b^2}{b^2+c^2}-\frac{c^2}{c^2+a^2}$$

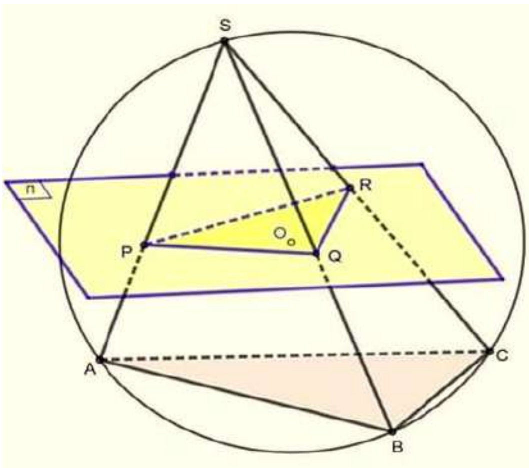
$$\Leftrightarrow \frac{b^2-c^2}{b^2+c^2}+\frac{c^2-a^2}{c^2+a^2}+\frac{a^2-b^2}{a^2+b^2}+\frac{R-2r}{2r} \geq 0$$

$$\Leftrightarrow \frac{b^2+c^2-2c^2}{b^2+c^2}+\frac{c^2+a^2-2a^2}{c^2+a^2}+\frac{a^2+b^2-2b^2}{a^2+b^2}+\frac{R-2r}{2r} \geq 0$$

$$\Leftrightarrow 3+\frac{R-2r}{2r}\ge\frac{2c^2}{b^2+c^2}+\frac{2a^2}{c^2+a^2}+\frac{2b^2}{a^2+b^2}$$
$$\Leftrightarrow 3+\frac{R-2r}{2r}+\frac{2b^2}{b^2+c^2}+\frac{2c^2}{c^2+a^2}+\frac{2a^2}{a^2+b^2}$$
$$\ge\left(\frac{2c^2}{b^2+c^2}+\frac{2b^2}{b^2+c^2}\right)+\left(\frac{2a^2}{c^2+a^2}+\frac{2c^2}{c^2+a^2}\right)+\left(\frac{2b^2}{a^2+b^2}+\frac{2a^2}{a^2+b^2}\right)$$
$$\Leftrightarrow\frac{R-2r}{2r}+2\sum_{cyc}\frac{a^2}{a^2+b^2}\overset{(*)}{\ge}3$$
$$\text{Now, LHS of }(*)\overset{\text{Bergstrom}}{\ge}\frac{R-2r}{2r}+\frac{2(4s^2)}{2\sum_{cyc}a^2}\overset{?}{\ge}3\Leftrightarrow\frac{R-2r}{2r}\overset{?}{\ge}3-\frac{2s^2}{s^2-4Rr-r^2}$$
$$\Leftrightarrow\frac{R-2r}{2r}\overset{?}{\ge}\frac{s^2-12Rr-3r^2}{s^2-4Rr-r^2}\Leftrightarrow(R-4r)s^2\ge(R-8r)(4Rr+r^2)$$
$$\Leftrightarrow(R-2r)s^2-2rs^2\overset{(**)}{\ge}(R-8r)(4Rr+r^2)$$
$$\text{Now, LHS of }(*)\overset{\text{Gerretsen}}{\ge}(R-2r)(16Rr-5r^2)-2r(4R^2+4Rr+3r^2)$$
$$\overset{?}{\ge}(R-8r)(4Rr+r^2)\Leftrightarrow 2R^2-7Rr+6r^2\overset{?}{\ge}0\Leftrightarrow(R-2r)(2R-3r)\overset{?}{\ge}0$$
$$\to\text{true}\because R\overset{\text{Euler}}{\ge}2r\Rightarrow(**)\Rightarrow(*)\text{ is true}\therefore\text{ in any }\Delta\,ABC,$$
$$\frac{a^2+b^2}{b^2+c^2}+\frac{b^2+c^2}{c^2+a^2}+\frac{c^2+a^2}{a^2+b^2}+\frac{R}{2r}\ge 1+\frac{b^2+c^2}{a^2+b^2}+\frac{c^2+a^2}{b^2+c^2}+\frac{a^2+b^2}{c^2+a^2},$$
with equality iff Δ ABC is equilateral (QED)

890. $SABC$ − tetrahedron, O − center of circumsphere,

$$\sphericalangle BSC=\sphericalangle CSA=\theta=60^\circ$$
$$SP=SQ=SR,\ \text{plane}(P,Q,R)=(\pi),\ O\in(\pi),\ \text{Volume}(SPQR)=V.$$
$$\text{Prove: }SA+SB+SC=4\cdot\sqrt[6]{72}\cdot\sqrt[3]{V}.$$



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

$$From\ 01.05.2022, we\ have:$$

$$SA\left(\frac{3}{SP}+\frac{-1}{SQ}+\frac{-1}{SR}\right)+SB\left(\frac{-1}{SP}+\frac{3}{SQ}+\frac{-1}{SR}\right)+SC\left(\frac{-1}{SP}+\frac{-1}{SQ}+\frac{3}{SR}\right)=4$$

$$SP=SQ=SR=d\Rightarrow SA+SB+SC=4d$$

$$V=\frac{d^3}{6}\cdot\frac{1}{\sqrt{2}}\Rightarrow d=\sqrt[6]{72}\cdot\sqrt[3]{V}\Rightarrow SA+SB+SC=4\cdot\sqrt[6]{72}\cdot\sqrt[3]{V}.$$

891. Prove that:

$$\sqrt[3]{\sin\frac{\pi}{14}}-\sqrt[3]{\sin\frac{3\pi}{14}}+\sqrt[3]{\sin\frac{5\pi}{14}}=\sqrt[3]{-\frac{5}{2}+\frac{3}{2}\sqrt[3]{7}}$$

Proposed by Vasile Mircea Popa-Romania

Solution 1 by Pham Duc Nam-Vietnam

Let: $x=\sqrt[3]{\sin\frac{\pi}{14}}, y=-\sqrt[3]{\sin\frac{3\pi}{14}}, z=\sqrt[3]{\sin\frac{5\pi}{14}}$

$$*\ \sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}=\sin\frac{\pi}{14}\cos\frac{4\pi}{14}\cos\frac{2\pi}{14}=\frac{2\sin\frac{\pi}{14}\cos\frac{\pi}{14}\cos\frac{4\pi}{14}\cos\frac{2\pi}{14}}{2\cos\frac{\pi}{14}}$$

$$=\frac{\sin\frac{2\pi}{14}\cos\frac{2\pi}{14}\cos\frac{4\pi}{14}}{2\cos\frac{\pi}{14}}=\frac{\sin\frac{4\pi}{14}\cos\frac{4\pi}{14}}{4\cos\frac{\pi}{14}}=\frac{\sin\frac{8\pi}{14}}{8\cos\frac{\pi}{14}}=\frac{\cos\frac{\pi}{14}}{8\cos\frac{\pi}{14}}=\frac{1}{8}$$

$$\Rightarrow xyz=-\sqrt[3]{\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}}=-\frac{1}{2}\Rightarrow (xyz)^2=\frac{1}{4}$$

$$*\ \sin\frac{\pi}{14}-\sin\frac{3\pi}{14}+\sin\frac{5\pi}{14}=S\Rightarrow S\cos\frac{\pi}{14}=\cos\frac{\pi}{14}\left(\sin\frac{\pi}{14}-\sin\frac{3\pi}{14}+\sin\frac{5\pi}{14}\right)$$

$$=\frac{1}{2}\sin\frac{\pi}{7}-\frac{1}{2}\left(\sin\frac{\pi}{7}+\cos\frac{3\pi}{14}\right)+\frac{1}{2}\left(\cos\frac{\pi}{14}+\cos\frac{3\pi}{14}\right)\Rightarrow S=\frac{1}{2}$$

$$\Rightarrow x^3+y^3+z^3=\frac{1}{2}$$

$$*\ \sin\frac{\pi}{14}\sin\frac{5\pi}{14}-\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}-\sin\frac{\pi}{14}\sin\frac{3\pi}{14}$$

$$=\frac{1}{2}\left(\cos\frac{2\pi}{7}-\cos\frac{3\pi}{7}+\cos\frac{4\pi}{7}-\cos\frac{\pi}{7}+\cos\frac{2\pi}{7}-\cos\frac{\pi}{7}\right)$$

$$=\frac{1}{2}\left(2\cos\frac{2\pi}{7}-\cos\frac{3\pi}{7}+\cos\frac{4\pi}{7}-2\cos\frac{\pi}{7}\right)$$

$$=\frac{1}{2}\left(4\cos^2\frac{\pi}{7}-2-4\cos^3\frac{\pi}{7}+3\cos\frac{\pi}{7}+8\cos^4\frac{\pi}{7}-8\cos^2\frac{\pi}{7}+1-2\cos\frac{\pi}{7}\right)$$

$$=\frac{1}{2}\left(8\cos^4\frac{\pi}{7}-4\cos^3\frac{\pi}{7}-4\cos^2\frac{\pi}{7}+\cos\frac{\pi}{7}-1\right)$$

$$=\frac{1}{2}\left(\cos\frac{\pi}{7}\left(8\cos^3\frac{\pi}{7}-4\cos^2\frac{\pi}{7}-4\cos\frac{\pi}{7}+1\right)-1\right)$$

Known:Minimal polynomial which root is $\cos\left(\frac{\pi}{7}\right)$ is: $8x^3-4x^2-4x+1$

$$\Rightarrow 8\cos^3\frac{\pi}{7}-4\cos^2\frac{\pi}{7}-4\cos\frac{\pi}{7}+1=0$$

$$\Rightarrow \sin\frac{\pi}{14}\sin\frac{5\pi}{14}-\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}-\sin\frac{\pi}{14}\sin\frac{3\pi}{14}=-\frac{1}{2}$$

$$\Rightarrow (xy)^3+(yz)^3+(xz)^3=-\frac{1}{2}$$

$$*\ x^3+y^3+z^3=(x+y)^3+z^3-3xy(x+y)$$

$$=(x+y+z)\left((x+y)^2-z(x+y)+z^2\right)-3xy(x+y)\Rightarrow x^3+y^3+z^3-3xyz$$

$$=(x+y+z)\left((x+y+z)^2-3(xy+yz+xz)\right)$$

$\Leftrightarrow 2=s(s^2-3p)$,where: $s=x+y+z, p=xy+xz+yz$

$*$ Replace: $x=xy, y=yz, z=xz \Rightarrow (xy)^3+(yz)^3+(xz)^3-3x^2y^2z^2$

$$=(xz+yz+xy)\left((xz+yz+xy)^2-3xyz(x+y+z)\right)\Leftrightarrow -\frac{5}{4}=p\left(p^2+\frac{3}{2}s\right)$$

$$\Rightarrow p^3=-\frac{3}{2}ps-\frac{5}{4}$$

$$*\ 2=s(s^2-3p)\Leftrightarrow 3ps=s^3-2\Rightarrow p^3=-\frac{1}{2}(s^3-2)-\frac{5}{4}=-\frac{s^3}{2}-\frac{1}{4}$$

$$\Rightarrow 27p^3s^3=\left(s^3-2\right)^3\Leftrightarrow 27\left(-\frac{s^3}{2}-\frac{1}{4}\right)s^3=\left(s^3-2\right)^3\Leftrightarrow 4s^9+30s^6+75s^3-32=0$$

$$\Leftrightarrow\left(s^3+\frac{5}{2}\right)^3=\frac{189}{8}=\left(\frac{3}{2}\sqrt[3]{7}\right)^3$$

$$\Rightarrow s^3=\frac{3}{2}\sqrt[3]{7}-\frac{5}{2}\Rightarrow s=\sqrt[3]{\frac{3}{2}\sqrt[3]{7}-\frac{5}{2}}\Rightarrow x+y+z=\sqrt[3]{\sin\frac{\pi}{14}}-\sqrt[3]{\sin\frac{3\pi}{14}}+\sqrt[3]{\sin\frac{5\pi}{14}}$$

$$=\boxed{\sqrt[3]{\frac{3}{2}\sqrt[3]{7}-\frac{5}{2}}}$$

Solution 2 by Hikmat Mammadov-Azerbaijan

$$LHS=\sqrt[3]{\cos\frac{\pi}{7}}+\sqrt[3]{\cos\frac{3\pi}{7}}+\sqrt[3]{\cos\frac{5\pi}{7}}$$

$$\left(e^{i\frac{\pi}{7}}\right)^7+1=e^{i\pi}+1=0$$

$$If\ z=e^{i\frac{\pi}{7}}\ or\ z=e^{i\frac{3\pi}{7}}\ or\ z=e^{i\frac{5\pi}{7}}$$

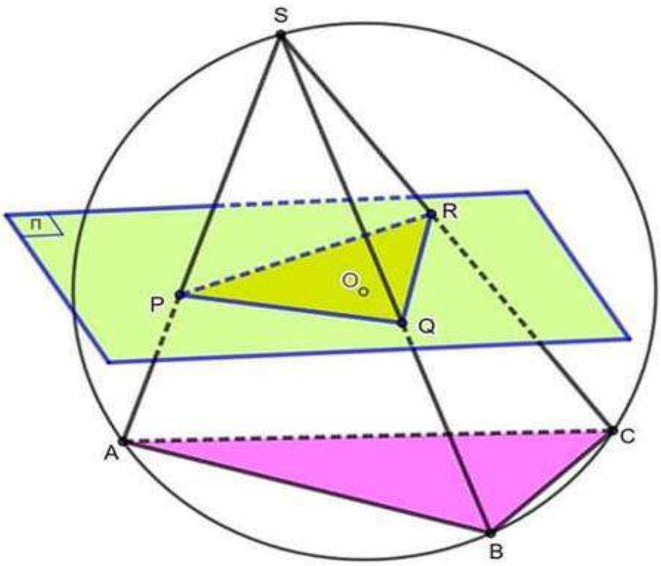
$$0=z^7+1=(z+1)\left(z^6-z^5+z^4-z^3+z^2-z+1\right)$$

