ROMANIAN MATHEMATICAL SOCIETY

Mehedinţi Branch

R. M. M. - 34

ROMANIAN MATHEMATICAL MAGAZINE

AUTUMN EDITION 2022

ISSN 2501-0099

ROMANIAN MATHEMATICAL SOCIETY

Mehedinți Branch

ROMANIAN MATHEMATICAL MAGAZINE

R.M.M.

Nr.34-AUTUMN EDITION 2022

ROMANIAN MATHEMATICAL SOCIETY

Mehedinți Branch

CONTENT

A SIMPLE PROOF FOR SCHWEITZER'S INEQUALITIES AND APPLICATIONS

By Daniel Sitaru, Claudia Nănuți – Romania

Abstract: In this paper it is proved Schweitzer's inequality in general form and particular cases. Also we give a few applications.

SCHWEITZER'S INEQUALITY:

If $0 < \alpha \le a_1, a_2, \dots, a_n \le \beta; n \in \mathbb{N}; n \ge 2$ then:

$$(a_1 + a_2 + \cdots a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \le \frac{(\alpha+\beta)n^2}{4\alpha\beta}$$

Proof: Lemma 1: If $x, y \in \mathbb{R}$ then: $4xy \le (x+y)^2$

Proof: $(x-y)^2 \ge 0 \Rightarrow x^2 - 2xy + y^2 \ge 0 \Rightarrow x^2 + 2xy + y^2 \ge 4xy$

$$\Rightarrow (x+y)^2 \ge 4xy \Rightarrow 4xy \le (x+y)^2$$

Lemma 2: If $0 < \alpha \le x \le \beta$ then:

$$\frac{x}{\sqrt{\alpha\beta}} + \frac{\sqrt{\alpha\beta}}{x} \le \frac{\alpha+\beta}{\sqrt{\alpha\beta}}$$

Proof:

$$\alpha \le x \le \beta \Rightarrow x - \alpha \ge 0; y - \beta \le 0 \Rightarrow$$

$$(x-\alpha)(y-\beta) \le 0 \Rightarrow x^2 - (\alpha+\beta)x + \alpha\beta \le 0$$

$$x^2 + \alpha\beta \le (\alpha+\beta)x \Rightarrow x + \frac{\alpha\beta}{x} \le \alpha+\beta \Rightarrow \frac{x}{\sqrt{\alpha\beta}} + \frac{\sqrt{\alpha\beta}}{x} \le \frac{\alpha+\beta}{\sqrt{\alpha\beta}}$$

Main proof:

$$4(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) =$$

$$= 4 \cdot \frac{1}{\sqrt{\alpha\beta}} \cdot \sqrt{\alpha\beta}(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) =$$

$$= 4\left(\frac{a_1}{\sqrt{\alpha\beta}} + \frac{a_2}{\sqrt{\alpha\beta}} + \cdots + \frac{a_n}{\sqrt{\alpha\beta}}\right)\left(\frac{\sqrt{\alpha\beta}}{a_1} + \frac{\sqrt{\alpha\beta}}{a_2} + \cdots + \frac{\sqrt{\alpha\beta}}{a_n}\right) \le$$

$$\overset{Lemma\ 1}{\le} \left(\frac{a_1}{\sqrt{\alpha\beta}} + \frac{\sqrt{\alpha\beta}}{a_1} + \frac{a_2}{\sqrt{\alpha\beta}} + \frac{\sqrt{\alpha\beta}}{a_2} + \cdots + \frac{a_n}{\sqrt{\alpha\beta}} + \frac{\sqrt{\alpha\beta}}{a_n}\right)^2 \le$$

$$\overset{Lemma\ 2}{\le} \left(\underbrace{\frac{\alpha+\beta}{\sqrt{\alpha\beta}} + \frac{\alpha+\beta}{\sqrt{\alpha\beta}} + \cdots + \frac{\alpha+\beta}{\sqrt{\alpha\beta}}}_{for\ "n"\ times}\right)^2 = \left(\frac{n(\alpha+\beta)}{\sqrt{\alpha\beta}}\right)^2 = \frac{(\alpha+\beta)^2 n^2}{\alpha\beta}$$

Case $n = 2$: If $0 < \alpha \le a, b \le \beta$ then:

$$(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \le \frac{(\alpha+\beta)^2}{\alpha\beta} \quad (1)$$

Case $n = 3$: If $0 < \alpha \le a, b, c \le \beta$ then:

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \le \frac{9(\alpha+\beta)^2}{4\alpha\beta} \quad (2)$$

Case $n = 4$: If $0 < \alpha \le a, b, c, d \le \beta$ then:

$$(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) \le \frac{4(\alpha+\beta)^2}{\alpha\beta} \quad (3)$$

Corollary 1: If $0 < \alpha \le x, y \le \beta$ then:

$$\left(\sqrt{xy}+\frac{x+y}{2}\right)\left(\frac{1}{\sqrt{xy}}+\frac{2}{x+y}\right) \le \frac{(\alpha+\beta)^2}{\alpha\beta}$$

Proof: We take in (1):

$$a = \sqrt{xy}; b = \frac{x+y}{2}$$

Corollary 2: If $0 < \alpha \le x, y \le \beta$ then:

$$\left(\sqrt{xy}+\sqrt{\frac{x^2+y^2}{2}}\right)\left(\frac{1}{\sqrt{xy}}+\sqrt{\frac{2}{x^2+y^2}}\right) \le \frac{(\alpha+\beta)^2}{\alpha\beta}$$

Proof: We take in (1):

$$a = \sqrt{xy}; b = \sqrt{\frac{x^2+y^2}{2}}$$

Corollary 3: If $0 < \alpha \le x, y \le \beta$ then:

$$\left(\frac{2xy}{x+y}+\sqrt{xy}\right)\left(\frac{1}{\sqrt{xy}}+\frac{x+y}{2xy}\right) \le \frac{(\alpha+\beta)^2}{\alpha\beta}$$

We take in (1): $a = \frac{2xy}{x+y}; b = \sqrt{xy}$

Corollary 4: If $0 < \alpha \le x, y \le \beta$ then:

$$\left(\frac{2xy}{x+y}+\frac{x+y}{2}\right)\left(\frac{x+y}{2xy}+\frac{2}{x+y}\right) \le \frac{(\alpha+\beta)^2}{\alpha\beta}$$

We take in (1):

$$a = \frac{2xy}{x+y}; b = \frac{x+y}{2}$$

Corollary 5: If $0 < \alpha \le x, y \le \beta$ then:

$$\left(\frac{x+y}{2}+\sqrt{\frac{x^2+y^2}{2}}\right)\left(\frac{2}{x+y}+\sqrt{\frac{2}{x^2+y^2}}\right)\le\frac{(\alpha+\beta)^2}{\alpha\beta}$$

We take in (1):

$$a=\frac{x+y}{2};b=\sqrt{\frac{x^2+y^2}{2}}$$

Corollary 6: If $0<\alpha\le x,y,z\le\beta$ then:

$$\left(\frac{3xyz}{xy+yz+zx}+\sqrt[3]{xyz}+\frac{x+y+z}{3}\right)\left(\frac{xy+yz+zx}{3xyz}+\frac{1}{\sqrt[3]{xyz}}+\frac{3}{x+y+z}\right)\le\frac{9(\alpha+\beta)^2}{4\alpha\beta}$$

Proof: We take in (2):

$$a=\frac{3xyz}{xy+yz+zx};b=\sqrt[3]{xyz};c=\frac{x+y+z}{3}$$

Corollary 7:If $0<\alpha\le x,y,z\le\beta$ then:

$$\left(\frac{3xyz}{xy+yz+zx}+\sqrt[3]{xyz}+\sqrt{\frac{x^2+y^2+z^2}{3}}\right)\left(\frac{xy+yz+zx}{3xyz}+\frac{1}{\sqrt[3]{xyz}}+\sqrt{\frac{3}{x^2+y^2+z^2}}\right)\le\frac{9(\alpha+\beta)^2}{4\alpha\beta}$$

Proof: We take in (2):

$$a=\frac{3xyz}{xy+yz+zx};b=\sqrt[3]{xyz};c=\sqrt{\frac{x^2+y^2+z^2}{3}}$$

Corollary 8: If $0<\alpha\le x,y,z\le\beta$ then:

$$\left(\sqrt[3]{xyz}+\frac{x+y+z}{3}+\sqrt{\frac{x^2+y^2+z^2}{3}}\right)\left(\frac{1}{\sqrt[3]{xyz}}+\frac{3}{x+y+z}+\sqrt{\frac{3}{x^2+y^2+z^2}}\right)\le\frac{9(\alpha+\beta)^2}{4\alpha\beta}$$

Proof: We take in (2):

$$a=\sqrt[3]{xyz};b=\frac{x+y+z}{3};c=\sqrt{\frac{x^2+y^2+z^2}{3}}$$

Corollary 9: If $0<\alpha\le x,y,z,t\le\beta$ then:

$$(m_h+m_g+m_a+m_q)\left(\frac{1}{m_h}+\frac{1}{m_g}+\frac{1}{m_a}+\frac{1}{m_q}\right)\le\frac{4(\alpha+\beta)^2}{\alpha\beta}$$

$$m_h=\frac{4xyzt}{xy+xz+xt+yz+yt+zt};m_g=\sqrt[4]{xyzt};$$

$$m_a=\frac{x+y+z+t}{4};m_q=\sqrt{\frac{x^2+y^2+z^2+t^2}{4}}$$

Proof: We take in (3):

$$a = m_h; b = m_g; c = m_a; d = m_q$$

REFERENCE:

[1] ROMANIAN MATHEMATICAL MAGAZINE: www. ssmrmh.ro

AMAZING IDENTITIES AND INEQUALITIES WITH MEDIANS

By Bogdan Fuştei-Romania

In ΔABC the following relationship holds:

$$\boldsymbol{\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} = \sqrt{\frac{r_b r_c}{bc}}}$$

$$\begin{cases}(b+c)^2 = b^2 + c^2 + 2bc \\ (b-c)^2 = b^2 + c^2 - 2bc\end{cases} \Rightarrow (b+c)^2 + (b-c)^2 = 4bc$$

$$((b+c)^2 + (b-c)^2)\cos^2\frac{A}{2} = 4bc \cdot \frac{r_b r_c}{bc} = 4r_b r_c; (1)$$

$$\boldsymbol{m_a^2 = \frac{2(b^2+c^2)-a^2}{4} \Rightarrow 4m_a^2 = 2(b^2+c^2)-a^2}$$

$$r_b r_c = s(s-a) = \frac{(a+b+c)(b+c-a)}{4} \Rightarrow 4r_b r_c = (b+c)^2 - a^2$$

$$4r_b r_c + (b-c)^2 = 2(b^2+c^2) - a^2 \Rightarrow 4m_a^2 = 4r_b r_c + (b-c)^2; (2)$$

