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Problem. Prove

$$\int_0^\infty \frac{x^2\cosh(x)}{\sinh^2(x)}\,dx = 3\zeta(2).$$

Solution. We have

$$\begin{aligned}\int_0^\infty \frac{x^2\cosh(x)}{\sinh^2(x)}\,dx &= 2\int_0^\infty \frac{x^2\left(e^x+e^{-x}\right)}{\left(e^x-e^{-x}\right)^2}\,dx = 2\int_0^\infty \frac{x^2e^{-x}\left(1+e^{-2x}\right)}{\left(1-e^{-2x}\right)^2}\,dx\\ &= 2\int_0^\infty \frac{x^2e^{-x}}{\left(1-e^{-2x}\right)^2}\,dx + 2\int_0^\infty \frac{x^2e^{-3x}}{\left(1-e^{-2x}\right)^2}\,dx\end{aligned}$$

Now, using the expansion series $\dfrac{1}{(1-y)^2} = \displaystyle\sum_{n=0}^\infty ny^{n-1}$, we obtain

$$\begin{aligned}\int_0^\infty \frac{x^2\cosh(x)}{\sinh^2(x)}\,dx &= 2\int_0^\infty x^2e^{-x}\left(\sum_{n=0}^\infty ne^{-2(n-1)x}\right)dx + 2\int_0^\infty x^2e^{-3x}\left(\sum_{n=0}^\infty ne^{-2(n-1)x}\right)dx\\ &= 2\sum_{n=0}^\infty n\left(\int_0^\infty x^2e^{-(2n-1)x}\,dx\right) + 2\sum_{n=0}^\infty n\left(\int_0^\infty x^2e^{-(2n+1)x}\,dx\right)\end{aligned}$$

By the fact that for all $a>0$, $b>0$, $\displaystyle\int_0^\infty x^ae^{-bx}\,dx = \frac{\Gamma(a+1)}{b^{a+1}}$, we get

$$\begin{aligned}\int_0^\infty \frac{x^2\cosh(x)}{\sinh^2(x)}\,dx &= 2\sum_{n=0}^\infty n\frac{\Gamma(3)}{(2n-1)^3} + 2\sum_{n=0}^\infty n\frac{\Gamma(3)}{(2n+1)^3}\\ &= 4\sum_{n=0}^\infty \frac{n}{(2n-1)^3} + 4\sum_{n=0}^\infty \frac{n}{(2n+1)^3}\\ &= 4\sum_{n=1}^\infty \frac{n}{(2n-1)^3} + 4\sum_{n=0}^\infty \frac{n}{(2n+1)^3}\\ &= 4\sum_{n=0}^\infty \frac{n+1}{(2n+1)^3} + 4\sum_{n=0}^\infty \frac{1}{(2n+1)^3}\\ &= 4\sum_{n=0}^\infty \frac{2n+1}{(2n+1)^3} = 4\sum_{n=0}^\infty \frac{1}{(2n+1)^2}\\ &= 4\left(\frac{3}{4}\zeta(2)\right) = 3\zeta(2).\end{aligned}$$