$$Since\ z \neq -1\ and\ z \neq 0.$$
$$\frac{z^7+1}{(z+1)z^3} = 0 \Rightarrow z^3 - z^2 + z - 1 - z^{-2} + z^{-3} = 0$$
$$\left(z+\frac{1}{z}\right)^3 = z^3 - z^{-3} + 3(z+z^{-1}) \Rightarrow z^3 + z^{-3} = (z+z^{-1})^2 - 2$$
$$z^3 - z^2 + z - 1 + z^{-1} - z^{-2} + z^{-3} = (z^3+z^{-3}) - (z^2+z^{-2}) + (z+z^{-1}) - 1 = 0$$
$$(z+z^{-1})^3 - 3(z+z^{-1}) - (z+z^{-1})^2 + 2 + (z+z^{-1}) - 1 = 0$$
$$Let\ \omega = z + z^{-1} \Rightarrow \omega^3 - \omega^2 - 2\omega + 1 = g(\omega) = 0$$
$$g(\omega)\ has\ e^{i\frac{\pi}{7}} + e^{-1\frac{\pi}{7}} = 2\cos\frac{\pi}{7}\ as\ a\ root.$$
$$Also, 2\cos\frac{3\pi}{7}\ and\ 2\cos\frac{5\pi}{7}\ are\ roots. Call\ these\ roots: \alpha^3, \beta^3, \gamma^3$$
$$Goal: \frac{\alpha+\beta+\gamma}{\sqrt[3]{2}}$$
$$Vieta's formulas: \omega^3 - \omega^2 - 2\omega + 1 = 0$$
$$\alpha^3 + \beta^3 + \gamma^3 = 1$$
$$\alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\alpha^3 = -2$$
$$\alpha^3\beta^3\gamma^3 = -1 \Rightarrow \alpha\beta\gamma = -1$$
$$\alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\alpha^3 = 3(\alpha\beta\beta\gamma\gamma\alpha) + (\alpha\beta+\beta\gamma+\gamma\alpha)\left((\alpha\beta+\beta\gamma+\gamma\alpha)^2 - 3\alpha\beta\gamma(\alpha+\beta+\gamma)\right)$$
$$(\alpha\beta+\beta\gamma+\gamma\alpha)^2 + 3(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha) = -5$$
$$(\alpha+\beta+\gamma)^3 - 3(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha) = 4$$
$$Galois: \frac{\Psi}{\sqrt[3]{2}}$$
$$\Psi = \alpha+\beta+\gamma \Rightarrow \Phi = \alpha\beta+\beta\gamma+\gamma\alpha$$
$$\Phi^3 + 3\Psi\Phi = -5 \Rightarrow \Psi^3 - 3\Psi\Phi = 4$$
$$(\Phi^3 + 3\Psi\Phi)(\Psi^3 - 3\Psi\Phi) = -20$$
$$(\Psi\Phi)^3 - 9(\Phi\Psi)^2 + 3\Psi\Phi(\Psi^3 - \Phi^3) = -20$$
$$Also, (\Psi^3 - 3\Phi\Psi)(\Psi^3 + 3\Psi\Phi) = 9 = \Psi^3 - \Phi^3 - 6\Phi\Psi$$
$$(\Psi\Phi)^3 - 9(\Psi\Phi)^2 + 27(\Psi\Phi) + 20 = 0 \Rightarrow (\Psi\Phi+3)^3 - 7 = 0 \Rightarrow \Psi\Phi = \sqrt[3]{7} - 3$$
$$\Phi^3 = 4 + 3\Psi\Phi = 3\sqrt[3]{7} - 5$$
$$Therefore,$$
$$\sqrt[3]{sin\frac{\pi}{14}} - \sqrt[3]{sin\frac{3\pi}{14}} + \sqrt[3]{sin\frac{5\pi}{14}} = \sqrt[3]{-\frac{5}{2} + \frac{3}{2}\sqrt[3]{7}}$$

892. $SABC$ **− tetrahedron,** O **− center of circumsphere**

$$\sphericalangle BSC = \sphericalangle CSA = \sphericalangle ASB = \theta = 60^\circ, SP = SQ = SR = d,$$

$$\text{plane}(P,Q,R) = (\pi), O \in (\pi). \text{ Prove: } d \geq \frac{3}{4}\cdot\sqrt[3]{SA\cdot SB\cdot SC}$$



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

$$\text{From } 04.05.22 \text{ we have:}$$

$$SA + SB + SC = 4 \cdot \sqrt[6]{72} \cdot \sqrt[3]{V}$$

$$\text{Is } SA + SB + SC \geq 3 \cdot \sqrt[3]{SA \cdot SB \cdot SC}$$

$$\text{Hence, } 3\sqrt[3]{SA \cdot SB \cdot SC} \leq 4 \cdot \sqrt[6]{72} \cdot \sqrt[3]{V}, \text{ then}$$

$$3 \cdot \sqrt[3]{SA \cdot SB \cdot SC} \leq 4 \cdot \sqrt[6]{72} \cdot \sqrt[3]{\frac{d^3}{6\sqrt{2}}} \text{ and then}$$

$$d \geq \frac{3}{4} \cdot \sqrt[3]{SA \cdot SB \cdot SC}$$

893. *In ΔABC the following relationship holds :*

$$\sum_{cyc} \frac{1}{a \sin\frac{A}{2}} \cdot \sum_{cyc} \frac{a}{\sin\frac{A}{2}} \geq \frac{4\sqrt{3}F}{r^2}$$

Proposed by Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{Since : } \quad a = 4R\sin\frac{A}{2}\cos\frac{A}{2} \quad (\text{and analogs}), \quad \text{then we have :}$$

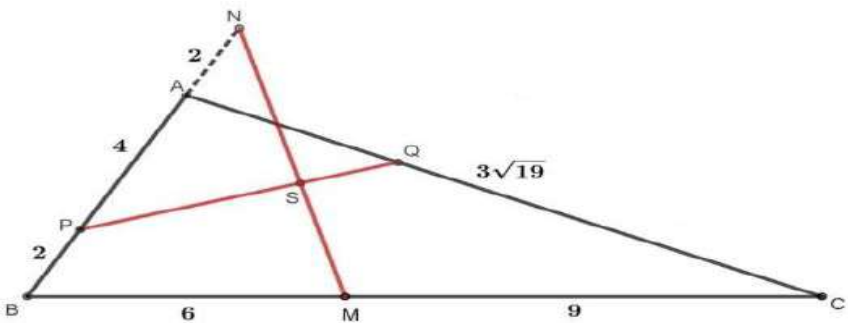
$$LHS=\sum_{cyc}\frac{1}{4R\sin\frac{A}{2}\cos\frac{A}{2}\cdot\sin\frac{A}{2}}\cdot\sum_{cyc}\frac{4R\sin\frac{A}{2}\cos\frac{A}{2}}{\sin\frac{A}{2}}=\sum_{cyc}\frac{1}{\sin^2\frac{A}{2}\cos\frac{A}{2}}\cdot\sum_{cyc}\cos\frac{A}{2}=$$

$$=\sum_{cyc}\left(\frac{1}{\cos\frac{A}{2}}+\frac{\cos\frac{A}{2}}{\sin^2\frac{A}{2}}\right)\cdot\sum_{cyc}\cos\frac{A}{2}=\sum_{cyc}\frac{1}{\cos\frac{A}{2}}\cdot\sum_{cyc}\cos\frac{A}{2}+\sum_{cyc}\frac{\cos\frac{A}{2}}{\sin^2\frac{A}{2}}\cdot\sum_{cyc}\cos\frac{A}{2}\geq$$

$$\overset{CBS}{\geq}3^2+\left(\sum_{cyc}\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}\right)^2=9+\left(\frac{s}{r}\right)^2=\frac{4\sqrt{3}s}{r}+\left(\frac{s}{r}-3\sqrt{3}\right)\left(\frac{s}{r}-\sqrt{3}\right)\overset{Mitrinovic}{\geq}\frac{4\sqrt{3}s}{r}=\frac{4\sqrt{3}F}{r^2}.$$

So the proof is completed. Equality holds iff ΔABC is equilateral.