From (1),(2) it follows that:

$$4m_a^2 = (b+c)\cos^2\frac{A}{2} + (b-c)^2 - (b-c)^2\cos^2\frac{A}{2}$$

Therefore, we get a new identity:

$$\boldsymbol{4m_a^2 = (b+c)^2\cos^2\frac{A}{2} + (b-c)^2\sin^2\frac{A}{2}}$$

Next, $4m_a^2 \ge (b+c)^2\cos^2\frac{A}{2} \Rightarrow m_a \ge \frac{b+c}{2}\cos\frac{A}{2}$

$$bc = r_b r_c + r r_a (and\ analogs), m_a g_a \ge m_a w_a \ge r_b r_c$$

$$(b-c)^2 = b^2 + c^2 - 2bc = n_a^2 + g_a^2 - 2r_b r_c, (n_a - g_a)^2 = n_a^2 + g_a^2 - 2n_a g_a \Rightarrow$$

$|b-c| \ge n_a - g_a$. So, we get:

$$4m_a^2 = (b+c)^2\cos^2\frac{A}{2} + (n_a - g_a)^2\sin^2\frac{A}{2}$$

But $\begin{cases}4m_a^2 = n_a^2 + g_a^2 + 2r_b r_c \\ (n_a + g_a)^2 = n_a^2 + g_a^2 + 2n_a g_a\end{cases} \Rightarrow n_a + g_a \ge 2m_a (and\ analogs).$

Hence, $n_a - g_a \geq 2(m_a - g_a), (and\ analogs)$.

Therefore, we get a new inequality:

$$\boldsymbol{4m_a^2 \geq (b+c)^2 cos^2\frac{A}{2} + 4(m_a - g_a)^2 sin^2\frac{A}{2}}$$

Next, $g_a \leq AI + r, w_a = AI + \frac{w_a r}{h_a} \Rightarrow g_a \leq w_a \Rightarrow n_a - g_a \geq 2(m_a - w_a), (and\ analogs)$.

$$4m_a^2 \geq (b+c)^2 cos^2\frac{A}{2} + 4(m_a - w_a)^2 sin^2\frac{A}{2}$$

$$4m_a^2 \geq (b+c)^2 cos^2\frac{A}{2} + (n_a - w_a)^2 sin^2\frac{A}{2}$$

We know that: $4m_a^2 = (b+c)^2 cos^2\frac{A}{2} + (b-c)^2 sin^2\frac{A}{2}$,

$$\sqrt{\frac{x^2+y^2}{2}} \geq \frac{x+y}{2} \Rightarrow \sqrt{x^2+y^2} \geq \frac{1}{\sqrt{2}}(x+y)$$

Let us denote $x = (b+c)cos\frac{A}{2}, y = \frac{|b-c|sinA}{2}$, then we have:

$$2m_a \geq \frac{1}{\sqrt{2}}\left((b+c)cos\frac{A}{2} + \frac{|b-c|sinA}{2}\right)$$

Therefore, we get a new inequality:

$$\boldsymbol{m_a \geq \frac{1}{2\sqrt{2}}\left((b+c)cos\frac{A}{2} + \frac{|b-c|sinA}{2}\right)}$$

$$w_a = \frac{2bc}{b+c}cos\frac{A}{2}, a = 4Rsin\frac{A}{2}cos\frac{A}{2}$$

$$m_a w_a \geq \frac{1}{2\sqrt{2}} \cdot 2\left((b+c)cos^2\frac{A}{2} \cdot \frac{bc}{b+c} + \frac{|b-c|sinA}{2} \cdot \frac{bc}{b+c}cos\frac{A}{2}\right)$$

$$m_a w_a \geq \frac{1}{\sqrt{2}}\left(bccos^2\frac{A}{2} + \frac{|b-c|}{b+c} \cdot bc \cdot \frac{a}{4R}\right), m_a w_a \geq \frac{1}{\sqrt{2}}\left(bc \cdot \frac{s(s-a)}{bc} + \frac{|b-c|}{b+c} \cdot \frac{abc}{4R}\right)$$

$$m_a w_a \geq \frac{1}{\sqrt{2}}\left(s(s-a) + \frac{|b-c|}{b+c} \cdot F\right)$$

Therefore, we get a new inequality:

$$\boldsymbol{\sum_{cyc} m_a w_a \geq \frac{1}{\sqrt{2}}\left(s^2 + F\sum_{cyc}\frac{|b-c|}{b+c}\right)}$$

$$s^2 = n_a^2 + 2r_a h_a, 3(m_a w_a + m_b w_b + m_c w_c) \geq \frac{1}{\sqrt{2}}\left(3s^2 + 3F\sum_{cyc}\frac{|b-c|}{b+c}\right)$$

Therefore, we get a new inequality:

$$\mathbf{3(m_a w_a + m_b w_b + m_c w_c) \geq \frac{1}{\sqrt{2}}\left(\sum_{cyc} n_a^2 + 2\sum_{cyc} r_a h_a + 3F\sum_{cyc}\frac{|b-c|}{b+c}\right)}$$

But $x^2+y^2+z^2 \geq xy+yz+zx, \forall x,y,z \in \mathbb{R} \Rightarrow n_a^2+n_b^2+n_c^2 \geq n_a n_b + n_b n_c + n_c n_a$

Hence, we get:

$$\mathbf{3(m_a w_a + m_b w_b + m_c w_c) \geq \frac{1}{\sqrt{2}}\left(\sum_{cyc} n_a n_b + 2\sum_{cyc} r_a h_a + 3F\sum_{cyc}\frac{|b-c|}{b+c}\right)}$$

We know that: $AI = \frac{r}{\sin\frac{A}{2}} = \frac{s-a}{\cos\frac{A}{2}}$ $(and\ analogs)$.

$$m_a \cdot AI \geq \frac{1}{2\sqrt{2}}\left[(b+c)\cos\frac{A}{2}\cdot\frac{s-a}{\cos\frac{A}{2}} + \frac{|b-c|\sin A}{2}\cdot\frac{r}{\sin\frac{A}{2}}\right]$$

Therefore, we get a new inequality:

$$\mathbf{m_a \cdot AI \geq \frac{1}{2\sqrt{2}}[(b+c)(s-a) + r|b-c|]}$$

But $(b+c)(s-a) = \frac{(b+c)(b+c-a)}{2} = \frac{(b+c)^2 - a(b+c)}{2}$

$$\sum_{cyc}(b+c)(s-a) = \frac{(b+c)^2 - a(b+c) + (a+c)^2 - b(a+c) + (a+b)^2 - c(a+b)}{2}$$

$$= a^2+b^2+c^2$$

Therefore, we get a new inequality:

$$\mathbf{\sum_{cyc} m_a \cdot AI \geq \frac{1}{2\sqrt{2}}\left(a^2+b^2+c^2 + r\sum_{cyc}|b-c|\right)}$$

$$AI^2 = bc - 4Rr, bc = 2Rh_a, \qquad AI^2 = 2R(h_a - 2r) \Rightarrow AI = \sqrt{2R(h_a-2r)}$$

Therefore, we get a new inequality:

$$\mathbf{\sum_{cyc} m_a \cdot \sqrt{h_a - 2r} \geq \frac{1}{4\sqrt{R}}\left(a^2+b^2+c^2 + r\sum_{cyc}|b-c|\right)}$$

We prove that:

$$m_a \geq \frac{1}{2\sqrt{2}}\left[(b+c)\cos\frac{A}{2} + \frac{|b-c|\sin A}{2}\right]$$

$$2m_a \geq \frac{1}{\sqrt{2}}\left[(b+c)cos\frac{A}{2} + \frac{|b-c|sinA}{2}\right], n_a + g_a \geq 2m_a$$

Therefore, we get a new inequality:

$$\boldsymbol{n_a g_a \geq \frac{1}{\sqrt{2}}\left[(b+c)cos\frac{A}{2} + \frac{|b-c|sinA}{2}\right]}$$

But $n_a g_a \geq m_a w_a \Rightarrow \frac{n_a g_a}{w_a} \geq m_a$.Therefore, we get a new inequality:

$$\boldsymbol{\frac{n_a g_a}{w_a} \geq \frac{1}{2\sqrt{2}}\left[(b+c)cos\frac{A}{2} + \frac{|b-c|sinA}{2}\right]}$$

REFERENCE:

ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

ABOUT AN INEQUALITY BY FLORICĂ ANASTASE-II

By Marin Chirciu-Romania

1) In ΔABC the following relationship holds:

$$\boldsymbol{\left(1+\frac{1}{a}tan\frac{A}{2}\right)\left(1+\frac{1}{b}tan\frac{B}{2}\right)\left(1+\frac{1}{c}tan\frac{C}{2}\right) \geq \left(1+\frac{9}{2}\cdot\frac{r}{s^2}\right)^3}$$

Proposed by Florică Anastase-Romania

Solution: Using Huygens inequality we get:

$$LHS = \prod_{cyc}\left(1+\frac{1}{a}tan\frac{A}{2}\right) \geq \left(1+\sqrt[3]{\frac{1}{abc}tan\frac{A}{2}tan\frac{B}{2}tan\frac{C}{2}}\right)^3 =$$

$$= \left(1+\sqrt[3]{\frac{1}{4Rrs}\cdot\frac{r}{s}}\right)^3 = \left(1+\sqrt[3]{\frac{1}{4Rs^2}}\right)^3 \overset{(1)}{\geq} \left(1+\frac{9}{2}\cdot\frac{r}{s^2}\right)^3 \geq RHD$$

$$(1) \Leftrightarrow \sqrt[3]{\frac{1}{4Rs^2}} \geq \frac{9}{2}\cdot\frac{r}{s^2} \Leftrightarrow \frac{1}{4Rs^2} \geq \left(\frac{9}{2}\cdot\frac{r}{s^2}\right)^3 \Leftrightarrow \frac{1}{4Rs^2} \geq \frac{729r^3}{8s^6} \Leftrightarrow$$

$2s^4 \geq 729Rr^3$, which follows from $s^2 \geq 16Rr - 5r^2 (Gerretsen)$.
Remains to prove that $2(16Rr - 5r^2) \geq 729Rr^3 \Leftrightarrow$
$512R^2 - 1049Rr + 50r^2 \geq 0 \Leftrightarrow (R-2r)(512R - 25r) \geq 0$, which is true from $R \geq 2r (Euler)$. Equality holds if and only if triangle is equilateral.

