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# R M M

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**DANIEL SITARU**

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## PROBLEMS FOR JUNIORS

**JP.481.** For  $x, y, z \in (0, 1)$ ,  $x + y + z = 1$  prove that:

$$\sum_{cyc} \frac{y+z}{x+z} \cdot \frac{x+2y-3xy}{x+2z-3xz} \geq 3$$

*Proposed by Florică Anastase - Romania*

**JP.482.** If  $x, y, z \in [0, \frac{\pi}{2}]$  then:

$$(\sin x)^{\cos^2 x} \cdot (\cos x)^{\sin^2 x} + (\sin y)^{\cos^2 y} \cdot (\cos y)^{\sin^2 y} + \\ + (\sin z)^{\cos^2 z} \cdot (\cos z)^{\sin^2 z} \leq \frac{3\sqrt{2}}{2}$$

*Proposed by Daniel Sitaru - Romania*

**JP.483.** If  $0 < a, b, c \leq 1$  then:

$$(a+b-ab) \cdot a^b + (b+c-bc) \cdot b^c + (c+a-ca) \cdot c^a \geq a+b+c$$

*Proposed by Daniel Sitaru - Romania*

**JP.484.** If  $a, b, c > 0$ , then:

$$\frac{(4a^2+3)(4b^2+3)}{(a+b+1)^2} + \frac{(4b^2+3)(4c^2+3)}{(b+c+1)^2} + \frac{(4c^2+3)(4a^2+3)}{(c+a+1)^2} \geq 12$$

*Proposed by Daniel Sitaru - Romania*

**JP.485.** If  $x, y, z > 0$ ,  $x + y + z = 3$  then:

$$\frac{x^2 + 6xy + y^2}{\sqrt{xy}} + \frac{y^2 + 6yz + z^2}{\sqrt{yz}} + \frac{z^2 + 6zx + x^2}{\sqrt{xz}} \geq 24$$

*Proposed by Daniel Sitaru - Romania*

**JP.486.** If  $a, b, c > 0$  with  $\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} = \frac{64}{9}$ , then prove that:

$$\frac{a^4 + b^3}{a^3 + b^3} + \frac{b^4 + c^3}{b^3 + c^3} + \frac{c^4 + a^3}{c^3 + a^3} \geq \frac{21}{8}$$

*Proposed by Titu Zvonaru - Romania*

**JP.487.** In acute  $\Delta ABC$ ,  $A'$ ,  $B'$ ,  $C'$  are the contact points by the altitudes with the circle circumscribed of the triangle  $ABC$ . Prove that:

$$\sqrt[3]{(BA' + A'C)(CB' + B'A)(AC' + C'B)} \geq 4r$$

*Proposed by Marian Ursărescu - Romania*

**JP.488.** In  $\Delta ABC$  the following relationship holds:

$$\sum_{cyc} \frac{a+b}{ab} \cdot h_c \geq \frac{a+b+c}{R}$$

*Proposed by Marian Ursărescu - Romania*

**JP.489.** If  $a, b, c > 0$ ,  $a + b + c = 3$  then:

$$\frac{(5a+b)(5a+4b)}{8a+b} + \frac{(5b+c)(5b+4c)}{8b+c} + \frac{(5c+a)(5c+4a)}{8c+a} \geq 18$$

*Proposed by Daniel Sitaru - Romania*

**JP.490.** If  $0 < x, y, z \leq \frac{\pi}{2}$  then:

$$\frac{(1 + \sin^2 x)^2 (1 + \sin^2 y)^2 (1 + \sin^2 z)^2}{(\sin^2 x + \sin^2 y)(\sin^2 y + \sin^2 z)(\sin^2 z + \sin^2 x)} \geq 8$$

*Proposed by Daniel Sitaru - Romania*

**JP.491.** If  $A, B, C$  are the angles of a triangle, solve in real numbers  $x, y, z$  the system:

$$\begin{cases} \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \sqrt{xy + yz + zx} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2} \\ \frac{xyz}{(x+y)(y+z)(z+x)} = \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \end{cases}$$