894.



$$\frac{AQ}{QC}=\frac{1}{2},\ \sphericalangle PSN=?,\ \frac{MS}{SN}=?,\ \frac{QS}{SP}=?$$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

$$\cos B=\frac{15^2+6^2-\left(3\sqrt{19}\right)^2}{2\cdot15\cdot6}=\frac{1}{2}\Rightarrow\theta=60^\circ$$

$$QQ_1\parallel AB\Rightarrow\frac{BQ_1}{BC}=\frac{AQ}{AC}\Rightarrow\frac{BQ_1}{15}=\frac{1}{3}\Rightarrow BQ_1=q_1=5$$

$$QQ_2\parallel BC\Rightarrow\frac{BQ_2}{BA}=\frac{CQ}{CA}\Rightarrow\frac{BQ_2}{6}=\frac{2}{3}\Rightarrow BQ_2=q_2=4$$

$$Plagiogonal\ system: BC\equiv Bx, BA\equiv By$$

$$B(0,0),M(6,0),P(0,2),N(0,8),Q(q_1,q_2)=Q(5,4)$$

$$\lambda_{PQ}=\frac{4-2}{5-0}=\frac{2}{5}=\lambda_2,\qquad\lambda_{MN}=\frac{-8}{6}=\frac{-4}{3}=\lambda_1$$

$$(\lambda_2+\lambda_1)\cos\theta+\lambda_2\lambda_1+1=\left(\frac{2}{5}+\frac{-4}{3}\right)\cdot\frac{1}{2}+\frac{2}{5}\cdot\frac{4}{3}+1=0\Rightarrow PQ\perp MN$$

$$\frac{SQ}{SP}=\frac{MC}{BM}\cdot\frac{AQ}{AC}\cdot\frac{AB}{NP}+\frac{AN}{NP}\left(1-\frac{AQ}{AC}\cdot\frac{BC}{BM}\right)$$

$$\frac{SQ}{PS}=\frac{9}{6}\cdot\frac{1}{3}\cdot\frac{6}{6}+\frac{-2}{6}\left(1-\frac{1}{3}\cdot\frac{15}{6}\right)\Rightarrow\frac{SQ}{PS}=\frac{4}{9}$$

$$\frac{MS}{SN}=\frac{BM}{BC}\cdot\frac{QC}{AQ}\cdot\frac{AP}{NP}+\frac{MC}{BC}\cdot\frac{PB}{NP}$$

$$\frac{MS}{SN}=\frac{6}{15}\cdot\frac{2}{1}\cdot\frac{4}{6}+\frac{9}{15}\cdot\frac{2}{6}\Rightarrow\frac{MS}{SN}=\frac{11}{15}$$

895.

If $x,y,z\in R, x+y, y+z, z+x, xy+yz+zx>0$ then in ΔABC holds:

$$\sum_{cyc}(y+z)\left(\frac{n_a}{h_a}-\left(\sqrt{\frac{n_b}{h_b}}-\sqrt{\frac{n_c}{h_c}}\right)^2\right)\ge 2\sqrt{\left(2\sum_{cyc}\frac{n_an_b}{h_ah_b}-\frac{1}{4r^2}\sum_{cyc}a^2+\frac{4R}{r}-2\right)\sum_{cyc}xy}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamd Amine Ben Ajiba-Tanger-Morocco

We have:

$$\left(\sqrt{x+y}+\sqrt{y+z}\right)^2=x+2y+z+2\sqrt{y^2+xy+yz+zx}>x+2y+z+2|y|\ge z+x.$$

$$Then:\ \sqrt{x+y}+\sqrt{y+z}>\sqrt{z+x}\ \ (and\ analogs)$$

So $\sqrt{x+y},\sqrt{y+z},\sqrt{z+x}$ can be the sides of a triangle Δ with area S such that:

$$16S^2=2\sum_{cyc}\sqrt{x+y}^2\sqrt{y+z}^2-\sum_{cyc}\sqrt{x+y}^4=4\sum_{cyc}xy\ \Rightarrow\ 2S=\sqrt{xy+yz+zx}.$$

$$Now, we\ have:\ \left(\frac{n_a}{h_a}\right)^2=\frac{a^2}{4s^2r^2}\left(s(s-a)+\frac{s(b-c)^2}{a}\right)=\frac{1}{4r^2}\left(\frac{a^2(s-a)}{s}+\frac{a(b-c)^2}{s}\right)=$$

$$=\frac{1}{4r^2}\left(\frac{(s-a)[a^2-(b-c)^2]}{s}+(b-c)^2\right)=\frac{1}{4r^2}\left(\frac{4(s-a)(s-b)(s-c)}{s}+(b-c)^2\right)=\frac{4r^2+(b-c)^2}{4r^2}.\ Then:$$

$$\frac{n_a}{h_a}=\sqrt{1+\frac{(b-c)^2}{4r^2}}\ \ (and\ analogs)$$

$$\Rightarrow\left(\sqrt{\frac{n_a}{h_a}}+\sqrt{\frac{n_b}{h_b}}\right)^2>\frac{n_a}{h_a}+\frac{n_b}{h_b}=\sqrt{1^2+\left(\frac{b-c}{2r}\right)^2}+\sqrt{1^2+\left(\frac{c-a}{2r}\right)^2}\overset{Minkowski}{\geq}\sqrt{(1+1)^2+\left(\frac{b-c}{2r}+\frac{c-a}{2r}\right)^2}=$$

$$=\sqrt{4+\frac{(a-b)^2}{4r^2}}>\sqrt{1+\frac{(a-b)^2}{4r^2}}=\frac{n_c}{h_c}\Rightarrow\sqrt{\frac{n_a}{h_a}}+\sqrt{\frac{n_b}{h_b}}>\sqrt{\frac{n_c}{h_c}}\ (and\ analogs)$$

$$Let\ \alpha=\sqrt{\frac{n_a}{h_a}},\beta=\sqrt{\frac{n_b}{h_b}},\gamma=\sqrt{\frac{n_c}{h_c}}\Rightarrow$$

$$\frac{n_a}{h_a}-\left(\sqrt{\frac{n_b}{h_b}}-\sqrt{\frac{n_c}{h_c}}\right)^2=(\alpha+\beta-\gamma)(\alpha-\beta+\gamma)>0\ (and\ analogs).$$

$$Now, by\ Oppenheim's\ inequality\ in\ triangle\ \Delta, we\ have, for\ any\ u,v,w>0:$$

$$u.\sqrt{y+z}^2+v.\sqrt{z+x}^2+w.\sqrt{x+y}^2\geq 4S\sqrt{uv+vw+wu}$$
$$=2\sqrt{(xy+yz+zx)(uv+vw+wu)}.$$

$$For\ u=\frac{n_a}{h_a}-\left(\sqrt{\frac{n_b}{h_b}}-\sqrt{\frac{n_c}{h_c}}\right)^2,\ v=\frac{n_b}{h_b}-\left(\sqrt{\frac{n_c}{h_c}}-\sqrt{\frac{n_a}{h_a}}\right)^2,\ w$$
$$=\frac{n_c}{h_c}-\left(\sqrt{\frac{n_a}{h_a}}-\sqrt{\frac{n_b}{h_b}}\right)^2, we\ have:$$

$$uv+vw+wu=\sum_{cyc}uv=\prod_{cyc}(\alpha+\beta-\gamma).\sum_{cyc}(\alpha+\beta-\gamma)=\prod_{cyc}(\alpha+\beta-\gamma).\sum_{cyc}\alpha=$$