2) In ΔABC the following relationship holds:

$$\boldsymbol{\left(1+\frac{1}{a}cot\frac{A}{2}\right)\left(1+\frac{1}{b}cot\frac{B}{2}\right)\left(1+\frac{1}{c}cot\frac{C}{2}\right) \geq \left(1+\frac{27}{2}\cdot\frac{r}{s^2}\right)^3}$$

Marin Chirciu

Solution: Using Huygens inequality we get:

$$LHS=\prod_{cyc}\left(1+\frac{1}{a}cot\frac{A}{2}\right)\ge\left(1+\sqrt[3]{\frac{1}{abc}cot\frac{A}{2}cot\frac{B}{2}cot\frac{C}{2}}\right)^3=$$

$$=\left(1+\sqrt[3]{\frac{1}{4Rrs}\cdot\frac{s}{r}}\right)^3=\left(1+\sqrt[3]{\frac{1}{4Rr^2}}\right)^3\overset{(1)}{\ge}\left(1+\frac{9}{2}\cdot\frac{r}{s^2}\right)^3\ge RHD$$

$$(1)\Leftrightarrow\sqrt[3]{\frac{1}{4Rr^2}}\ge\frac{9}{2}\cdot\frac{r}{s^2}\Leftrightarrow\frac{1}{4Rs^2}\ge\left(\frac{9}{2}\cdot\frac{r}{s^2}\right)^3\Leftrightarrow\frac{1}{4Rr^2}\ge\frac{27^3r^3}{8s^6}\Leftrightarrow$$

$2s^6\ge 27^3Rr^5$, which follows from $s^2\ge 16Rr-5r^2(Gerretsen)$.

Remains to prove that $2(16Rr-5r^2)^3\ge 27^3Rr^5\Leftrightarrow$

$8192R^3-7680R^2r-17283Rr^2-250r^3\ge 0\Leftrightarrow$

$(R-2r)(8192R^2+8740Rr+125r^2)\ge 0$, which is true from $R\ge 2r(Euler)$.

Equality holds if and only if triangle is equilateral.

3) In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{1}{a}cot\frac{A}{2}\ge 3\sum_{cyc}\frac{1}{a}tan\frac{A}{2}$$

Marin Chirciu

Solution: Using identities in triangle:

$$\sum_{cyc}\frac{1}{a}tan\frac{A}{2}=\frac{1}{4R}\left[1+\left(\frac{4R+r}{s}\right)^2\right];\quad\sum_{cyc}\frac{1}{a}cot\frac{A}{2}=\frac{s^2+r^2-8Rr}{4Rr^2}$$

Inequality becomes as: $\frac{s^2+r^2-8Rr}{4Rr^2}\ge 3\cdot\frac{1}{4R}\left[1+\left(\frac{4R+r}{s}\right)^2\right]\Leftrightarrow$

$s^2(s^2-2r^2-8Rr)\ge 3r^2(4R+r)^2$, which follows from $s^2\ge 16Rr-5r^2\ge\frac{r(4R+r)^2}{R+r}$

Remains to prove that: $\frac{r(4R+r)^2}{R+r}(16Rr-5r^2-2r^2-8Rr)\ge 3r^2(4R+r)^2\Leftrightarrow$

$R\ge 2r(Euler)$.

Equality holds if and only if triangle is equilateral.

4) In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{1}{bc}cot\frac{B}{2}cot\frac{C}{2}\ge 9\sum_{cyc}\frac{1}{bc}tan\frac{B}{2}tan\frac{C}{2}$$

Marin Chirciu

Solution: Using identities:

$$\sum_{cyc}\frac{1}{bc}cot\frac{B}{2}cot\frac{C}{2}=\frac{2R-r}{2Rr^2};\quad\sum_{cyc}\frac{1}{bc}tan\frac{B}{2}tan\frac{C}{2}=\frac{4R+r}{2Rs^2}$$

Inequality becomes as: $\frac{2R-r}{2Rr^2}\ge 9\cdot\frac{4R+r}{2Rs^2}\Leftrightarrow s^2(2R-r)\ge 9r^2(4R+r)$, which it follows from $s^2\ge 16Rr-5r^2(Gerretsen)$.

Remains to prove that: $(16Rr-5r^2)(2R-r)\ge 9r^2(4R+r)\Leftrightarrow$

$16R^2-31Rr-2r^2\ge 0\Leftrightarrow(R-2r)(16R+r)\ge 0$, which is true from $R\ge 2r(Euler)$.

Equality holds if and only if triangle is equilateral.

REFERENCE:

ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

LOGARITHMIC INTEGRALS BY LOGARITHMIC SERIES

By Narendra Bhandari –Nepal

Nepal Academy for Science and Technology

Abstract: In this article the logarithmic integrals of the following two classes in closed forms

$$\int_0^{\frac{\pi}{2}} \ln(a^2\cos^2 x + b^2\sin^2 x)\,dx = \pi\ln\left(\frac{a+b}{2}\right) \quad (1)$$

$$\int_0^{\frac{\pi}{2}} \ln\left(p^4\cos^4 x + \frac{q^4}{16}\sin^4 2x\right)dx = 2\pi\ln\left(\frac{p}{4}\right) + \frac{\pi}{2}\mathcal{A}(p,q) \quad (2)$$

where $\mathcal{A}(p,q) = \ln\left(1+\sqrt{1+\frac{q^4}{p^4}}\right) + 2\ln\left(\sqrt{2}+\sqrt{1+\sqrt{1+\frac{a^4}{p^4}}}\right)$ for all $a, b > 0$,

$p > q > 0$ are evaluated using the Maclaurin series of $\log(1+y)$ for $y \in (-1,1]$.

Introduction

The aforementioned formal integral, [1] is a classical integral that can be found in book, *Integrals, series and products* (see page no 532, section 4.226) and latter integral, [2] is a variant version (due to motivation) of the former integral. The common technique to solve these integrals is Feynman technique however, this paper presents the evaluation of these integrals by series of $\ln(1+y)$ around $y = 0$ that boils down to alternating sum with central binomial coefficients.

Theorems and Proofs.

Theorem 1. For all $a, b > 0$, the following integral equality holds.

$$\int_0^{\frac{\pi}{2}} \ln(a^2\cos^2 x + b^2\sin^2 x)\,dx = \pi\ln\left(\frac{a+b}{2}\right)$$

Before we develop the proof of Theorem 1 we need the following lemmas.

Lemma 1.1. For $|x| < \frac{1}{4}$, the generating function of central binomial coefficients $\binom{2n}{n}$ for $n \geq 0$ integer is given by

$$\sum_{n=0}^{\infty}\binom{2n}{n}x^n = \frac{1}{\sqrt{1-4x}}$$

Proof: Consider the function $f(x) = \frac{1}{\sqrt{1+x}}$ for all $x \in (-1,1]$ and by generalized binomial theorem we write $f(x)$ as

$$\frac{1}{\sqrt{1+x}} = \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}x^n = \sum_{n=0}^{\infty}\left[\prod_{k=0}^{n}\left(-k-\frac{1}{2}\right)\right]\frac{x^n}{n!} = \sum_{n=0}^{\infty}(-1)^n\frac{(2n-1)!!}{2^n n!}x^n$$

since $(2n-1)!! = \frac{(2n)!}{2^n n!}$ and replacing x by $-4x$ we have then

$$\sum_{n=0}^{\infty}(-1)^n\frac{(2n-1)!!}{4^n n!}(-4x)^n = \sum_{n=0}^{\infty}\binom{2n}{n}x^n = \frac{1}{\sqrt{1-4x}}$$

Lemma 1.2. Let $n \geq 0$ be integer then the following equality holds.

$$\int_0^{\frac{\pi}{2}} \sin^{2n} u\, du = \int_0^{\frac{\pi}{2}} \cos^{2n} u\, du = \frac{\pi}{2\cdot 4^n}\binom{2n}{n}$$

Proof: Due to Euler's integral of the first kind, Beta function

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$$

with the substitution of $t = \sin^2 u$ we obtain

$$B(x,y) = 2\int_0^{\frac{\pi}{2}} \sin^{2x-1} u \cos^{2y-1} u\, du$$

To obtain the desired integral we either set $x = \frac{1}{2}$ or $y = \frac{1}{2}$ and if $y = \frac{1}{2}$ then $x = \frac{2n+1}{2}$ and vice versa.

$$B\left(\frac{2n+1}{2}, \frac{1}{2}\right) = \int_0^{\frac{\pi}{2}} \sin^{2n} u\, du = \int_0^{\frac{\pi}{2}} \cos^{2n} u\, du$$

since $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ and using the relation we obtain

$$\int_0^{\frac{\pi}{2}} \sin^{2n} u\, du = \int_0^{\frac{\pi}{2}} \cos^{2n} u\, du = \frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(n+1)} = \frac{\pi}{2\cdot 4^n}\binom{2n}{n}$$

since we used the relation $\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^n}(2n-1)!! = \frac{(2n)!}{4^n n!}\sqrt{\pi}$

Proof of Theorem 1

Since $\sin\left(\frac{\pi}{2} - x\right) = \cos x$ and thus we write

$$\int_0^{\frac{\pi}{2}} \ln(a^2\cos^2 x + b^2\sin^2 x)\, dx = \int_0^{\frac{\pi}{2}} \ln(a^2\sin^2 x + b^2\cos^2 x)\, dx$$

Now for $a > b > 0$ we write $a^2 = (a^2 - b^2) + b^2 = k + b^2$ so we write the integral

$$\mathcal{L}(k,b) = \int_0^{\frac{\pi}{2}} \ln(b^2 + k\cos^2 x)dx = \pi\ln b + \int_0^{\frac{\pi}{2}} \ln\left(1 + \frac{k}{b^2}\cos^2 x\right)dx$$

Now $|\cos^2 x| < 1$ for all $x \in \left(0, \frac{\pi}{2}\right)$ and $\frac{a^2-b^2}{b^2} < 1$ implies $\left|\frac{k}{b^2}\cos^2 x\right| < 1$ and hence

$$\mathcal{J}(k,b) = \int_0^{\frac{\pi}{2}} \ln\left(1 + \frac{k}{b^2}\cos^2 x\right)dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\left(\frac{k}{b^2}\right)^n \int_0^{\frac{\pi}{2}} \cos^{2n} x\, dx$$

Now by Lemma 1.2, the latter expression boils down to the following infinite sum.

$$\mathcal{J}(k,b) = \frac{\pi}{2}\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{4^n n}\left(\frac{k}{b^2}\right)\binom{2n}{n} = \frac{\pi}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\left(\frac{k}{4b^2}\right)^n\binom{2n}{n}$$

To obtain the last series we exploit the Lemma 1.1.

$$\sum_{n=0}^{\infty}\binom{2n}{n}x^n=\frac{1}{\sqrt{1-4x}}\Rightarrow\sum_{n=1}^{\infty}\binom{2n}{n}x^n=\frac{1}{\sqrt{1-4x}}-1$$

Now by dividing by x and integrating from 0 to y we get

$$\sum_{n=1}^{\infty}\binom{2n}{n}y^n=\int_0^y\frac{1}{x}\left(\frac{1}{\sqrt{1-4x}}-1\right)dx=-2\ln\left(\frac{\sqrt{1-4y}+1}{2}\right)$$

Now setting $y=-\frac{k}{4b^2}=-\frac{a^2-b^2}{b^2}$

$$\mathcal{I}(k,b)=\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(\frac{k}{4b^2}\right)^n\binom{2n}{n}=\pi\ln\left(\frac{a+b}{2}\right)-\pi\ln b$$

and hence

$$\mathcal{L}(a,b)=\int_0^{\frac{\pi}{2}}\ln(a^2\cos^2 x+b^2\sin^2 x)\,dx=\pi\ln\left(\frac{a+b}{2}\right)$$

and for the case of $b>a$ we note that $\sin\left(\frac{\pi}{2}-x\right)=\cos x$ and replacing a by b for k and vice versa the desired same result is obtained.