*Proposed by Cristian Miu - Romania*

**JP.492.** In  $\Delta ABC$  the following relationship holds:

$$2F \leq 2Rr + (3\sqrt{3} - 4)r^2 + \frac{(4R+r)(2R+(3\sqrt{3}-4)r)}{9}$$

*Proposed by Tran Quoc Anh - Vietnam*

**JP.493.** Find  $x, y, z > 0$  such that  $x + y + z = 9$  and

$$\frac{x^3 + y^3}{(x+y)^3} + \frac{y^3 + z^3}{(y+z)^3} + \frac{z^3 + x^3}{(z+x)^3} = \frac{4}{3}$$

*Proposed by Daniel Sitaru - Romania*

**JP.494.** If  $a, b, c > 0, a + b + c = 3$  then:

$$\frac{ab + bc + ca}{(a+b)(b+c)(c+a)} \leq \frac{3}{8}$$

*Proposed by Daniel Sitaru - Romania*

**JP.495.** If  $t > 0$  then in any  $\Delta ABC$  triangle with the area  $F$  the following inequality holds:

$$(r_a^2 + t)(r_b^2 + t)(r_c^2 + t) \geq \frac{27\sqrt{3}}{4}t^2F$$

*Proposed by D.M. Bătinețu-Giurgiu - Romania*

## PROBLEMS FOR SENIORS

**SP.481.** If  $x, y, z > 0$  then in  $\Delta ABC$  the following relationship holds:

$$\frac{xa^2}{r_a} + \frac{yb^2}{r_b} + \frac{cz^2}{r_c} \geq \frac{2\sqrt{3}(xy + yz + zx)}{2R - r}$$

*Proposed by D.M. Bătinețu-Giurgiu - Romania*

**SP.482.** In  $\Delta ABC, D, E \in (BC)$  such that  $[BD] \equiv [EC]$ . Prove that:

$$AD^2 + DE^2 + EA^2 + 1.5(BC^2 - DE^2) \geq 2(s^2 + r^2 - 5Rr)$$

*Proposed by Gheorghe Molea - Romania*

**SP.483.** If  $x, y, z > 0, x^2 + y^2 + z^2 = \sqrt{3}$ , then:

$$\sqrt{x^4 + x^2y^2 + y^4} + \sqrt{y^4 + y^2z^2 + z^4} + \sqrt{z^4 + z^2x^2 + x^4} \geq 3$$

*Proposed by Daniel Sitaru - Romania*

**SP.484.** If  $a, b, c > 0$  and  $a^x + 2b^x + 3c^x \leq a + 2b + 3c, (\forall)x \in \mathbb{R}$ , then:

$$a^a \cdot b^{2b} \cdot c^{3c} = 1$$

*Proposed by Daniel Sitaru - Romania*

**SP.485.** If  $a, b, c, d > 0$  and  $a^{\log x} + b^{\log x} + c^{\log x} + d^{\log x} \geq 4$ ;  $(\forall)x \in (0, \infty)$ , then:  $abcd = 1$ .

*Proposed by Daniel Sitaru - Romania*

**SP.486.** If  $A, B, C \in M_3(\mathbb{R})$  are such that:

$AB = BA = BC = CB = CA = AC = O_3$ , then:

$$\det(I_3 + 3A + 4B + 5C + 9A^2 + 16B^2 + 25C^2) \geq 0$$

*Proposed by Daniel Sitaru - Romania*

**SP.487.** Find:

$$\Omega = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n^{2a} + \sqrt[3]{(k+1)^2(k^2+1)^2}}, \quad a \in \mathbb{R}, a > 0$$

*Proposed by Florică Anastase, Ionuț Bină - Romania*

**SP.488.** Let  $A \in M_2(\mathbb{R})$  invertible such that  $\det(A^2 - 2A + 2I_2) = 0$ .  
Find  $\text{Tr}(A^{-1})$ .