$$=2\sum_{cyc}\alpha^2\beta^2-\sum_{cyc}\alpha^4=2\sum_{cyc}\frac{n_an_b}{h_ah_b}-\sum_{cyc}\left(\frac{n_a}{h_a}\right)^2$$
$$=2\sum_{cyc}\frac{n_an_b}{h_ah_b}-\frac{1}{4F^2}\sum_{cyc}a^2\left(s(s-a)+\frac{s(b-c)^2}{a}\right)=$$

$$=2\sum_{cyc}\frac{n_an_b}{h_ah_b}-\frac{1}{4r^2}\sum_{cyc}a^2+\frac{s}{4F^2}\sum_{cyc}a(a+b-c)(a-b+c)$$
$$=2\sum_{cyc}\frac{n_an_b}{h_ah_b}-\frac{1}{4r^2}\sum_{cyc}a^2+\frac{abc}{4Fr}\sum_{cyc}4\sin^2\frac{A}{2}=$$

$$=2\sum_{cyc}\frac{n_an_b}{h_ah_b}-\frac{1}{4r^2}\sum_{cyc}a^2+\frac{4R}{r}.\left(1-\frac{r}{2R}\right)=2\sum_{cyc}\frac{n_an_b}{h_ah_b}-\frac{1}{4r^2}\sum_{cyc}a^2+\frac{4R}{r}-2.$$

Therefore,

$$\sum_{cyc}(y+z)\left(\frac{n_a}{h_a}-\left(\sqrt{\frac{n_b}{h_b}}-\sqrt{\frac{n_c}{h_c}}\right)^2\right)\ge 2\sqrt{\left(2\sum_{cyc}\frac{n_an_b}{h_ah_b}-\frac{1}{4r^2}\sum_{cyc}a^2+\frac{4R}{r}-2\right)\sum_{cyc}xy}.$$

896. *In ΔABC the following relationship holds:*

$$\sqrt{\frac{{h_a}^2{r_a}^2+{h_b}^2{r_b}^2+{h_c}^2{r_c}^2}{r(4R+r)}}\ge\frac{F}{R}\left(\frac{br_c}{cr_b}+\frac{cr_b}{br_c}\right)$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $x=s-a,\ \ y=s-b,\ \ z=s-c$. *We have:*

$$a(s-a)+b(s-b)-c(s-c)=(y+z)x+(z+x)y-(x+y)z=$$

$$=2xy>0\ \ (and\ analogs)$$

Then

$$a(s-a),b(s-b),c(s-c)$$

can be the sides of a triangle Δ *with area* S *such that:*

$$16S^2=\sum_{cyc}a(s-a).\prod_{cyc}[a(s-a)+b(s-b)-c(s-c)]$$

$$=2r(4R+r)\prod_{cyc}2(s-a)(s-b)=$$

$$=2r(4R+r).8(sr^2)^2=16s^2r^5(4R+r).\ Then:\ \ S=Fr\sqrt{r(4R+r)}.$$

$$Lemma:\ \ In\ any\ \Delta ABC, we\ have:\ \frac{\sqrt{a^2b^2+b^2c^2+c^2a^2}}{2F}\ge\frac{b}{c}+\frac{c}{b}\quad(*)$$

$$Proof:\ (*)\ \Leftrightarrow\ 2bc\sqrt{a^2b^2+b^2c^2+c^2a^2}$$
$$\ge(b^2+c^2)\sqrt{2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)}$$

$$\overset{squaring}{\Leftrightarrow}\quad 4b^2c^2(a^2b^2+b^2c^2+c^2a^2)$$
$$\ge(2b^2c^2+b^4+c^4)[2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)]$$

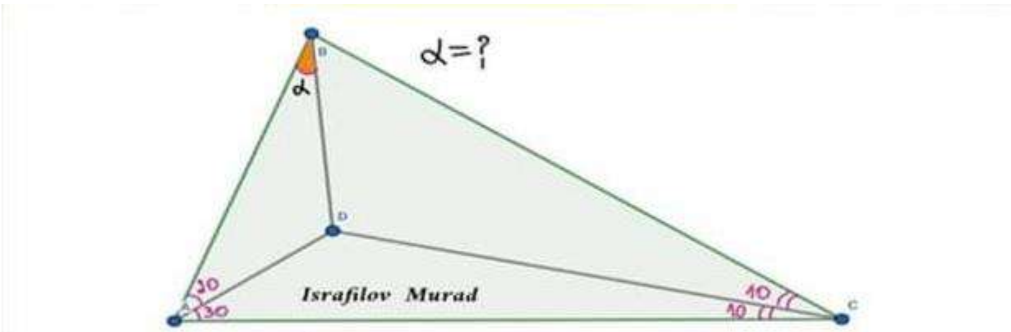
$$\Leftrightarrow\ 0 \geq -a^4(b^2+c^2)^2 + 2(b^4+c^4)(a^2b^2+c^2a^2) - (b^4+c^4)^2$$
$$= -[a^2(b^2+c^2)-(b^4+c^4)]^2$$

Which is true and the proof of the lemma is complete. Applying this lemma in triangle Δ, *we have* :

$$\frac{b(s-b)}{c(s-c)} + \frac{c(s-c)}{b(s-b)} \leq \frac{\sqrt{\sum_{cyc}\big(b(s-b)\big)^2\big(c(s-c)\big)^2}}{2S} =$$

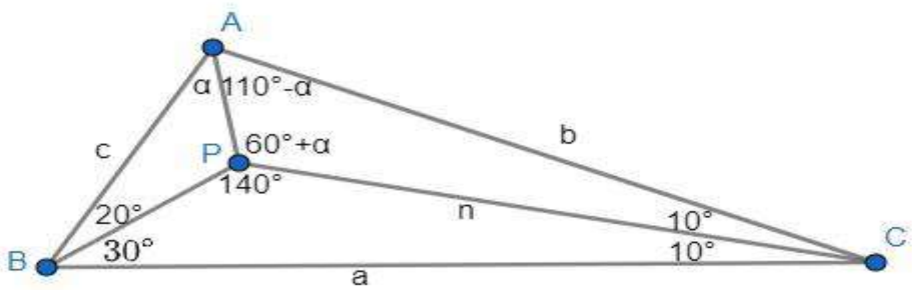
$$\frac{\sqrt{\sum_{cyc}\frac{(4RF)^2(Fr)^2}{\big(a(s-a)\big)^2}}}{2Fr\sqrt{r(4R+r)}} = \frac{R}{F}\cdot\sqrt{\frac{\sum_{cyc} {h_a}^2{r_a}^2}{r(4R+r)}}.$$

$$\textit{Therefore,}\ \sqrt{\frac{{h_a}^2{r_a}^2+{h_b}^2{r_b}^2+{h_c}^2{r_c}^2}{r(4R+r)}} \geq \frac{F}{R}\left(\frac{br_c}{cr_b}+\frac{cr_b}{br_c}\right).$$

897. 