Theorem 2. For all $p>q>0$ the following integral equality holds

$$\int_0^{\frac{\pi}{2}}\ln\left(p^4\cos^4 x+\frac{q^4}{16}\sin^4 2x\right)dx=2\pi\ln\left(\frac{p}{4}\right)+\frac{\mathcal{A}(p,q)}{2}\pi$$

where $\mathcal{A}(p,q)=\ln\left(1+\sqrt{1+\frac{q^4}{p^4}}\right)+2\ln\left(\sqrt{2}+\left(1+\sqrt{1+\frac{q^4}{p^4}}\right)^{\frac{1}{2}}\right)$

To work with Theorem 2 we need the following lemma.

Lemma 2.1. For $|x|<\frac{1}{16}$, the generating function for the coefficients $\binom{4n}{n}$ for all $n\geq 0$ is given by

$$\sum_{n=0}^{\infty}\binom{4n}{2n}(-x)^n=\frac{1}{\sqrt{2}}\sqrt{\frac{1+\sqrt{1+16x}}{1+16x}}$$

Proof. Let the function $\mathcal{F}(x)=\frac{1}{\sqrt{1-4x}}=\sum_{n=0}^{\infty}\binom{2n}{n}x^n$ by Lemma 1.1. and it is easy too see that

$$\sum_{n=0}^{\infty}\binom{4n}{2n}(-x^2)^n=\mathcal{R}\left(\sum_{n=0}^{\infty}\binom{2n}{n}(ix)^n\right)=\mathcal{R}\left(\frac{1}{\sqrt{1-4ix}}\right)$$

Now to evaluate the real part, let $\frac{1}{\sqrt{1-4ix}}=re^{i\theta}$ and $\mathcal{R}(\mathcal{F}(ix))=r\cos\theta$. Here $\cos 2\theta=r^2$ and $\sin 2\theta=4r^2x$ and hence $\theta=\frac{1}{2}\arctan 4x$ and $r=\frac{1}{\sqrt[4]{1+16x^2}}$.

Therefore,

$$\mathcal{R}(\mathcal{F}(ix))=\frac{\cos\left(\frac{1}{2}\arctan 4x\right)}{\sqrt{1+16x^2}}=\frac{1}{\sqrt{2}}\frac{\sqrt{1+\cos\arctan 4x}}{\sqrt[4]{1+16x^2}}=\frac{1}{\sqrt{2}}\sqrt{\frac{1+\sqrt{1+16x^2}}{\sqrt{1+16x^2}}}$$

we used $\cos(\arctan x)=\frac{1}{\sqrt{1+x^2}}$ and on replacing x by $4x$ and simplification gives us the equality right hand side. Moreover, replacing x^2 by x yields the desired result

$$\sum_{n=0}^{\infty}\binom{4n}{2n}(-x)^n=\frac{1}{\sqrt{2}}\sqrt{\frac{1+\sqrt{1+16x}}{1+6x}}$$

which completes the proof.

Remark.

$$\sum_{n=0}^{\infty}\binom{4n}{2n}x^{2n}=\frac{\mathcal{F}(x)+\mathcal{F}(-x)}{2}=\frac{1}{2}\left(\frac{1}{\sqrt{1-4x}}+\frac{1}{\sqrt{1+4x}}\right)$$

Lemma 2.2. We show that

$$\int_0^{\frac{\pi}{2}}\ln(\cos x)\,dx=\int_0^{\frac{\pi}{2}}\ln(\sin x)\,dx=-\frac{\pi}{2}\ln 2$$

Using the integral property

$$\int_a^b f(x)\,dx=\int_a^b f(b+a-x)\,dx$$

We directly get the result

$$\int_0^{\frac{\pi}{2}}\ln(\cos x)\,dx=\int_0^{\frac{\pi}{2}}\ln(\sin x)\,dx$$

For all $0\le x<\frac{\pi}{2}$ we use the Fourier series of $\ln(\cos x)$ we have

$$\int_0^{\frac{\pi}{2}}\ln(\cos x)\,dx=\int_0^{\frac{\pi}{2}}\left(-\ln 2-\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\cos(2kx)\right)=-\frac{\pi}{2}\ln 2-0=-\frac{\pi}{2}\ln 2$$

Since

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\int_0^{\frac{\pi}{2}}\cos(2kx)\,dx=\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\sin(2k\pi)=0$$

as $\sin(2\pi k)=0$ for all k (integers).

Proof of Theorem 2

Since

$$\int_0^{\frac{\pi}{2}}\ln\left(p^4\cos^4 x+\frac{q^4}{16}\sin^4 2x\right)dx=\int_0^{\frac{\pi}{2}}\ln(\cos^4 x)\,dx+\int_0^{\frac{\pi}{2}}\ln(p^4+q^4\sin^4 x)\,dx$$

Since the formal integral

$$\int_0^{\frac{\pi}{2}}\ln(\cos^4 x)\,dx=4\int_0^{\frac{\pi}{2}}\ln(\cos x)\,dx=-2\pi\ln 2$$

by Lemma 2.2. Note that

$$\int_0^{\frac{\pi}{2}} \ln(p^4 + q^4 \sin^4 x)\, dx = 2\pi \ln p + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(\frac{q^4}{p^4}\right)^n \int_0^{\frac{\pi}{2}} \sin^{4n} x\, dx$$

By the Lemma 1.2. we have

$$\mathcal{S}(p,q) = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left(\frac{q^4}{16p^4}\right)^n \binom{4n}{2n}$$

We now evaluate last sum by the use of the Lemma 2.1. by dividing x and integrating from 0 to z.

$$\sum_{n=1}^{\infty} \frac{(-z)^n}{n} \binom{4n}{2n} = \int_0^z \frac{1}{x} \left(\sqrt{\frac{1+\sqrt{1+16x}}{1+16x}} - \sqrt{2} \right) \frac{dx}{\sqrt{2}}$$

We evaluate the indefinite integral of latter result by making substitution $1 + 16x = y$ gives us

$$\frac{1}{\sqrt{2}} \int \left(-\frac{\sqrt{1+\sqrt{y}} + \sqrt{2y}}{(1-y)\sqrt{y}} \right) dy \overset{u=\sqrt{y}}{=} -\frac{2}{\sqrt{2}} \int \frac{\sqrt{1+u} - \sqrt{2u}}{1-u^2}$$

further substitute $\sqrt{1+u} = w$ gives us

$$-\frac{4}{\sqrt{2}} \int \frac{w^2 - \sqrt{2}w^3 + \sqrt{2}w}{(w^2-1)^2 - 1} dw = \frac{4}{\sqrt{2}} \int \frac{(\sqrt{2}w+1)(w-\sqrt{2})}{w(w+\sqrt{2})(w-\sqrt{2})} = \frac{4}{\sqrt{2}} \int \frac{\sqrt{2}w+1}{w(w+\sqrt{2})}$$

and last integral on RHS $\int \frac{\sqrt{2}w+1}{w(w+\sqrt{2})} = \sqrt{2} \ln\left(\frac{2}{\sqrt{2}} + w\right)$ and making undo of each substitution made with simplification we yield

$$\int \frac{1}{x} \left(\sqrt{\frac{1+\sqrt{1+16x}}{1+16x}} - \sqrt{2} \right) \frac{dx}{\sqrt{2}} = -2 \ln\left(\sqrt{1+\sqrt{1+16x}} + \frac{\sqrt{1+16x}+1}{\sqrt{2}} \right) + C$$

and applying the limits we get

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \binom{4n}{2n} z^n = \underbrace{2 \ln\left(\sqrt{1+\sqrt{1+16z}} + \frac{\sqrt{1+16z}+1}{\sqrt{2}} \right)}_{M} - 3\ln 2$$

Also

$$M = \ln\left(1+\sqrt{1+16z}\right) + 2\ln\left(\sqrt{2} + \sqrt{1+\sqrt{1+16x}}\right) - \ln 2$$

for $z = \frac{q^4}{16p^4}$ we obtain the closed form of

$$\mathcal{S}(p,q) = \underbrace{\frac{\pi}{2} \left(\ln\left(1 + \sqrt{1+\frac{q^4}{p^4}}\right) + 2\ln\left(\sqrt{2} + \sqrt{1+\sqrt{1+\frac{q^4}{p^4}}}\right) \right)}_{\mathcal{A}(p,q)} - 2\pi \ln 2$$

Combining the result we obtain the result

$$\int_0^{\frac{\pi}{2}} \ln\left(p^4\cos^4 x + \frac{q^4}{16}\sin^4 x\right)dx = 2\pi\ln\left(\frac{p}{4}\right) + \frac{\mathcal{A}(p,q)}{2}\pi$$

as required which completes the proof.

Remarkable Result from study

As we proved that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\binom{4n}{2n} z^n = M - 3\ln 2$$

It is interesting to note that sum on left hand attains the hypergeometric expression, namely

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n}\binom{4n}{2n} z^n = \frac{3}{8}\,{}_4F_3\left(1,1,\frac{5}{4},\frac{7}{4};\frac{3}{2},2,2;-z\right)z$$

In other words $\frac{3}{8}\,{}_4F_3\left(1,1,\frac{5}{4},\frac{7}{4};\frac{3}{2},2,2;-z\right)z$

$$= \ln\left(1+\sqrt{1+16z}\right) + 2\ln\left(\sqrt{2}+\sqrt{1+\sqrt{1+16z}}\right) - 3\ln 2$$

References:

[1] I.S Gradshteyn and I.M Ryznik, ***Tables of integrals, series and products***, 7th edition, edited by Alan Jeffrey and Daniel Zwillinger.