*Proposed by Marian Ursărescu - Romania*

**SP.489.** Let  $p, q \in \mathbb{N}^*$  - odd numbers with  $p \neq q$ . Find all functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  continuous such that  $f(\sqrt[p]{px + qy}) = f(\sqrt[q]{qx + py})$ ,  $\forall x, y \in \mathbb{R}$ .

*Proposed by Marian Ursărescu - Romania*

**SP.490.** If  $x, y, z, t > 0$ ,  $x+y+z+t = 4$  then:  $xy + yz + zt + tx \leq 4$ .

*Proposed by Daniel Sitaru - Romania*

**SP.491.** If  $x \geq 0$  then:

$$(x+1)^{x+1} \cdot (x^2+1)^{x^2+1} \leq e^{x^2+x} \cdot \sqrt{e^{x^4+x^2}}$$

*Proposed by Daniel Sitaru - Romania*

**SP.492.** If  $ABC$ , is a triangle,  $\omega$  - Brocard's point and  $x = \sin 2A + \sin 2B$ ,  $y = \sin 2B + \sin 2C$ ,  $z = \sin 2C + \sin 2A$ , then prove that:

$$\frac{1}{3} \min\{x, y, z\} \leq \tan \omega \leq \frac{1}{3} \max\{x, y, z\}$$

*Proposed by Cristian Miu - Romania*

**SP.493.** If  $a_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}}$ ;  $n \geq 1$  and

$$\lim_{n \rightarrow \infty} \frac{a_n}{\sqrt{n+1}} \cdot e^{2\sqrt{n}} = x > 0$$

then find:

$$\Omega(x) = \lim_{n \rightarrow \infty} (e^{a_{n+1}} - e^{a_n}) \cdot x_n$$

*Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu - Romania*

**SP.494.** Let  $ABC$  be a triangle with inradius  $R$  and circumradius  $R$ . Prove that:

$$3\left(\frac{R}{2r}\right)^{-1} \leq \sum_{cyc} \frac{(x+y)\sin A}{x\sin B + y\sin C} \leq 3\left(\frac{R}{2r}\right), x, y > 0$$

Proposed by George Apostolopoulos - Greece

**SP.495.** If  $0 \leq a \leq b$  then:

$$\int_a^b \frac{1 + \tan^3 x}{\sqrt{1 - \tan x + \tan^2 x}} dx \geq \tan b - \tan a$$

Proposed by Daniel Sitaru - Romania

## UNDERGRADUATE PROBLEMS

**UP.481.** Let  $t \geq 0$  and  $(a_n)_{n \geq 1}$  sequence of real numbers strictly positive such that

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n \cdot n^{t+1}} = a > 0$$

Find:

$$\Omega(a, t) = \lim_{n \rightarrow \infty} \frac{\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}}{(\sqrt[n]{(2n-1)!!})^t}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru - Romania

**UP.482.** Let  $f : \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$  a continuous function such that

$$\lim_{x \rightarrow \infty} \frac{f(x+1)}{xf(x)} = a > 0$$

and exist

$$\lim_{x \rightarrow \infty} \frac{(f(x))^{\frac{1}{x}}}{x}.$$

Find:

$$\Omega(a) = \lim_{x \rightarrow \infty} \left( (x+1)^2 \cdot (f(x+1))^{-\frac{1}{x+1}} - x^2 \cdot (f(x))^{-\frac{1}{x}} \right)$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru - Romania

**UP.483.** Prove that:

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{8k^2 + 4k + 1} = \frac{1}{2} \mathcal{T} \left( \psi \left( \frac{1+i}{8} \right) - \psi \left( \frac{1+i}{4} \right) \right)$$

Proposed by Fao Ler - Iraq

**UP.484.** Find:

$$\Omega = \int_0^\infty \frac{x\sqrt{x} \log x}{x^4 + x^2 + 1} dx$$

Proposed by Vasile Mircea Popa - Romania

**UP.485.** Find:

$$\Omega = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\cos^{-1} x}{\sqrt{8x^4 - 6x^2 + 1}} dx$$