Proposed by Israfilov Murad-Baku-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India



$$\text{Sine } law \text{ on } \Delta\, \mathrm{ACP} \Rightarrow \frac{\mathrm{n}}{\sin(110°-\alpha)} = \frac{\mathrm{b}}{\sin(60°+\alpha)}$$

$$\Rightarrow \frac{\mathrm{n}}{\mathrm{b}} \overset{(*)}{=} \frac{\sin(70°+\alpha)}{\sin(60°+\alpha)}$$

$$\text{Sine } law \text{ on } \Delta\, \mathrm{BCP} \Rightarrow \frac{\mathrm{n}}{\sin 30°} = \frac{a}{\sin 140°} \Rightarrow \mathrm{n} \overset{(**)}{=} \frac{a}{2\sin 40°}$$

$$and \text{ Sine } law \text{ on } \Delta\, \mathrm{ABC} \Rightarrow \frac{\mathrm{b}}{\sin 50°} = \frac{a}{\sin 110°} \Rightarrow \mathrm{b} \overset{(***)}{=} \frac{a\sin 50°}{2\sin 70°}$$

$$\therefore (**), (***) \Rightarrow \frac{\mathrm{n}}{\mathrm{b}} = \frac{\sin 70°}{2\sin 40° \sin 50°} = \frac{\cos 20°}{\cos 10° - \cos 90°}$$

$$\Rightarrow \frac{\mathrm{n}}{\mathrm{b}} \overset{(****)}{=} = \frac{\cos 20°}{\cos 10°} \therefore (***), (****) \Rightarrow 2\sin(70°+\alpha)\cos 10° = 2\sin(60°+\alpha)\cos 20°$$

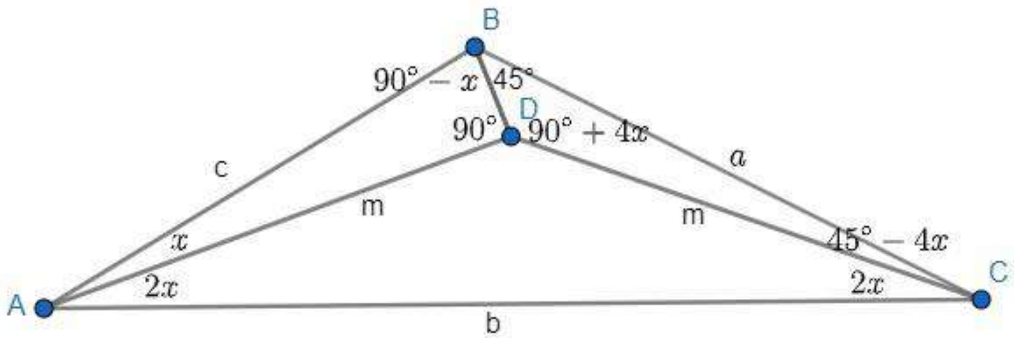
$$\Rightarrow \sin(80°+\alpha) + \sin(60°+\alpha) = \sin(80°+\alpha) + \sin(40°+\alpha)$$

$$\Rightarrow \sin(60°+\alpha) - \sin(40°+\alpha) = 0 \Rightarrow 2\cos(50°+\alpha)\sin 10° = 0$$

$$\Rightarrow \cos(50°+\alpha) = 0 \; and \because 50° < 50°+\alpha < 160°$$

$$\therefore 50°+\alpha = 90° \Rightarrow \boxed{\alpha = 40°} \; (ans)$$

898.



$$\text{Prove}: \frac{\mathrm{AD}}{\mathrm{BC}} = \sqrt{\frac{\varphi}{2}}$$

Proposed by Murat Oz-Turkiye

Solution 1 by Miguel Angel Perez Ortega Lopez-Mexico

$$b = c \Rightarrow \begin{cases} S_A = \dfrac{2b^2 - a^2}{2} \\ S_B = S_C = \dfrac{a^2}{2} \end{cases}$$

$$BD: b^2x + a(a+b)z = 0$$

$$AD: 2b^2y + (2b^2 - a^2)z = 0$$

$$\{D\} = BD \cap AD = (2a(a+b); 2b^2 - a^2; -2b^2)$$

$$CD: (2b^2 - a^2)x - 2a(a+b)y = 0$$

$$S = S\cot\frac{\pi}{4} = \frac{AD \cdot CD}{\begin{vmatrix} 1 & 1 & 1 \\ 0 & 2b^2 & 2b^2 - a^2 \\ 2b^2 - a^2 & -2a(a+b) & 0 \end{vmatrix}} = \frac{a(a-2b)(a^2+2ab+2b^2)}{2(2b^2-a^2)}$$

$$0 = S^2 - \left[\frac{a(a-2b)(a^2+2ab+2b^2)}{2(2b^2-a^2)}\right]^2 =$$

$$= \frac{a^2(4b^2-a^2)}{4} - \left[\frac{a(a-2b)(a^2+2ab+2b^2)}{2(2b^2-a^2)}\right]^2 =$$

$$= \frac{a^3(2b-a)(a^4+2a^3b-2a^2b^2-8ab^3-4b^4)}{2(2b^2-a^2)^2}$$

$$\left(\frac{a}{b}\right)^4 + 2\left(\frac{a}{b}\right)^3 - 2\left(\frac{a}{b}\right)^2 - 8\left(\frac{a}{b}\right) - 4 = 0 \Rightarrow \frac{a}{b} = \phi - 1 + \sqrt{\phi}$$

$$CD^2 = -\frac{a^3 - 4ab^2 - 4b^3}{a + 2b}$$

$$\frac{AB^2}{CD^2} = -\frac{b^2(a+2b)}{a^3 - 4ab^2 - 4b^3} = -\frac{\frac{a}{b} + 2}{\left(\frac{a}{b}\right)^3 - 4\left(\frac{a}{b}\right)^2 - 4} =$$

$$= \frac{6 + 4\left(\frac{a}{b}\right) - \left(\frac{a}{b}\right)^2 - \left(\frac{a}{b}\right)^3}{4}$$

$$\frac{AB^2}{CD^2} = \frac{6 + 4(\phi - 1 + \sqrt{\phi}) - (\phi - 1 + \sqrt{\phi})^2 - (\phi - 1 + \sqrt{\phi})^3}{4} =$$

$$= \frac{2 + 3\sqrt{\phi} + 5\phi + 3\phi\sqrt{\phi} - \phi^2\sqrt{\phi} - \phi^3}{4}$$

$$\phi^2 - \phi - 1 = 0 \Rightarrow (\sqrt{\phi})^4 - (\sqrt{\phi})^2 - 1 = 0$$

$$\frac{AB^2}{CD^2} = \frac{(\sqrt{\phi})^2}{2} \Rightarrow \frac{AB}{CD} = \sqrt{\frac{\phi}{2}}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\text{Sine } law \text{ on } \Delta\,\mathrm{BCD} \Rightarrow \frac{a}{\sin(90^\circ+4x)}=\frac{\mathrm{m}}{\sin 45^\circ}=\frac{\mathrm{n}}{\sin(45^\circ-4x)}\ (\mathrm{n}=\mathrm{BD})$$

$$\Rightarrow \frac{\mathrm{m}}{a}\overset{(\bullet)}{=}\frac{1}{\sqrt{2}\cos 4x}\ and\ \frac{\mathrm{n}}{\mathrm{m}}\overset{(*)}{=}\sqrt{2}\sin(45^\circ-4x)$$

$$\text{Again, sine } law \text{ on } \Delta\,\mathrm{ABD} \Rightarrow \frac{\mathrm{m}}{\sin(90^\circ-x)}=\frac{\mathrm{n}}{\sin x}\Rightarrow \frac{\mathrm{n}}{\mathrm{m}}\overset{(**)}{=}\frac{\sin x}{\cos x}\ \therefore (*),(**)$$

$$\Rightarrow \frac{\sin x}{\cos x}=\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos 4x-\frac{1}{\sqrt{2}}\sin 4x\right)\Rightarrow \sin x=\cos x.\cos 4x-\cos x.\sin 4x$$