[2] https://mathsworld.wolfram.com/CentralBinomialCoefficient.html

[3] https://en.m.wikipedia.org/wiki/Wallis

LEGENDRE FORMULA IN TERMS OF THE PRIME COUNTING FUNCTION-REVISITED

By Mohammed Bouras-Morocco

University Sidi Mohamed Ben Abdellah, Faculty of Science, Fès-Morocco

Abstract: Let $\boldsymbol{\pi(x)}$ be the prime counting function that gives the number of primes less than or equal to x, Legendre conjectured an approximation which was very similar to $\boldsymbol{\pi(x)}$. We propose an exact representation for the Legendre formula which is valid for infinitely many naturals numbers x. Our proof relies on using the expansion Taylor series at $\boldsymbol{x} \to \infty$ for some new representations of the Legendre formula

We present the first representation :

$$\boldsymbol{\pi(x) = \frac{1}{\sqrt[x]{x.e^{-a}} - 1} + \frac{1}{2}}$$

And the second representation :

$$\boldsymbol{\pi(x) = \frac{1}{\sqrt[x]{x} - 1 - \frac{a}{x}} + \frac{1}{2}}$$

$\boldsymbol{where\ the\ Legendre\ constant\ \ a = 1.08633}$,„

Keywords : Prime number, prime counting function, Legendre formula, Taylor series.

1. Introduction

This paper concerns some representations for the Legendre formula and the Gauss function for the prime counting function $\pi(x)$.
The prime density function, the Gauss conjecture, states that :

$$\pi(x) \sim \frac{x}{\ln(x)} \quad as\ x \to \infty \qquad (1)$$

The Legendre's conjecture regarding the $\pi(x)$ states that :

$$\pi(x) = \frac{x}{\ln(x) - a} \qquad (2)$$

where the Legendre constant $a = 1.08633$,„

***Theorem* 1** : we will present now the first representation and the proof for the Legendre formula

$$\pi(x) = \frac{1}{\sqrt[x]{x.e^{-a}} - 1} + \frac{1}{2} \quad as\ x \to \infty$$

Proof theorem 1 : Taylor series *of* $\pi(x)$ *as* $x \to \infty$

I'm required to make a Taylor series expansion of a function $\pi(x)$ at $x \to \infty$ .In order to do this I introduce new variable $x = \frac{1}{y}$, so that $x \to +\infty$ is the same as $y \to 0^+$. Thus I can expand $\pi\left(\frac{1}{y}\right)$ at $y = 0$:

$$\pi\left(\frac{1}{y}\right) = \frac{1}{y(-\ln(y) - a)} - \frac{1}{2} - \frac{y(\ln(y) + a)}{12} + O(y^2)$$

Taylor expansion at infinity

$$\pi(x) = \frac{x}{\ln(x) - a} - \frac{1}{2} + \frac{\ln(x) - a}{12x} + O\left(\frac{1}{x^2}\right)$$

We have

$$\lim_{x\to\infty} \frac{\ln(x) - a}{12x} = 0$$

So that

$$\pi(x) = \frac{x}{\ln(x) - a} - \frac{1}{2} \quad as\ x \to \infty$$

Hence

$$\frac{1}{\sqrt[x]{x.e^{-a}} - 1} + \frac{1}{2} = \frac{x}{\ln(x) - a} \qquad (3)$$

The relation (3) give the new representation of the Legendre formula for $a = 1.08633$,„

***Theorem* 2 :** The second representation for the Legendre formula

$$\pi(x) = \frac{1}{\sqrt[x]{x} - 1 - \frac{a}{x}} + \frac{1}{2} \quad as\ x \to \infty$$

Proof theorem 2

We use the same method of proof 1 ,Taylor series *of* $\pi(x)$ *as* $x \to \infty$ and we find:

$$\pi(x) = \frac{x}{\ln(x) - a} - \frac{\ln(x)^2}{2(a - \ln(x))^2} + O\left(\frac{1}{x^2}\right)$$

We are looking for the Taylor series of $\pi(x)$, We have :

$$\frac{\ln(x)^2}{2(a - \ln(x))^2} \sim \frac{1}{2} \quad as\ x \to \infty$$

Hence

$$\frac{1}{\sqrt[x]{x}-1-\frac{a}{x}}=\frac{x}{\ln(x)-a}-\frac{1}{2} \quad as\ x\to\infty \qquad (4)$$

2. Main results :

Looking at the expression of $(1),(3)$ and (4) above, for $a=0$ we obtain a new representation for the Gauss conjecture :

$$\pi(x)\sim\frac{1}{\sqrt[x]{x}-1}+\frac{1}{2} \quad as\ x\to\infty$$

Looking at the expression of (2) , (3) and (4) above, for $a=1.08633$,,, we obtain some new representations for the legendre formula :

$$\pi(x)=\frac{1}{\sqrt[x]{x.e^{-a}}-1}+\frac{1}{2}$$

And

$$\pi(x)=\frac{1}{\sqrt[x]{x}-1-\frac{a}{x}}+\frac{1}{2}$$

This completes the proof. The table shows how the functions $\frac{1}{\sqrt[x]{x}-1-\frac{a}{x}}+\frac{1}{2}$, $\frac{1}{\sqrt[x]{x.e^{-a}}-1}+\frac{1}{2}$ and the Legendre formula compart at powers of 10. $(for\ a=1.08633,,,)$

x	$\frac{1}{\sqrt[x]{x}-1-\frac{a}{x}}+\frac{1}{2}$	$\frac{1}{\sqrt[x]{x.e^{-a}}-1}+\frac{1}{2}$	$\frac{x}{\ln(x)-a}$
10	7	8	8
10^2	28	28	28
10^3	171	171	171
10^4	1230	1230	1230
10^5	9590	9590	9590
10^6	78 559	78 559	78 559
10^7	665 257	665 257	665 257
10^8	5 768 892	5 768 892	5 768 892
10^9	50 924 441	50 924 441	50 924 441
10^{10}	455 798 466	455 798 466	455 798 466
10^{11}	4 125 054 147	4 125 054 147	4 125 054 147
10^{12}	37 672 316 307	37 672 316 307	37 672 316 307
10^{13}	346 653 178 885	346 653 178 885	346 653 178 885
10^{14}	3 210 287 167 276	3 210 287 167 276	3 210 287 167 276
10^{15}	29 893 179 954 460	29 893 179 954 460	29 893 179 954 460
10^{16}	279 680 917 170 575	279 680 917 170 575	279 680 917 170 575
10^{17}	2 627 594 920 124 090	2 627 594 920 124 090	2 627 594 920 124 090
10^{18}	24 776 883 130 563 108	24 776 883 130 563 108	24 776 883 130 563 108
10^{19}	234 396 314 864 306 897	234 396 314 864 306 897	234 396 314 864 306 897
10^{20}	2 223 933 570 740 069 490	2 223 933 570 740 069 490	2 223 933 570 740 069 490

Table 1

We can see above at table 1 that the new representations gives exactly the values of Legendre's formula

REFERENCES :

[1]: $R.$ *Farhadian and* $R.$ *Jakimczuk, One more disproof for the Legendre's conjecture reading the prime counting function, Notes on Number theory and Discrete Mathematics, DOI*: 10.7546/*nntdm*.2018.24.3.84 − 91

[2]: İ. *Okumuş and E. Çelik, Improved of Approximating Function* $Li(x)$, *International Journal of Engineering and Applied Sciences* (*IJEAS*), *ISSN*: 2394 − 3661, *Volume* − 5, *Issue* − 4, *April* 2018

[3]: *T. Kotnik, The prime counting function and its analytic approximations, Springer science, Slovenia* (2007).

[4]: *A. E. Patkowski, A note on the Gram Series , International Mathematical Forum* (2008).

[5]: *J. R. Sousa, An Exact Formula for the prime counting function, Reaserchgate, S. Paulo − Brazil* (2019).

[6]: *M. Hassani, On the means of the values of prime counting function, Iranian journal of mathematical sciences and informatics, Iran* (2018).

REFINEMENTS OF EULER'S INEQUALITY

By Marius Drăgan, Neculai Stanciu – Romania

Abstract. In this short note we present new refinements of Euler's inequality.

MSC Subject Classification: 51M16, 26D05

Keywords and phrases: Jensen inequality, Mitrinovic inequality, Gerretsen inequality, Euler inequality, geometric identities, geometric inequalities.

In any triangle ABC with usual notations: $a = BC, b = CA, c = AB, R$ circumradius, r inradius and s semiperimeter of triangle the following chain of inequalities are true:

$$\frac{2}{R} \leq \frac{3\sqrt{3}}{\sqrt{4R^2+4Rr+3r^2}} \leq \frac{3\sqrt{3}}{s} \leq \sum \frac{1}{a\cos\frac{A}{2}} \leq \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{s^2+r^2+4Rr}}{sr} \leq \frac{1}{r}\sqrt{\frac{15R-3r}{16R-5r}} \leq \frac{1}{r}$$

Proof. The inequality $\frac{3\sqrt{3}}{s} \leq \sum \frac{1}{a\cos\frac{A}{2}}$ is equivalent to $\sum \frac{1}{\sin\frac{A}{2}\cos^2\frac{A}{2}} \geq \frac{12R\sqrt{3}}{s}$

By AM-GM inequality we obtain that $\sum \frac{1}{\sin\frac{A}{2}\cos^2\frac{A}{2}} \geq 3\sqrt[3]{\frac{1}{\left(\prod \sin\frac{A}{2}\right)\left(\prod \cos^2\frac{A}{2}\right)}}$

Using identities $\prod \sin\frac{A}{2} = \frac{r}{4R}$ and $\prod \cos\frac{A}{2} = \frac{s}{4R}$, we have to show

$\frac{1}{\left(\prod \sin\frac{A}{2}\right)\left(\prod \cos\frac{A}{2}\right)^2} \geq \frac{64\cdot 3\sqrt{3}\cdot R^3}{s^3} \Leftrightarrow s \geq 3\sqrt{3}r$, i.e. Mitrinovic's inequality.