Proposed by Vasile Mircea Popa - Romania

**UP.486.** If  $0 < a \leq b \leq 1 \leq c \leq d$  then:

$$\begin{aligned} & \tan^{-1} a + \tan^{-1} b + \tan^{-1} c + \tan^{-1} d \leq \\ & \leq 3 \tan^{-1} \left( \frac{a+b+1}{3} \right) + \tan^{-1}(c+d-1) \end{aligned}$$

Proposed by Daniel Sitaru - Romania

**UP.487.** Prove that:

$$\Omega = - \int_0^1 \frac{t \log^3 t}{(1+t)^2} dt = \frac{7\pi^4}{120} - \frac{9}{2}\zeta(3)$$

Proposed by Said Attaoui - Algerie

**UP.488.** If  $0 < a \leq b$  then:

$$\left( \int_a^b \frac{x^2 + 1}{x^3 + 1} dx \right) \left( \int_a^b \frac{\sqrt{x}}{x^3 + 1} dx \right) \leq 2(\sqrt{b} - \sqrt{a})(\tan^{-1} b - \tan^{-1} a)$$

Proposed by Daniel Sitaru - Romania

**UP.489.** Find:

$$\Omega = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{\left[ \sum_{k=1}^n \sqrt[2k+1]{\frac{2k+1}{2k-1}} \right] + \sqrt[3]{(i+1)^2(i^2+1)^2}},$$

 $a \in \mathbb{R}, a > 0, [*]$  - GIF

Proposed by Florică Anastase - Romania

**UP.490.** Find:

$$\Omega = \lim_{n \rightarrow \infty} \left( \prod_{i=1}^{2n} e^{(-1)^k k!(2n-k)!} \right)^{\frac{1}{(2n)!}}$$

Proposed by Florică Anastase, Ionuț Bină - Romania

**UP.491.** In  $\Delta ABC$ ,  $a, b, c$  - sides and  $h_a, h_b, h_c$  - altitudes. If

$$\Omega(x, \alpha) = \int \frac{\cos \alpha dx}{\sin x - \sin \alpha} + \int \frac{\sin \alpha dx}{\cos x - \cos \alpha}$$

then prove:

$$e^{\Omega(h_a^2, a)} + e^{\Omega(h_b^2, b)} + e^{\Omega(h_c^2, c)} \geq s \left( \frac{F}{R} + 1 \right)$$

*Proposed by Florică Anastase - Romania*

**UP.492.** Let  $(a_n)_{n \geq 1}$  be sequence of real numbers strictly positive such that:

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{n^2 a_n} = a > 0.$$

Find:

$$\Omega = \lim_{n \rightarrow \infty} \left( \sqrt[n+1]{\frac{a_{n+1}}{(n+1)!}} - \sqrt[n]{\frac{a_n}{n!}} \right)$$

*Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru - Romania*

**UP.493.** Let  $t \geq 0$  and  $(a_n)_{n \geq 1}, (b_n)_{n \geq 1}$  sequences of real numbers strictly positive such that

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n \cdot n^{t+1}} = a > 0, \lim_{n \rightarrow \infty} \frac{b_{n+1}}{b_n \cdot n^t} = b > 0.$$

Find:

$$\Omega(a, b, t) = \lim_{n \rightarrow \infty} \frac{\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}}{\sqrt[n]{b_n}}$$

*Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru - Romania*

**UP.494.** If  $f, g : \mathbb{R} \rightarrow (0, \infty)$ ,  $f, g$  -derivable,  $f', g'$  - continuous,  $0 < a \leq b$  then:

$$16 \int_a^b (f'(x) + g'(x))(f^3(x) + g^3(x)) dx \geq (f(b) + g(b))^4 - (f(a) + g(a))^4$$

*Proposed by Daniel Sitaru - Romania*

**UP.495.** Find a closed form:

$$\Omega = \sum_{n=1}^{\infty} \frac{7^n}{7^{2n} + 40 \cdot 7^n + 175}$$

*Proposed by Daniel Sitaru - Romania*

MATHEMATICS DEPARTMENT, "THEODOR COSTESCU" NATIONAL ECONOMIC, COLLEGE DROBETA TURNU - SEVERIN, ROMANIA