$$\Rightarrow \sin x(1+4\cos 2x.\cos^2x)=\cos x.(2\cos^2 2x-1)$$

$$\Rightarrow (1-\cos^2x)\left(1+4\cos^2x.(2\cos^2x-1)\right)^2=\cos^2x(2(2\cos^2x-1)^2-1)^2$$

$$\Rightarrow (1-\cos^2x)(8\cos^4x-4\cos^2x+1)^2=\cos^2x.(8\cos^4x-8\cos^2x+1)^2$$

$$\Rightarrow \mathrm{t}(8\mathrm{t}^2-8\mathrm{t}+1)^2-(1-\mathrm{t})(8\mathrm{t}^2-4\mathrm{t}+1)^2=0\ (\mathrm{t}=\cos^2x)$$

$$\Rightarrow 128\mathrm{t}^5-256\mathrm{t}^4+176\mathrm{t}^3-56\mathrm{t}^2+10\mathrm{t}-1=0$$

$$\Rightarrow (2\mathrm{t}-1)(64\mathrm{t}^4-96\mathrm{t}^3+40\mathrm{t}^2-8\mathrm{t}+1)=0$$

$$\Rightarrow \boxed{64\mathrm{t}^4-96\mathrm{t}^3+40\mathrm{t}^2-8\mathrm{t}+1\overset{(***)}{=}0}\ \left(\because \cos^2x\neq\frac{1}{2}\ as\ x<\frac{45^\circ}{4}\right)$$

$$\text{Now, } 64\left(\mathrm{t}^2-\mathrm{t}.\frac{3+\sqrt{5}}{4}+\frac{3+\sqrt{5}}{16}\right)\left(\mathrm{t}^2-\mathrm{t}.\frac{3-\sqrt{5}}{4}+\frac{3-\sqrt{5}}{16}\right)$$

$$=64\left(\mathrm{t}^4-\mathrm{t}^3.\frac{3-\sqrt{5}}{4}+\mathrm{t}^2.\frac{3-\sqrt{5}}{16}-\mathrm{t}^3.\frac{3+\sqrt{5}}{4}+\mathrm{t}^2.\frac{9-5}{4}-2\mathrm{t}.\frac{9-5}{64}+\mathrm{t}^2.\frac{3+\sqrt{5}}{16}+\frac{9-5}{256}\right)$$

$$=64\mathrm{t}^4-96\mathrm{t}^3+40\mathrm{t}^2-8\mathrm{t}+1$$

$$\therefore \boxed{64\mathrm{t}^4-96\mathrm{t}^3+40\mathrm{t}^2-8\mathrm{t}+1\overset{(****)}{=}64\left(\mathrm{t}^2-\mathrm{t}.\frac{3+\sqrt{5}}{4}+\frac{3+\sqrt{5}}{16}\right)\left(\mathrm{t}^2-\mathrm{t}.\frac{3-\sqrt{5}}{4}+\frac{3-\sqrt{5}}{16}\right)}$$

$$\therefore (***),(****)\Rightarrow \left(\mathrm{t}^2-\mathrm{t}.\frac{3+\sqrt{5}}{4}+\frac{3+\sqrt{5}}{16}\right)\left(\mathrm{t}^2-\mathrm{t}.\frac{3-\sqrt{5}}{4}+\frac{3-\sqrt{5}}{16}\right)=0$$

$$\Rightarrow \mathrm{t}^2-\mathrm{t}.\frac{3+\sqrt{5}}{4}+\frac{3+\sqrt{5}}{16}=0\ (\because \mathrm{t\ is\ } real)\Rightarrow \mathrm{t}=\frac{3+\sqrt{5}\pm\sqrt{2(\sqrt{5}+1)}}{8}$$

$$\Rightarrow \boxed{\mathrm{t}=\frac{3+\sqrt{5}+\sqrt{2(\sqrt{5}+1)}}{8}}\ \left(\because x<\frac{45^\circ}{4}\Rightarrow \mathrm{t}=\cos^2x>\cos^2\frac{45^\circ}{4}\right)$$

$$\cos 4x=2\cos^2 2x-1=2(2\cos^2x-1)^2-1=8\mathrm{t}^2-8\mathrm{t}+1$$

$$=\frac{14+6\sqrt{5}+2\sqrt{5}+2+(6+2\sqrt{5}).\sqrt{2(\sqrt{5}+1)}-24-8\sqrt{5}-8.\sqrt{2(\sqrt{5}+1)}+8}{8}$$

$$=\frac{(\sqrt{5}-1).\sqrt{2(\sqrt{5}+1)}}{4}\Rightarrow\frac{1}{\sqrt{2}\cos 4x}=\frac{2}{(\sqrt{5}-1).\sqrt{\sqrt{5}+1}}$$

$$=\frac{(\sqrt{5}-1)\sqrt{\sqrt{5}+1}.\sqrt{\sqrt{5}+1}}{2(\sqrt{5}-1).\sqrt{\sqrt{5}+1}}=\frac{\sqrt{\frac{\sqrt{5}+1}{2}}}{\sqrt{2}}=\frac{\sqrt{\varphi}}{\sqrt{2}}\overset{\text{via }(\bullet)}{\Rightarrow}\frac{m}{a}=\sqrt{\frac{\varphi}{2}}\Rightarrow\frac{AD}{BC}=\sqrt{\frac{\varphi}{2}}\ (QED)$$

899. In any Δ ABC holds:

$$\frac{a^3 s_b}{b s_a^3}+\frac{b^3 s_c}{c s_b^3}+\frac{c^3 s_a}{a s_c^3}\geq 4$$

Proposed by Marin Chirciu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{1}{am_a}\sum_{cyc}a^2\geq 2\sqrt{3}\Leftrightarrow\frac{1}{a^2m_a^2}\geq\frac{12}{\left(\sum_{cyc}a^2\right)^2}$$

$$\Leftrightarrow\left(\sum_{cyc}a^2\right)^2-3a^2(2b^2+2c^2-a^2)\geq 0$$

$$\Leftrightarrow\left(\sum_{cyc}a^2\right)^2-3a^2\left(2\sum_{cyc}a^2-3a^2\right)\geq 0\Leftrightarrow\left(\sum_{cyc}a^2\right)^2-6a^2\sum_{cyc}a^2+9a^4\geq 0$$

$$\Leftrightarrow\left(\sum_{cyc}a^2-3a^2\right)^2\geq 0\Leftrightarrow(b^2+c^2-2a^2)^2\geq 0\rightarrow\text{true}\Rightarrow am_a\overset{(\bullet)}{\leq}\frac{1}{2\sqrt{3}}\sum_{cyc}a^2$$

$$\text{Now, }\frac{a^3 s_b}{b s_a^3}+\frac{b^3 s_c}{c s_b^3}+\frac{c^3 s_a}{a s_c^3}=\sum_{cyc}\frac{\left(\frac{a}{s_a}\right)^3}{\frac{b}{s_b}}\overset{\text{Holder}}{\geq}\frac{\left(\sum_{cyc}\frac{a}{s_a}\right)^3}{3\sum_{cyc}\frac{b}{s_b}}=\frac{1}{3}\left(\sum_{cyc}\frac{a}{s_a}\right)^2\overset{?}{\geq}4$$

$$\Leftrightarrow\sum_{cyc}\frac{a}{s_a}\overset{?}{\geq}2\sqrt{3}\Leftrightarrow\sum_{cyc}\frac{a^3(b^2+c^2)}{2abc.am_a}\underset{(*)}{\overset{?}{\geq}}2\sqrt{3}$$

$$\text{Via }(\bullet)\text{ and analogs, }\sum_{cyc}\frac{a^3(b^2+c^2)}{2abc.am_a}\geq\sum_{cyc}\frac{a^3(b^2+c^2)}{2abc.\frac{1}{2\sqrt{3}}.\sum_{cyc}a^2}$$

$$\overset{A-G}{\geq}2\sqrt{3}.\sum_{cyc}\frac{a^3.2bc}{2abc.\sum_{cyc}a^2}=2\sqrt{3}.\frac{2abc.\sum_{cyc}a^2}{2abc.\sum_{cyc}a^2}=2\sqrt{3}\Rightarrow(*)\text{ is true}$$