Next, we have the facts: $\sum \frac{1}{a\cos\frac{A}{2}} = \frac{\sum bc\cos\frac{B}{2}\cos\frac{C}{2}}{abc\prod \cos\frac{A}{2}}, bc\cos\frac{B}{2}\cos\frac{C}{2} =$

$= bc\sqrt{\frac{s(s-b)}{ac}\cdot\frac{s(s-c)}{ab}} = \frac{bcs}{a\sqrt{bc}}\sqrt{(s-b)(s-c)} \overset{MA-MG}{\leq} \frac{s\sqrt{bc}}{a}\cdot\frac{s-b+s-c}{2} = \frac{s\sqrt{bc}}{2}$ and

$$abc\prod\cos\frac{A}{2}=abc\sqrt{\frac{s(s-a)}{bc}\cdot\frac{s(s-b)}{ac}\cdot\frac{s(s-c)}{ab}}=s^2r$$

By the facts from above, Jensen's inequality and Gerretsen's inequality ($16Rr-5r^2\le s^2$)

we obtain $\sum\frac{1}{a\cos\frac{A}{2}}\le\frac{s\sum\sqrt{bc}}{2s^2r}\overset{Jensen}{\le}\frac{\sqrt{3\sum bc}}{2sr}=\frac{\sqrt{3(s^2+r^2+4Rr)}}{2sr}=\frac{\sqrt{3}}{2r}\sqrt{1+\frac{r^2+4Rr^2}{s^2}}\overset{Gerretsen}{\le}$

$$\overset{Gerretsen}{\le}\frac{\sqrt{3}}{2r}\sqrt{1+\frac{r^2+4Rr^2}{16Rr-5r^2}}=\frac{1}{r}\sqrt{\frac{15R-3r}{16R-5r}}$$

Since, $\frac{2}{R}\le\frac{3\sqrt{3}}{s}\Leftrightarrow s\le\frac{3\sqrt{3}}{2}R$, i.e. Mitrinovic's inequality;

$\frac{1}{r}\sqrt{\frac{15R-3r}{16R-5r}}\le\frac{1}{r}\Leftrightarrow R\ge 2r$, i.e. Euler's inequality; and

$$\frac{2}{R}\le\frac{3\sqrt{3}}{\sqrt{4R^2+4Rr+3r^2}}\Leftrightarrow 11R^2-16Rr-12R^2\ge 0\Leftrightarrow(11R+6r)(R-2r)\ge 0$$

we obtain:

$$\frac{2}{R}\le\frac{3\sqrt{3}}{\sqrt{4R^2+4Rr+3r^2}}\le\frac{3\sqrt{3}}{s}\le\sum\frac{1}{a\cos\frac{A}{2}}\le\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{s^2+r^2+4Rr}}{sr}\le\frac{1}{r}\sqrt{\frac{15R-3r}{16R-5r}}\le\frac{1}{r}$$

which represent refinements of Euler's inequality.

Remark. The inequality $\frac{2}{R}\le\sum\frac{1}{a\cos\frac{A}{2}}\le\frac{1}{r}$ represent the problem 12168 from The American

Mathematical Monthly (AMM), Vol. 127, No. 3, March 2020, p. 274, proposed by professor

Martin Lukarevski, University "Goce Delcev", Stip, North Macedonia.

ABOUT A RMM INEQUALITY-IX

By Marin Chirciu

1) In ΔABC the following inequality holds:

$$\frac{a^4}{r_ar_b}+\frac{b^4}{r_br_c}+\frac{c^4}{r_cr_a}\ge\frac{16F}{\sqrt{3}}$$

D.M. Bătinețu-Giurgiu, Flaviu Cristian Verde

Solution: We have

$$Ms=\sum\frac{a^4}{r_ar_b}=\frac{1}{r_ar_br_c}\sum a^4r_c=\frac{1}{r_ar_br_c}\sum\frac{a^4}{\frac{1}{r_c}}\overset{Holder}{\ge}\frac{1}{rp^2}\cdot\frac{(\sum a)^4}{9\sum\frac{1}{r_a}}=$$

$=\frac{1}{rp^2}\cdot\frac{(2p)^4}{9\cdot\frac{1}{r}}=\frac{16p^2}{9}\overset{(1)}{\geq}\frac{16pr}{\sqrt{3}}=\frac{16F}{\sqrt{3}}=Md$, where (1) $\Leftrightarrow p\geq 3r\sqrt{3}$, (Mitrinovic inequality)

Equality holds if and only if the triangle is equilateral.

Remark: In the same way:

2) In ΔABC the following relationship holds:

$$48r^2\leq\frac{a^4}{r_br_c}+\frac{b^4}{r_cr_a}+\frac{c^4}{r_ar_b}\leq\frac{16}{r}(R^3-5r^3)$$

Marin Chirciu

Solution: We prove:

Lemma.

3) In ΔABC the following relationship holds:

$$\sum\frac{a^4}{r_br_c}=\frac{4}{r}[p^2(R-2r)+r^2(5R+2r)]$$

Proof.

$$\sum\frac{a^4}{r_br_c}=\sum\frac{a^4}{\frac{S}{s-b}\frac{S}{p-c}}=\frac{1}{S^2}\sum s^4(s-b)(s-c)=$$

$$=\frac{1}{s^2r^2}\cdot 4rp^2[p^2(R-2r)+r^2(5R+2r)]=$$

$=\frac{4}{r}[s^2(R-2r)+r^2(5R+2r)]$, which follows from

$$\sum a^4\,(s-b)(s-c)=4rp^2[p^2(R-2r)+r^2(5R+2r)]$$

Back to the main problem. RHS inequality.

$$\sum\frac{a^4}{r_br_c}=\frac{4}{r}[p^2(R-2r)+r^2(5R+2r)]\overset{Gerretsen}{\leq}$$

$$\leq\frac{4}{r}[(4R^2+4Rr+3r^2)(R-2r)+r^2(5R+2r)]=$$

$$=\frac{4}{r}(4R^3-4R^2r-4r^3)=\frac{16}{r}(R^3-R^2r-r^3)\overset{Euler}{\leq}\frac{16}{r}(R^3-5r^3)$$

Equality holds if and only if the triangle is equilateral.

LHS inequality

$$\sum\frac{a^4}{r_br_c}=\frac{4}{r}[p^2(R-2r)+r^2(5R+2r)]\overset{Gerretsen}{\geq}$$

$$\geq\frac{4}{r}[(16Rr-5r^2)(R-2r)+r^2(5R+2r)]=$$

$$=4(16R^2-32Rr+12r^2)\overset{Euler}{\geq}48r^2$$

Equality holds if and only if the triangle is equilateral.

Remark. In the same way:

4) In ΔABC the following relationship holds:

$$\mathbf{24Rr\leq\frac{a^4}{h_bh_c}+\frac{b^4}{h_ch_a}+\frac{c^4}{h_ah_b}\leq 4R^2\left(\frac{2R}{r}-1\right)}$$

Marin Chirciu

Solution: We prove

Lemma.

5) In ΔABC the following relationship holds:

$$\mathbf{\sum\frac{a^4}{h_bh_c}=\frac{2R}{r}(p^2-3r^2-6Rr)}$$

Proof.

$$\sum\frac{a^4}{h_bh_c}=\sum\frac{a^4}{\frac{2S}{b}\frac{2S}{c}}=\frac{1}{4S^2}\sum a^4bc=\frac{1}{4p^2r^2}\cdot abc\sum a^3=$$

$$=\frac{1}{4p^2r^2}\cdot 4Rrp\cdot 2p(p^2-3r^2-6Rr)=$$

$=\frac{2R}{r}(p^2-3r^2-6Rr)$, which follows from $\sum a^3=2p(p^2-3r^2-6Rr)$

Back to the main problem. RHS inequality

$$\sum\frac{a^4}{h_bh_c}=\frac{2R}{r}(s^2-3r^2-6Rr)\overset{Gerretsen}{\leq}\frac{2R}{r}(4R^2+4Rr+3r^2-3r^2-6Rr)=$$

$$=\frac{2R}{r}(4R^2-2Rr)=\frac{4R^2}{r}(2R-r)=4R^2\left(\frac{2R}{r}-1\right)$$

Equality holds if and only if the triangle is equilateral.

LHS inequality

$$\sum\frac{a^4}{h_bh_c}=\frac{2R}{r}(p^2-3r^2-6Rr)\overset{Gerretsen}{\geq}\frac{2R}{r}(16Rr-5r^2-3r^2-6Rr)=$$

$$=\frac{2R}{r}(10Rr-8r^2)=4R(5R-4r)\overset{Euler}{\geq}24Rr$$

Equality holds if and only if the triangle is equilateral.

Remark.

Between the sums $\sum\frac{a^4}{h_bh_c}$ and $\sum\frac{a^4}{r_br_c}$ the following relationship holds:

6) In ΔABC the following relationship holds:

$$\sum \frac{a^4}{h_b h_c} \le \sum \frac{a^4}{r_b r_c}$$

Marin Chirciu

Solution: Using the sums $\sum \frac{a^4}{h_b h_c} = \frac{2R}{r}(p^2 - 3r^2 - 6Rr)$ and $\sum \frac{a^4}{r_b r_c} = \frac{4}{r}[p^2(R-2r) + r^2(5R+2r)]$, the inequality can be written:

$$\frac{2R}{r}(p^2 - 3r^2 - 6Rr) \le \frac{4}{r}[p^2(R-2r) + r^2(5R+2r)] \Leftrightarrow$$

$$\Leftrightarrow R(p^2 - 3r^2 - 6Rr) \le 2[p^2(R-2r) + r^2(5R+2r)] \Leftrightarrow$$

$$\Leftrightarrow p^2(R-4r) + r(6R^2 + 13Rr + 4r^2) \ge 0$$

We distinguish the following cases:

Case 1). If $(R-4r) \ge 0$, the inequality is obvious.

Case 2). If $(R-4r) < 0$, the inequality can be rewritten:

$r(6R^2 + 13Rr + 4r^2) \ge p^2(4r - R)$, which follows from Gerretsen's inequality:

$$p^2 \le 4R^2 + 4Rr + 3r^2$$

It remains to prove that:

$$r(6R^2 + 13Rr + 4r^2) \ge (4R^2 + 4Rr + 3r^2)(4r - R) \Leftrightarrow 2R^3 - 3R^2 r - 4r^3 \ge 0 \Leftrightarrow$$

$\Leftrightarrow (R-2r)(2R^2 + Rr + 2r^2) \ge 0$, obviously from Euler's inequality $R \ge 2r$.

Equality holds if and only if the triangle is equilateral.

Reference:

ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

ABOUT A RMM INEQUALITY-VIII

By Marin Chirciu

1) If $a, b, c > 0$ and $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = 1$ then $a + b + c \ge 6$

Daniel Sitaru – Romania

Solution: Using Bergström's inequality we obtain:

$1 = \frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} \ge \frac{(1+1+1)^2}{\sum(a+1)} = \frac{9}{\sum a + 3}$, wherefrom $1 \ge \frac{9}{\sum a + 3} \Leftrightarrow \sum a \ge 6$.

Equality holds if and only if $a = b = c = 2$.

Remark The problem can be developed.

