A SIMPLE PROOF FOR BOHR'S INEQUALITY AND APPLICATIONS

BY DANIEL SITARU-ROMANIA

Abstract: *In this paper its proved Bohr's inequality and are presented a few applications.*

BOHR'S INEQUALITY:

If $z_1, z_2, ..., z_n \in \mathbb{C}, n \in \mathbb{N}, n \geq 2, a_1, a_2, ..., a_n > 0$ and $\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n} = \frac{1}{a}$ then:

(1) $\boldsymbol{a|z_1 + z_2 + ... + z_n|^2 \leq a_1|z_1|^2 + a_2|z_2|^2 + ... + a_n|z_n|^2}$

Proof.

We will use the mathematical induction:

For $n = 2 : a_1, a_2 > 0; z_1, z_2 \in \mathbb{C}$ we must prove that:

$$a|z_1 + z_2|^2 \leq a_1|z_1|^2 + a_2|z_2|^2;\ \frac{1}{a_1} + \frac{1}{a_2} = \frac{1}{a}$$

$$\text{Denote: } \frac{1}{a_1} = u; \frac{1}{a_2} = v; \frac{1}{a} = u + v$$

$$z_1 = x + iy; z_2 = p + iq; x, y, p, q \in \mathbb{R}$$

$$\frac{|z_1 + z_2|^2}{\frac{1}{a}} \leq \frac{|z_1|^2}{\frac{1}{a_1}} + \frac{|z_2|^2}{\frac{1}{a_2}}$$

$$\frac{|z_1 + z_2|^2}{u + v} \leq \frac{|z_1|^2}{u} + \frac{|z_2|^2}{v}$$

$$uv|z_1 + z_2|^2 \leq (u + v)v|z_1|^2 + (u + v)u|z_2|^2$$

$$uv|x + iy + p + iq|^2 \leq (uv + v^2)|x + iy|^2 + (uv + u^2)|p + iq|^2$$

$$uv((x + p)^2 + (y + q)^2) \leq (uv + v^2)(x^2 + y^2) + (uv + u^2)(p^2 + q^2)$$

$$uvx^2 + uvp^2 + 2uvpx + uvy^2 + uvq^2 + 2uvuq \leq$$

$$\leq uvx^2 + uvy^2 + x^2v^2 + v^2y^2 + uvp^2 + uvq^2 + u^2p^2 + u^2q^2$$

$$x^2v^2 + v^2y^2 + u^2p^2 + u^2q^2 - 2uvxp - 2uvyq \geq 0$$

$$(xv - up)^2 + (yv - uq)^2 \geq 0$$

Suppose that $P(n)$ is true:

$$z_1, z_2, ..., z_n \in \mathbb{C}, n \in \mathbb{N}, n \geq 2, a_1, a_2, ..., a_n > 0$$

$$\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_n}=\frac{1}{a}$$

$$a|z_1+z_2+...+z_n|^2 \le a_1|z_1|^2+a_2|z_2|^2+...+a_n|z_n|^2$$

$$\frac{|z_1+z_2+...+z_n|^2}{\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_n}} \le \frac{|z_1|^2}{\frac{1}{a_1}}+\frac{|z_2|^2}{\frac{1}{a_2}}+...+\frac{|z_n|^2}{\frac{1}{a_n}}$$

$$P(n+1): \frac{|z_1+z_2+...+z_n+z_{n+1}|^2}{\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_n}+\frac{1}{a_{n+1}}} \le \frac{|z_1|^2}{\frac{1}{a_1}}+\frac{|z_2|^2}{\frac{1}{a_2}}+...+\frac{|z_n|^2}{\frac{1}{a_n}}+\frac{|z_{n+1}|^2}{\frac{1}{z_{n+1}}}$$

Denote: $u=\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_n}; v=\frac{1}{a_{n+1}}; w_1=z_1+z_2+...+z_n; w_2=z_{n+1}$

$$\frac{|z_1+z_2+...+z_n+z_{n+1}|^2}{\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_n}+\frac{1}{a_{n+1}}}=\frac{|w_1+w_2|^2}{u+v} \overset{P(2)}{\le}$$

$$\le \frac{|w_1|^2}{u}+\frac{|w_2|^2}{v}=\frac{|z_1+z_2+...+z_n|^2}{\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_n}}+\frac{|z_{n+1}|^2}{\frac{1}{a_{n+1}}} \overset{P(n)}{\le}$$

$$\le \frac{|z_1|^2}{\frac{1}{a_1}}+\frac{|z_2|^2}{\frac{1}{a_2}}+...+\frac{|z_n|^2}{\frac{1}{a_n}}+\frac{|z_{n+1}|^2}{\frac{1}{a_{n+1}}}$$

$$P(n) \longrightarrow P(n+1)$$

For $n=2$ the inequality of Bohr its equivalent with Bergstrom's inequality:

$$\frac{|z_1+z_2|^2}{\frac{1}{a_1}+\frac{1}{a_2}} \le \frac{|z_1|^2}{\frac{1}{a_1}}+\frac{|z_2|^2|}{\frac{1}{a_2}}; a_1,a_2>0; z_1,z_2\in\mathbb{C}$$

Equality holds for $\frac{z_1}{a_2}=\frac{z_2}{a_1}$.

Corollary 1:

If $z\in\mathbb{C}, a,b>0$ then:

$$\left(\frac{1}{a}+\frac{1}{b}\right)|1+z|^2 \le \frac{1}{a}+\frac{1}{b}|z|^2$$

Proof.

We take in (1): $n=2; z_1=1; z_2=z$.

Corollary 2:

If $z\in\mathbb{C}, a,b>0$ then:

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)|1+z+z^2|^2 \le \frac{1}{a}+\frac{1}{b}|z|^2+\frac{1}{c}|z|^4$$

Proof.

We take in (1): $n=3; z_1=1, z_2=z, z_3=z^2$.

Corollary 3:

If $z_1, z_2, z_3 \in \mathbb{C}$ then:

$$|z_1 + z_2 + z_3|^2 \leq 3(|z_1|^2 + |z_2|^2 + |z_3|^2)$$

Proof.

We take in (1): $n = 3; a_1 = a_2 = a_3 = 3.$

Corollary 4:

If $z_1, z_2, z_3, z_4 \in \mathbb{C}$ then:

$$|z_1 + z_2 + z_3 + z_4|^2 \leq 4(|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2)$$

Proof.

We take in (1): $n = 3; a_1 = a_2 = a_3 = a_4 = 4.$

References

[1] ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro