Integrals involving $\zeta(2)$ and $\zeta(4)$.

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$$J=\int_0^\infty\int_0^\infty \frac{(xt)^2e^{-(x+\sqrt{2}t)}}{(1+e^{-x}+e^{-\sqrt{2}t}+e^{-(x+\sqrt{2}t)})^2}\,dx\,dt=\frac{5}{\sqrt{2}}\zeta(4).$$

Solution. Observe that

$$\begin{aligned}1+e^{-x}+e^{-\sqrt{2}t}+e^{-(x+\sqrt{2}t)} &= 1+e^{-x}+e^{-\sqrt{2}t}(1+e^{-x})\\ &= (1+e^{-x})(1+e^{-\sqrt{2}t})\end{aligned}$$

So

$$\begin{aligned}J &= \int_0^\infty\int_0^\infty \frac{(xt)^2e^{-(x+\sqrt{2}t)}}{\left[(1+e^{-x})(1+e^{-\sqrt{2}t})\right]^2}\,dx\,dt\\ &= \int_0^\infty\int_0^\infty \frac{x^2t^2e^{-(x+\sqrt{2}t)}}{(1+e^{-x})^2(1+e^{-\sqrt{2}t})^2}\,dx\,dt\\ &= \left(\int_0^\infty \frac{x^2e^{-x}}{(1+e^{-x})^2}\,dx\right)\left(\int_0^\infty \frac{t^2e^{-\sqrt{2}t}}{(1+e^{-\sqrt{2}t})^2}\,dt\right)\end{aligned}$$

Now, let us evaluate for all $a>0$ the integral $J_a=\displaystyle\int_0^\infty \frac{x^2e^{-ax}}{(1+e^{-ax})^2}\,dx$. To do that, consider the integral

$I = \displaystyle\int_0^\infty \frac{xe^{-x}}{(1+e^{-x})}\,dx$, so since $e^{-x} < 1$, we have by applying the geometric series

$$\begin{aligned} I &= \int_0^\infty \frac{xe^{-x}}{(1+e^{-x})}\,dx = \int_0^\infty xe^{-x}\left(\sum_{n=0}^\infty (-1)^n e^{-nx}\right)\,dx \\ &= \sum_{n=0}^\infty (-1)^n \left(\int_0^\infty xe^{-(n+1)x}\,dx\right) \\ &= \sum_{n=0}^\infty \frac{(-1)^n}{(n+1)^2} \\ &\qquad \text{since for all } b > 0;\ c > 0 : \int_0^\infty x^c e^{-bx}\,dx = \frac{\Gamma(c+1)}{b^{c+1}}. \end{aligned}$$

Therefore

$$\begin{aligned} I &= \sum_{n=0}^\infty \frac{(-1)^n}{(n+1)^2} = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^2} \\ &= -\sum_{n=0}^\infty \frac{1}{(2n)^2} + \sum_{n=0}^\infty \frac{1}{(2n+1)^2} \\ &= -\sum_{n=0}^\infty \frac{1}{(2n)^2} + \zeta(2) - \sum_{n=1}^\infty \frac{1}{(2n)^2} \\ &= \frac{1}{2}\zeta(2). \end{aligned}$$

Now, making change of the variable $x \to ax$, $a > 0$ in I, we get

$$I_a := \int_0^\infty \frac{xe^{-ax}}{(1+e^{-ax})}\,dx = \frac{1}{2a^2}\zeta(2).$$

Differentiate wrt a, we obtain

$$\begin{aligned} \frac{\partial}{\partial a} I_a &= \frac{\partial}{\partial a}\left(\int_0^\infty \frac{xe^{-ax}}{(1+e^{-ax})}\,dx\right) \\ &= \int_0^\infty \frac{x\left(-xe^{-ax}(1+e^{-ax}) + xe^{-2ax}\right)}{(1+e^{-ax})^2}\,dx \\ &= -\int_0^\infty \frac{x^2e^{-ax}}{(1+e^{-ax})^2}\,dx \\ &= -J_a = -\frac{1}{a^3}\zeta(2). \end{aligned}$$

Thereby

$$J_a = \frac{1}{a^3}\zeta(2).$$

Thus

$$J = J_1 \times J_{\sqrt{2}} = \frac{1}{2\sqrt{2}}\zeta^2(2)$$

Recall that

$$\begin{aligned}\zeta^2(2) &= \left(\frac{\pi^2}{6}\right)^2 = \frac{\pi^4}{36} \\ &= \frac{\pi^4}{90}\frac{90}{36} = \frac{5}{2}\zeta(4).\end{aligned}$$

Finally

$$J = \frac{5}{\sqrt{2}}\zeta(4).$$

Remark 0.1 *We have*

$$\int_0^1\int_0^1 \frac{\log^2(u)\log^2(v)}{(1+u+v+uv)^2}\,du\,dv = \zeta^2(2).$$

Proof of Remark 0.1. Making the double changes of variables $u = e^{-x}$ and $v = e^{-\sqrt{2}t}$ in J, we get

$$\begin{aligned}J &= \int_0^\infty\int_0^\infty \frac{(xt)^2 e^{-(x+\sqrt{2}t)}}{(1+e^{-x}+e^{-\sqrt{2}t}+e^{-(x+\sqrt{2}t)})^2}\,dx\,dt \\ &= \int_0^1\int_0^1 \frac{2uv\log^2(u)\log^2(v)}{(1+u+v+uv)^2}\left(\frac{du}{u}\,\frac{dv}{\sqrt{2}v}\right) \\ &= \sqrt{2}\int_0^1\int_0^1 \frac{\log^2(u)\log^2(v)}{(1+u+v+uv)^2}\,du\,dv.\end{aligned}$$

Hence

$$\begin{aligned}\int_0^1\int_0^1 \frac{\log^2(u)\log^2(v)}{(1+u+v+uv)^2}\,du\,dv &= \frac{5}{2}\zeta(4) \\ &= \zeta^2(2).\end{aligned}$$

This leads

$$\begin{aligned}\zeta^2(2) &= \int_0^1\int_0^1 \frac{\log^2(u)\log^2(v)}{(1+u+v(1+u))^2}\,du\,dv \\ &= \int_0^1\int_0^1 \frac{\log^2(u)\log^2(v)}{[(1+u)(1+v)]^2}\,du\,dv \\ &= \int_0^1\int_0^1 \left(\frac{\log(u)}{(1+u)}\frac{\log(v)}{(1+v)}\right)^2\,du\,dv \\ &= \left(\int_0^1\left(\frac{\log(u)}{1+u}\right)^2\,du\right)^2.\end{aligned}$$

So

$$\int_0^1 \left(\frac{\log(u)}{1+u}\right)^2 \, du \quad = \quad \zeta(2).$$