$\therefore$ in any Δ ABC, $\frac{a^3 s_b}{bs_a^3}+\frac{b^3 s_c}{cs_b^3}+\frac{c^3 s_a}{as_c^3}\ge 4$, " = " iff Δ ABC is equilateral (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\ 4\sqrt{a^2+b^2+c^2}.m_a \overset{AM-GM}{\le} (a^2+b^2+c^2)+4{m_a}^2=3(b^2+c^2)$$

$$Then:\ s_a=\frac{2bc}{b^2+c^2}.m_a\le\frac{3bc}{2\sqrt{a^2+b^2+c^2}}\overset{AM-GM}{\le}\frac{\sqrt{3}bc}{2\sqrt[3]{abc}}\ (and\ analogs)$$

$$\Rightarrow\ s_a s_b s_c\le\frac{3\sqrt{3}}{8}.abc.$$

$$Therefore,\quad \frac{a^3 s_b}{b{s_a}^3}+\frac{b^3 s_c}{c{s_b}^3}+\frac{c^3 s_a}{a{s_c}^3}\overset{AM-GM}{\ge}3\sqrt[3]{\left(\frac{abc}{s_a s_b s_c}\right)^2}\ge 3\sqrt[3]{\left(\frac{8}{3\sqrt{3}}\right)^2}=4.$$

Equality holds iff ΔABC is equilateral.

Solution 3 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\ \frac{a^3 s_b}{b{s_a}^3}+\frac{b^3 s_c}{c{s_b}^3}+\frac{c^3 s_a}{a{s_c}^3}=\frac{\left(\frac{a^2}{{s_a}^2}\right)^2}{\frac{ab}{s_a s_b}}+\frac{\left(\frac{b^2}{{s_b}^2}\right)^2}{\frac{bc}{s_b s_c}}+\frac{\left(\frac{c^2}{{s_c}^2}\right)^2}{\frac{ca}{s_c s_a}}\ge$$

$$\overset{CBS}{\ge}\frac{\left(\frac{a^2}{{s_a}^2}+\frac{b^2}{{s_b}^2}+\frac{c^2}{{s_c}^2}\right)^2}{\frac{ab}{s_a s_b}+\frac{bc}{s_b s_c}+\frac{ca}{s_c s_a}}\ge\frac{a^2}{{s_a}^2}+\frac{b^2}{{s_b}^2}+\frac{c^2}{{s_c}^2}\overset{s_a\le m_a}{\ge}\frac{a^2}{{m_a}^2}+\frac{b^2}{{m_b}^2}+\frac{c^2}{{m_c}^2}.$$

Now if $a\ge b\ge c\ \Rightarrow\ m_a\le m_b\le m_c$ then by Chebyshev's inequality,

$$we\ have:\ \frac{a^2}{{m_a}^2}+\frac{b^2}{{m_b}^2}+\frac{c^2}{{m_c}^2}\ge\frac{1}{3}(a^2+b^2+c^2)\left(\frac{1}{{m_a}^2}+\frac{1}{{m_b}^2}+\frac{1}{{m_c}^2}\right)\ge$$

$$\overset{CBS}{\ge}\frac{1}{3}(a^2+b^2+c^2).\frac{3^2}{{m_a}^2+{m_b}^2+{m_c}^2}=\frac{3(a^2+b^2+c^2)}{\frac{3}{4}(a^2+b^2+c^2)}=4.$$

$$Therefore,\quad \frac{a^3 s_b}{b{s_a}^3}+\frac{b^3 s_c}{c{s_b}^3}+\frac{c^3 s_a}{a{s_c}^3}\ge 4.$$

$$Equality\ holds\ iff\ \Delta ABC\ is\ equilateral.$$

900. $In\ \ \Delta ABC\ the\ following\ relationship\ holds:$

$$\frac{b{m_a}^3}{a^3m_b}+\frac{c{m_b}^3}{b^3m_c}+\frac{a{m_c}^3}{c^3m_a}\ge\frac{9}{4}.$$

Proposed Daniel Sitaru-Romania

Solution 1 by Tapas Das-India

$$\sum_{cyc}\frac{bm_a^3}{a^3m_b}=\sum_{cyc}\frac{\left(\frac{m_a}{a}\right)^3}{\frac{m_b}{b}}\overset{Holder}{\ge}\frac{\left(\sum\frac{m_a}{a}\right)^3}{3\sum\left(\frac{m_a}{a}\right)}=\frac{1}{3}\left(\sum_{cyc}\frac{m_a}{a}\right)^2\ge$$

$$\ge\frac{1}{3}\left(\frac{3\sqrt{3}}{2}\right)^2=\frac{1}{3}\cdot\frac{27}{4}=\frac{9}{4}$$

$$\because m_a=\frac{1}{2}\sqrt{2(b^2+c^2)-a^2}\ (and\ analogs)$$

$$\sum_{cyc}\frac{m_a}{a}=\frac{1}{2}\sum_{cyc}\sqrt{2(a^2+b^2)-b^2}=\frac{1}{\sqrt{3}}\sum_{cyc}\frac{1}{\frac{2}{3}\cdot\sqrt{\frac{a^2}{2(b^2+c^2)-a^2}}}\ge$$

$$\ge\frac{1}{\sqrt{3}}\sum_{cyc}\frac{1}{\frac{1}{3}+\frac{a^2}{2(b^2+c^2)-a^2}}=\frac{\sqrt{3}}{2}\sum_{cyc}\frac{2b^2+2c^2-a^2}{2(a^2+b^2+c^2)}=\frac{3\sqrt{3}}{2}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\ \sum_{cyc}\frac{x^3}{y}=\sum_{cyc}\frac{x^4}{xy}\overset{CBS}{\ge}\frac{(x^2+y^2+z^2)^2}{xy+yz+zx}\ge x^2+y^2+z^2,\ \ \forall x,y,z>0.$$

$$For\ x=\frac{m_a}{a},\ \ y=\frac{m_b}{b},\ \ z=\frac{m_c}{c},\ \ we\ get:$$

$$\sum_{cyc}\frac{b{m_a}^3}{a^3m_b}\ge\sum_{cyc}\frac{{m_a}^2}{a^2}=\frac{1}{4}\sum_{cyc}\frac{2b^2+2c^2-a^2}{a^2}=\frac{1}{4}\left(2\sum_{cyc}\frac{b^2}{a^2}+2\sum_{cyc}\frac{c^2}{a^2}-3\right)\ge$$

$$\overset{AM-GM}{\geq} \frac{1}{4}(2.3+2.3-3)=\frac{9}{4},\ \ as\ desired.$$

$$Equality\ holds\ iff\ \Delta ABC\ is\ equilateral.$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru