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HUYGENS' AND WİLKER'S INEQUALITIES REVISITED

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ABSTRACT: This paper introduces some refinements of Huygens' inequality and a generalization of Wilker's inequality

Keywords: Huygens' inequality,Wilker's inequality

1. Introduction

Hugyens and Wilker's type inequalities are very useful in analysis. For example, we can say approximations of trigonometric functions with linear functions. More precise approximations can be made using the inequalities in this article.

The Huygens' inequality

$$\frac{2\sin x}{x}+\frac{\tan x}{x}>3, \qquad 0<x<\frac{\pi}{2}. \tag{1.1}$$

Or:

$$2\sin x+\tan x\geq 3x, \quad 0\leq x<\frac{\pi}{2}. \tag{1.2}$$

The Mitrinovic-Adamovic's inequality

$$\frac{\sin x}{x}>\sqrt[3]{\cos x}, \qquad 0<x<\frac{\pi}{2}. \tag{1.3}$$

And Wilker's inequality

$$\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}>2 \qquad 0<x<\frac{\pi}{2}. \tag{1.4}$$

We can see the proof of these inequalities in [1-4].

2. Preliminiaries and some useful lemmas:

Lemma 1 [5]. (AM-GM inequality) If $a_i > 0, i = \overline{1,n}$ then the following inequality is satisfied

$$\frac{\sum_{i=1}^{n} a_i}{n} \geq \left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}}, \qquad n \in N^* \tag{2.1}$$

Lemma 2 [6]. If $a_i > 0, i = \overline{1,n}$ then the following inequality is satisfied

$$\left(\sum_{i=1}^{n}(a_i + b_i)^2\right) \geq 4\left(\sum_{i=1}^{n}\left(\sqrt{a_i b_i}\right)\right)\left(\sum_{i=1}^{n}\left(\sqrt{\frac{a_i^2+b_i^2}{2}}\right)\right) \tag{2.2}$$

Lemma 3 If $0 \leq x < \frac{\pi}{2}$ then the following inequalities are satisfied

$$\sin x \leq x \leq \tan x \tag{2.3}$$

Proof: Let f: $R^+ \to R$ be a function defined by $f(x) = x - \sin x$, then $f'(x) = 1 - \cos x \geq 0$ for all $0 \leq x < \frac{\pi}{2}$.We obtain $f(x)$ is an increasing function on $[0, \frac{\pi}{2})$.Then we obtain $f(x) \geq 0$. This show that: $x - \sin x \geq 0$.Also we can show $x \leq \tan x$ in the same way.

Main result:

Theorem 1. If $0 \leq x < \frac{\pi}{2}$ then the following inequality is satisfied

$$2\sin x + \tan x \geq \sin x\left(1 + 2\left(\sqrt[4]{\frac{1+\cos^2 x}{2\cos^3 x}}\right)\right) \geq \sin x\left(1 + \frac{2}{\sqrt{\cos x}}\right) \geq 3\frac{\sin x}{\sqrt[3]{\cos x}} \geq 3x \tag{3.1}$$

Proof: i) From (1.3) we obtain that: $\frac{\sin x}{\sqrt[3]{\cos x}} \geq x$

ii) Now we show: $\sin x\left(1 + \frac{2}{\sqrt{\cos x}}\right) \geq 3\frac{\sin x}{\sqrt[3]{\cos x}}$.

When $x = 0$, right equals left $(0 = 0)$.We assume that $x \neq 0$. Then we apply Lemma 2,we obtain

$$1 + \frac{2}{\sqrt{\cos x}} = 1 + \frac{1}{\sqrt{\cos x}} + \frac{1}{\sqrt{\cos x}} \geq 3\frac{\sin x}{\sqrt[3]{\cos x}}$$

Now if we multiply the right and left of the inequality by sin x, ii) is proved.

iii) $2\sin x+\tan x\geq \sin x\left(1+2\left(\sqrt[4]{\frac{1+cos^2x}{2cos^3x}}\right)\right)$

When $x=0$, right equals left $(0=0)$.We assume that x ≠ 0. Then we apply Lemma 1,we obtain

$$2\sin x+\tan x=\sin x+\sin x+\tan x\geq \sin x+2\left(\sqrt[4]{\frac{\sin x\tan x(sin^2x+tan^2x)}{2}}\right)$$

$$=\sin x\left(1+2\left(\sqrt[4]{\frac{1+cos^2x}{2cos^3x}}\right)\right).$$ iii) is proved.

iiii) $\sin x\left(1+2\left(\sqrt[4]{\frac{1+cos^2x}{2cos^3x}}\right)\right)\geq \sin x\left(1+\frac{2}{\sqrt{\cos x}}\right)$

We apply Lemma 1 we get: $1+cos^2x\geq 2\cos x$.This show that iiii) is true.The proof of Theorem is complete.

Theorem 2. If $0<x<\frac{\pi}{2}$ and $s_n=\left(\frac{\sin x}{x}\right)^n+\left(\frac{\tan x}{x}\right)^{n-1}$ $,n\epsilon N\ and\ n\geq 2$ then the following inequality is satisfied

1) $\forall n\,, s_n>2$

2) $f>p, s_f>s_p$

Proof: Let's prove the theorem by mathematical induction.If $n=2$ we obtain (1.4) this show inequality is true for $n=2$.Now we show for $n=3$

If we apply Lemma 3, Lemma 1 and (1.4) respectively, we get

$$\left(\frac{\sin x}{x}\right)^3+\left(\frac{\tan x}{x}\right)^2=\frac{sin^3x+x\,tan^2x}{x^3}>\frac{sin^3x+\sin x\,tan^2x}{x^3}\geq\frac{2sin^3x\frac{1}{\cos x}}{x^3}>2.$$

Suppose the inequality is true for $n=k$.Then we must show $for\ n=k+1$.

If we apply (2.3) we obtain

$$\left(\frac{\sin x}{x}\right)^{n-1} > \left(\frac{\sin x}{x}\right)^{n} \text{ and } \frac{\tan x}{x} > \frac{\sin x}{x}$$

Also (1.1) show that

$$\frac{\tan x}{x} + \frac{\sin\ x}{x} - 2 > 0$$

Then we get:

$$s_{n+1} - s_n = \left(\frac{\sin x}{x}\right)^{n+1} + \left(\frac{\tan x}{x}\right)^{n} - \left(\frac{\sin x}{x}\right)^{n} - \left(\frac{\tan x}{x}\right)^{n-1} =$$

$$= \left(\frac{\tan x}{x}\right)^{n-1}\left[\frac{\tan x}{x} - 1\right] - \left(\frac{\sin x}{x}\right)^{n}\left[1 - \frac{\sin\ x}{x}\right] \geq$$

$$\geq \left(\frac{\tan x}{x}\right)^{n-1}\left[\frac{\tan x}{x} - 1\right] - \left(\frac{\sin x}{x}\right)^{n-1}\left[1 - \frac{\sin\ x}{x}\right] \geq$$

$$\geq \left(\frac{\tan x}{x}\right)^{n-1}\left[\frac{\tan x}{x} - 1\right] - \left(\frac{\tan x}{x}\right)^{n-1}\left[1 - \frac{\sin\ x}{x}\right] =$$

$$= \left(\frac{\tan x}{x}\right)^{n-1}\left[\frac{\tan x}{x} + \frac{\sin\ x}{x} - 2\right] > 0.$$

With this we proved both 1) and 2) . The proof of Theorem is complete.

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