2) If $a,b,c>0$ such that $\frac{1}{a+\lambda}+\frac{1}{b+\lambda}+\frac{1}{c+\lambda}=\frac{3}{\lambda+2}$ and $\lambda\ge 0$ then $a+b+c\ge 6$

Marin Chirciu

Solution: Using Bergström's inequality we obtain:
$\frac{3}{\lambda+2}=\frac{1}{a+\lambda}+\frac{1}{b+\lambda}+\frac{1}{c+\lambda}\ge\frac{(1+1+1)^2}{\sum(a+\lambda)}=\frac{9}{\sum a+3\lambda}$, wherefrom $\frac{3}{\lambda+2}\ge\frac{9}{\sum a+3\lambda}\Leftrightarrow\sum a\ge 6$.
Equality holds if and only if $a=b=c=2$.
Note: For $\lambda=1$ we obtain the problem proposed by Daniel Sitaru in RMM 11/2020.
Remark: The problem can be developed.

3) If $a_1,a_2,\dots,a_n>0$ such that $\frac{1}{a_1+\lambda}+\frac{1}{a_2+\lambda}+\dots+\frac{1}{a_n+\lambda}=\frac{n}{\lambda+2}$ and $\lambda\ge 0$ then

$$a_1+a_2+\dots+a_n\ge 2n$$

Marin Chirciu

Solution: Using Bergström's inequality we obtain:
$\frac{n}{\lambda+2}=\frac{1}{a_1+\lambda}+\frac{1}{a_2+\lambda}+\dots+\frac{1}{a_n+\lambda}\ge\frac{(1+1+\dots+1)^2}{\sum(a_1+\lambda)}=\frac{n^2}{\sum a_1+n\lambda}$, wherefrom
$\frac{n}{\lambda+2}\ge\frac{n^2}{\sum a_1+n\lambda}\Leftrightarrow\sum a\ge 2n$
Equality holds if and only if $a_1=a_2=\dots=a_n=2$.
Note: For $\lambda=1$ and $n=3$ we obtain the problem proposed by Daniel Sitaru in RMM 11/2020.
Remark: The problem can be developed.

4) If $a_1,a_2,\dots,a_n>0$ such that $\frac{1}{a_1+\lambda}+\frac{1}{a_2+\lambda}+\dots+\frac{1}{a_n+\lambda}=\frac{n}{\lambda+1}$ and $\lambda\ge 0$ then

$$a_1+a_2+\dots+a_n\ge n$$

Marin Chirciu

Solution: Using Bergström's inequality we obtain:

$$\frac{n}{\lambda+1}=\frac{1}{a_1+\lambda}+\frac{1}{a_2+\lambda}+\dots+\frac{1}{a_n+\lambda}\ge\frac{(1+1+\dots+1)^2}{\sum(a_1+\lambda)}=\frac{n^2}{\sum a_1+n\lambda}$$

wherefrom $\frac{n}{\lambda+1}\ge\frac{n^2}{\sum a_1+n\lambda}\Leftrightarrow\sum a\ge n$. Equality holds if and only if $a_1=a_2=\dots=a_n=1$.

Reference:
ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

ABOUT AN INEQUALITY BY GEORGE APOSTOLOPOULOS-IX

By Marin Chirciu

1) In ΔABC the following relationship holds:

$$9\sqrt{3}r^{\frac{3}{2}}\le m_a\sqrt{h_a}+m_b\sqrt{h_b}+m_c\sqrt{h_c}\le\frac{9\sqrt{6}}{4}R^{\frac{3}{2}}$$

George Apostolopoulos – Greece

Solution:

Using CBS inequality we obtain:

$$E^2=\left(\sum m_a\sqrt{h_a}\right)^2\overset{CBS}{\le}\sum m_a^2\sum h_a=\frac{3}{4}\sum a^2\sum h_a\overset{Leibniz}{\le}\frac{3}{4}\cdot 9R^2\cdot\sum h_a\overset{(1)}{\le}$$

$$\leq \frac{3}{4}\cdot 9R^2\cdot\frac{9R}{2}=\frac{3}{8}\cdot 81R=\frac{9\sqrt{6}}{4}R^{\frac{3}{2}}=M^2d, \text{ where } \sum h_a\leq\frac{9R}{2}$$

Equality holds if and only if the triangle is equilateral.

LHS inequality: Using Chebyshev's inequality for the same ordered triplets (m_a,m_b,m_c) and $(\sqrt{h_a},\sqrt{h_b},\sqrt{h_c})$ we obtain:

$$E=\sum m_a\sqrt{h_a}\overset{Chebyshev}{\geq}\frac{1}{3}\sum m_a\sum\sqrt{h_a}\overset{(2)}{\geq}\frac{1}{3}\cdot 9r\cdot\sum\sqrt{h_a}\overset{(3)}{\geq}$$

$\geq\frac{1}{3}\cdot 9r\cdot 3\sqrt{3r}=9\sqrt{3}r^{\frac{3}{2}}=Ms$, where (2) $\Leftrightarrow \sum m_a\geq 9r$ and (3) $\Leftrightarrow \sum\sqrt{h_a}\geq 3\sqrt{3r}$, which follows from means inequality, the identity $\prod h_a=\frac{2p^2r^2}{R}$ and Coșniță and Turtoiu $2p^2\geq 27Rr$. Equality holds if and only if the triangle is equilateral.

Remark: In the same way:

2) In ΔABC the following relationship holds:

$$\mathbf{9\sqrt{3}r^{\frac{3}{2}}\leq m_a\sqrt{w_a}+m_b\sqrt{w_b}+m_c\sqrt{w_c}\leq\frac{9\sqrt{6}}{4}R^{\frac{3}{2}}}$$

Marin Chirciu

Solution: RHS inequality.

Using CBS inequality we obtain:

$$E^2=\left(\sum m_a\sqrt{h_a}\right)^2\overset{CBS}{\leq}\sum m_a^2\sum w_a=\frac{3}{4}\sum a^2\sum w_a\overset{Leibniz}{\leq}\frac{3}{4}\cdot 9R^2\cdot\sum w_a\overset{(1)}{\leq}$$

$\leq\frac{3}{4}\cdot 9R^2\cdot\frac{9R}{2}=\frac{3}{8}\cdot 81R=\frac{9\sqrt{6}}{4}R^{\frac{3}{2}}=M^2d$, where (1) $\Leftrightarrow \sum w_a\leq\frac{9R}{2}$.

Equality holds if and only if the triangle is equilateral. LHS inequality:

Using Chebyshev inequality for the same ordered triplets (m_a,m_b,m_c) and $(\sqrt{w_a},\sqrt{w_b},\sqrt{w_c})$ we obtain:

$$E=\sum m_a\sqrt{w_a}\overset{Chebyshev}{\geq}\frac{1}{3}\sum m_a\sum\sqrt{w_a}\overset{(2)}{\geq}\frac{1}{3}\cdot 9r\cdot\sum\sqrt{w_a}\overset{(3)}{\geq}\frac{1}{3}\cdot 9r\cdot 3\sqrt{3r}=$$

$=9\sqrt{3}r^{\frac{3}{2}}=Ms$, where (2) $\Leftrightarrow \sum m_a\geq 9r$ and (3) $\Leftrightarrow \sum\sqrt{w_a}\geq 3\sqrt{3r}$, which follows from $\sum\sqrt{w_a}\geq\sum\sqrt{h_a}$, from $\sum\sqrt{h_a}\geq 3\sqrt{3r}$, means inequality, the identity $\prod h_a=\frac{2p^2r^2}{R}$ and Coșniță and Turtoiu $2p^2\geq 27Rr$. Equality holds if and only if the triangle is equilateral.

Remark: In the same way:

3) In ΔABC the following relationship holds:

$$\mathbf{9\sqrt{3}r^{\frac{3}{2}} \le m_a\sqrt{r_a} + m_b\sqrt{r_b} + m_c\sqrt{r_c} \le \frac{9\sqrt{6}}{4}R^{\frac{3}{2}}}$$

Marin Chirciu

Solution: RHS inequality: Using CBS inequality we obtain:

$$E^2 = \left(\sum m_a\sqrt{r_a}\right)^2 \overset{CBS}{\le} \sum m_a^2 \sum r_a = \frac{3}{4}\sum a^2 \sum w_a \overset{Leibniz}{\le} \frac{3}{4}\cdot 9R^2 \cdot \sum r_a \overset{(1)}{\le}$$

$\le \frac{3}{4}\cdot 9R^2 \cdot \frac{9R}{2} = \frac{3}{8}\cdot 81R^3 = \left(\frac{9\sqrt{6}}{4}R^{\frac{3}{2}}\right)^2 = M^2 d$, where (1) follows from

$$\sum r_a = 4R + r \overset{Euler}{\le} \frac{9R}{2}$$

Equality holds if and only if the triangle is equilateral.
LHS inequality: Using means inequality we obtain:

$$E = \sum m_a\sqrt{r_a} \overset{AGM}{\ge} 3\sqrt[3]{m_a m_b m_c\sqrt{r_a r_b r_c}} \overset{(2)}{\ge} 3\sqrt[3]{27r^3\sqrt{rp^2}} \overset{Mitrinovic}{\ge}$$

$\ge 3\sqrt[3]{27r^3\sqrt{27r^3}} = 3\cdot 3r\sqrt{3r} = 9\sqrt{3}\cdot r^{\frac{3}{2}} = Ms$, where (2) $\Leftrightarrow m_a m_b m_c \ge 27r^3$.
Equality holds if and only if the triangle is equilateral.
Remark: In the same way:
4) In ΔABC the following relationship holds:

$$\mathbf{9\sqrt{3}r^{\frac{3}{2}} \le m_a\sqrt{m_a} + m_b\sqrt{m_b} + m_c\sqrt{m_c} \le \frac{9\sqrt{6}}{4}R^{\frac{3}{2}}}$$

Marin Chirciu

Solution: RHS inequality: Using CBS inequality we obtain:

$$E^2 = \left(\sum m_a\sqrt{m_a}\right)^2 \overset{CBS}{\le} \sum m_a^2 \sum m_a = \frac{3}{4}\sum a^2 \sum w_a \overset{Leibniz}{\le} \frac{3}{4}\cdot 9R^2 \cdot \sum m_a \overset{(1)}{\le}$$

$\le \frac{3}{4}\cdot 9R^2 \cdot \frac{9R}{2} = \frac{3}{8}\cdot 81R^3 = \left(\frac{9\sqrt{6}}{4}R^{\frac{3}{2}}\right)^2 = M^2 d$, where (1) it follows from

$$\sum m_a \le 4R + r \overset{Euler}{\le} \frac{9R}{2}$$

Equality holds if and only if the triangle is equilateral.
RHS inequality: Using means inequality we obtain:

$$E = \sum m_a\sqrt{m_a} \overset{ACM}{\ge} 3\sqrt[3]{m_a m_b m_c\sqrt{m_a m_b m_c}} \overset{(2)}{\ge} 3\sqrt[3]{27r^3\sqrt{27r^3}} = 3\cdot 3r\sqrt{3r} =$$

$= 9\sqrt{3}\cdot r^{\frac{3}{2}} = Ms$, where (2) $\Leftrightarrow m_a m_b m_c \ge 27r^3$.

