NEW INEQUALITIES IN TRIANGLES

D.M. BĂTINEŢU-GIURGIU, CLAUDIA NĂNUŢI, FLORICĂ ANASTASE-ROMANIA

Abstract: *In this paper are presented new inequalities in triangles.*

Let $n \in \mathbb{N}^*, n \geq 2$ and triangles $T_k = A_kB_kC_k$ with areas F_k, semiperimeter s_k, lengths sides a_k, b_k, c_k and circumradii $R_k, k = \overline{1,n}$.

Theorem 1. If $x, y, z > 0$ then:

$$\prod_{cyc}\Big(\frac{(x+y)^2}{z^2 \cdot \sqrt[n]{(a_1a_2...a_n)^2}} + t\Big) \geq \frac{9t^2}{\sqrt[n]{R_1^2R_2^2...R_n^2}} \tag{1}$$

Proof. Let be $u, v, w > 0$, then:

$$(u^2+t)(v^2+t) \geq \frac{3}{4}t((u+v)^2+t) \text{ and} \tag{2}$$

$$(v^2+t)(w^2+t) \geq t(v+w)^2 \tag{3}$$

We have:

$$(u^2+t)(v^2+t) \geq \frac{3}{4}t((u+v)^2+t) \Leftrightarrow$$
$$4u^2v^2 + 4t(u^2+v^2) + 4t^2 \geq 3t(u^2+v^2) + 6tuv + 3t^2 \Leftrightarrow$$
$$4u^2v^2 - 4tuv + t^2 + t(u^2+v^2-2uv) \geq 0 \Leftrightarrow$$
$$(2uv-t)^2 + t(u-v)^2 \geq 0$$

Equality holds for $2uv = t$ and $u = v$, and hence:

$$(v^2+t)(w^2+t) \geq t(v+w)^2 \Leftrightarrow$$
$$v^2w^2 + t(v^2+w^2) + t^2 \geq t(v^2+w^2) \Leftrightarrow$$
$$v^2w^2 - 2tvw + t^2 \geq 0 \Leftrightarrow (vw-t)^2 \geq 0$$

Equality holds for $2uv = t$ and $u = v$. Therefore,

$$(u^2+t)(v^2+t)(w^2+t) \overset{(2)}{\geq} \frac{3}{4}t((u+v)^2+t)(w^2+t) \overset{(3)}{\geq} \frac{3}{4}t^2(u+v+w)^2 \tag{4}$$

If in (4) we take:

$$u = \frac{x+y}{z \cdot \sqrt[n]{a_1a_2...a_n}}; v = \frac{y+z}{x \cdot \sqrt[n]{b_1b_2...b_n}}; w = \frac{z+x}{y \cdot \sqrt[n]{c_1c_2...c_n}}$$

$$\prod_{cyc}\Big(\frac{(x+y)^2}{z^2 \cdot \sqrt[n]{(a_1a_2...a_n)^2}} + t\Big) \geq \frac{3}{4}t^2\prod_{cyc}\Big(\frac{x+y}{z \cdot \sqrt[n]{a_1a_2...a_n}}\Big)^2 \tag{5}$$

In [1]. has proved:

$$\sum_{cyc} \frac{x+y}{z \cdot \sqrt[n]{a_1 a_2 ... a_n}} \geq \frac{2\sqrt{3}}{\sqrt[n]{R_1^2 R_2^2 ... R_n^2}} \tag{6}$$

From (5) and (6), it follows:

$$\prod_{cyc} \left(\frac{(x+y)^2}{z^2 \cdot \sqrt[n]{(a_1 a_2 ... a_n)^2}} + t \right) \geq \frac{3}{4} t^2 \left(\frac{2\sqrt{3}}{\sqrt[n]{R_1^2 R_2^2 ... R_n^2}} \right)^2$$

$$\prod_{cyc} \left(\frac{(x+y)^2}{z^2 \cdot \sqrt[n]{(a_1 a_2 ... a_n)^2}} + t \right) \geq \frac{9t^2}{\sqrt[n]{R_1^2 R_2^2 ... R_n^2}}$$

□

Theorem 2. If $m, x, y, z \in [1, \infty)$ and $x + y + z = 3m$, then:

$$\prod_{cyc} \left(\frac{(x^x + y^x + z^x)^2}{\sqrt[n]{a_1 a_2 ... a_n)^2}} + t \right) \geq \frac{81}{4} \cdot \frac{m^{2m}}{\sqrt[n]{R_1^2 R_2^2 ... R_n^2}} \tag{7}$$

Proof. If in (4) we take:

$$u = \frac{x^x + y^x + z^x}{\sqrt[n]{a_1 a_2 ... a_n}}; v = \frac{x^y + y^y + z^y}{\sqrt[n]{b_1 b_2 ... b_n}}; w = \frac{x^z + y^z + z^z}{\sqrt[n]{c_1 c_2 ... c_n}}$$

we get:

$$\prod_{cyc} \left(\frac{(x^x + y^x + z^x)^2}{\sqrt[n]{a_1 a_2 ... a_n)^2}} + t \right) \geq \frac{3}{4} t^2 \left(\sum_{cyc} \frac{x^x + y^x + z^x}{\sqrt[n]{a_1 a_2 ... a_n}} \right)^2 \tag{8}$$

In [1]. has proved:

$$\sum_{cyc} \frac{x^x + y^x + z^x}{\sqrt[n]{a_1 a_2 ... a_n}} \geq \frac{3\sqrt{3} \cdot m^m}{\sqrt[n]{R_1 R_2 ... R_n}} \tag{9}$$

Therefore,

$$\prod_{cyc} \left(\frac{(x^x + y^x + z^x)^2}{\sqrt[n]{a_1 a_2 ... a_n)^2}} + t \right) \geq \frac{3}{4} t^2 \left(\frac{3\sqrt{3} \cdot m^m}{\sqrt[n]{R_1 R_2 ... R_n}} \right)^2$$

$$\prod_{cyc} \left(\frac{(x^x + y^x + z^x)^2}{\sqrt[n]{a_1 a_2 ... a_n)^2}} + t \right) \geq \frac{81}{4} \cdot \frac{m^{2m}}{\sqrt[n]{R_1^2 R_2^2 ... R_n^2}}$$

□

References

[1] D.M. Bătineţu-Giurgiu, Claudia Nănuţi-ABOUT FEW INEQUALITIES IN TRIANGLES, R.M.M.-38, Autumn Edition 2023, pages 20-22