Equality holds if and only if the triangle is equilateral.

Remark: In the same way:

5) In ΔABC the following inequality holds:

$$\mathbf{m_a\sqrt{s_a} + m_b\sqrt{s_b} + m_c\sqrt{s_c} \le \frac{9\sqrt{6}}{4}R^{\frac{3}{2}}}$$

Marin Chirciu

Solution: Using CBS inequality we obtain:

$$E^2=\left(\sum m_a\sqrt{s_a}\right)^2\overset{CBS}{\leq}\sum m_a^2\sum s_a=\frac{3}{4}\sum a^2\sum s_a\overset{Leibniz}{\leq}\frac{3}{4}\cdot 9R^2\cdot\sum s_a\overset{(1)}{\leq}$$

$\leq\frac{3}{4}\cdot 9R^2\cdot\frac{9R}{2}=\frac{3}{8}\cdot 81R^3=\left(\frac{9\sqrt{6}}{4}R^{\frac{3}{2}}\right)^2=M^2d$, where (1) it follows from

$$\sum s_a=\sum\frac{2bc}{b^2+c^2}m_a\leq\sum m_a\leq 4R+r\overset{Euler}{\leq}\frac{9R}{2}$$

Equality holds if and only if the triangle is equilateral.

Reference:

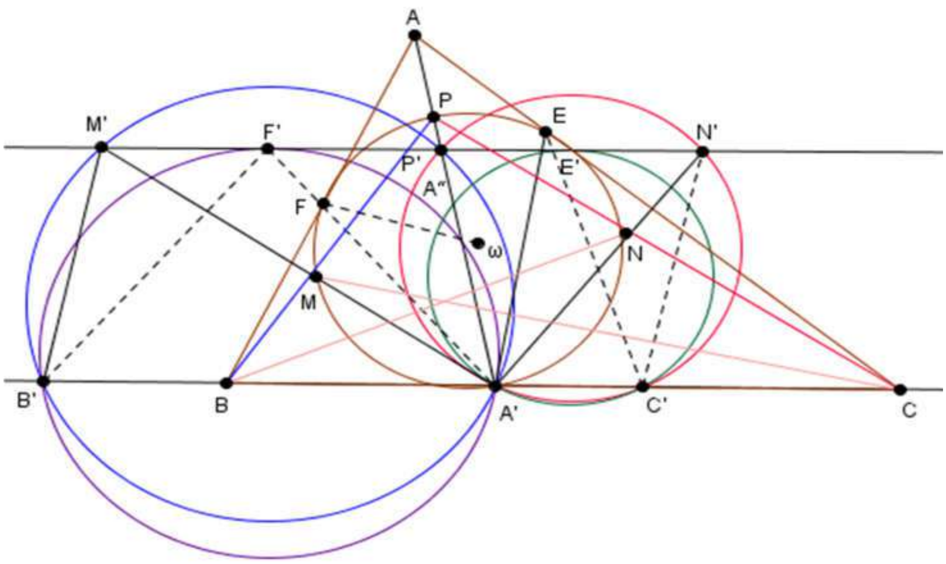
ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

AN AMAZING CONCURRENT PROBLEM

By Adrian Popa-Romania

Let be $\Delta ABC,\omega$ –the incircle, A' –the tangent point by sides BC with the circle ω. Let be $\{P\}=AA'\cap\omega,\{M\}=BP\cap\omega,\{N\}=CP\cap\omega$.Prove that AA',BN,CM are concurrent.

Solution 1:



Denote the inverse of the points:

$i(B)=B',i(C)=C',\ldots,i(M)=M',i(N)=N'$and $i(A)=A"$

If AA',BN,CM are concurrent, form Ceva's theorem, we have:

$$\frac{A'B}{A'C}\cdot\frac{NC}{NP}\cdot\frac{MP}{MB}=1.$$

From the power of the point B to ω we have:

$$BM\cdot CP=BA'^2\Rightarrow BM=\frac{BA'^2}{BP}.$$

From the power of the point C to ω we have:

$$CN\cdot CP=CA'^2\Rightarrow CN=\frac{CA'^2}{CP}.$$

Then, we get:

$$\frac{A'B}{A'C}\cdot\frac{CA'^2}{CN\cdot CP}\cdot\frac{MP\cdot BP}{BA'^2}=1\Rightarrow\frac{MP}{NP}\cdot\frac{CA'}{BA'}\cdot\frac{BP}{CP}=1.$$

$$\Delta BMA'\sim\Delta BA'P\left(\hat{B}\equiv\hat{B},\widehat{MA'B}\equiv\widehat{BPA'}\equiv\frac{\widehat{MA'}}{2}\right)\Rightarrow\frac{BP}{BA'}=\frac{PA'}{MA'}$$

$$\Delta CPA'\sim\Delta CA'N\left(\hat{C}\equiv\hat{C},\widehat{MA'B}\equiv\widehat{NA'C}\equiv\frac{\widehat{NA'}}{2}\right)\Rightarrow\frac{CA'}{CP}=\frac{NA'}{PA'}$$

Then, we get:

$$\frac{MP}{NP}\cdot\frac{CA'}{BA'}\cdot\frac{BP}{CP}=\frac{MP}{NP}\cdot\frac{A'N}{A'P}\cdot\frac{A'P}{A'M}=1\Rightarrow\frac{MP}{NP}=\frac{MA'}{NA'};(1)$$

Let be center inversion A' and power $k>0,k\in\mathbb{R}$. The circle ω –it's transformed into a straight line perpendicular to the diameter passing through to inversion $A',\omega A'\perp BC\Rightarrow d\parallel BC$.

Let be E,F –the point by tangent's of ω with AB,AC respectively, then $i(AB)$ is on circle tangent to d in F' and $i(AC)$ is on circle tangent to d in E'.

$$M'P'=\frac{k\cdot A'M\cdot A'P}{k}\Rightarrow MP=\frac{M'P'\cdot A'M\cdot A'P}{k};$$

$$N'P'=\frac{k\cdot NP}{A'N\cdot A'P}\Rightarrow NP=\frac{N'P'\cdot A'N\cdot A'P}{k}$$

From (1) we get:

$$\frac{MP}{NP}=\frac{M'P'\cdot A'M\cdot A'P}{k}\cdot\frac{k}{N'P'\cdot A'N\cdot A'P}=\frac{MA'}{NA'}\cdot\frac{M'P'}{N'P'}=\frac{MA'}{NA'}$$

We must show that

$$\frac{M'P'}{N'P'}=1$$

$$\begin{cases}i(AB)=(O_1)\\ i(AC)=(O_2)\end{cases}\Rightarrow(O_1)\cap(O_2)=A"=i(A)\Rightarrow AB\cap AC=\{A\}\Rightarrow A,A',A"\text{ –collinear.}$$

$$\begin{cases}P\in AA'\Rightarrow P'=i(P)\in AA'\\ P\in\omega\Rightarrow i(P)=P'\in d\end{cases}\Rightarrow\begin{cases}P\in d\\ (O_1)\text{tangent to } d \text{ in } F'\Rightarrow P'F'=P'E'.\\ (O_2)\text{tangent to } d \text{ in } E'\end{cases}$$

The line BP it transforms into the passing circle through it points A', B', P', M' because $M \in BP \Rightarrow M' \in i(BP) \Rightarrow M'P'A'B'$ −inscriptible and $M'P' \parallel A'B' \Rightarrow M'P'A'B'$ −isosceles trapeze, then $M'B' \equiv P'A'; (a)$

The line CP it transforms into the passing it points A', C', P', N' because $N \in CP \Rightarrow N' \in i(CP) \Rightarrow N'P'A'C'$ −inscriptible and $N'P' \parallel A'C' \Rightarrow N'P'A'C'$ −isosceles trapeze, then $N'C' \equiv P'A'; (b)$ .From (a),(b) we have $M'B' \equiv C'N'$.

$$\begin{cases} A' - inversion\ pole \\ B' = i(B) \\ F' = i(F) \end{cases} \Rightarrow \Delta A'FB \sim \Delta A'B'F' \Rightarrow \frac{A'B}{A'F'} = \frac{BF}{B'F'} \Rightarrow \frac{A'B}{BF} = \frac{A'F'}{B'F'}; A'B \equiv BF \Rightarrow \frac{A'F'}{B'F'} =$$

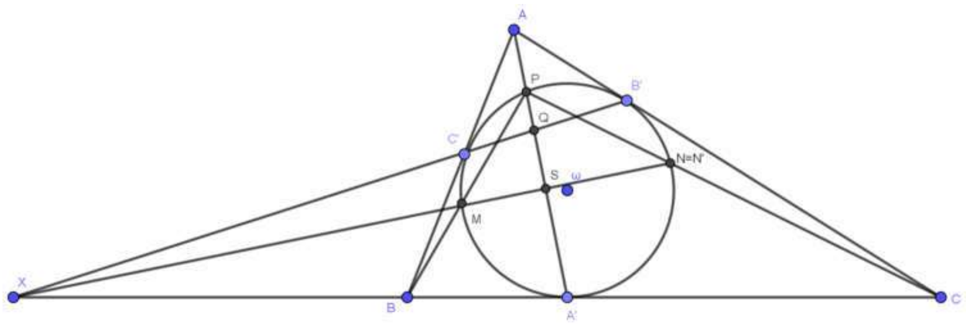
$1 \Rightarrow A'F' \equiv B'F' \Rightarrow M'F' \equiv F'P'$.

$$\begin{cases} A' - inversion\ pole \\ C' = i(C) \\ E' = i(E) \end{cases} \Rightarrow \Delta AEC \sim \Delta A'C'E' \Rightarrow \frac{A'C}{A'E'} = \frac{CE}{E'C'} \Rightarrow \frac{A'C}{EC} = \frac{A'E'}{E'C'}; A'C \equiv EC \Rightarrow \frac{A'E'}{C'E'} =$$

$1 \Rightarrow A'E' \equiv C'E' \Rightarrow N'E' \equiv E'P'$.

So, $\begin{cases} M'F' \equiv F'P' \\ P'E' \equiv E'N' \\ F'P' \equiv P'E' \end{cases} \Rightarrow M'F' + F'P' \equiv P'E' + E'N' \Rightarrow MP' \equiv N'P'$.

Solution 2



$$\begin{cases} A \in Ext(\omega) \\ AB' - \text{tangent to } (\omega) \\ AC' - \text{tangent to } (\omega) \end{cases} \Rightarrow B'C' - \text{polar point } A \text{ to } (\omega).$$

$$\begin{cases} A \in Ext(\omega) \\ BC - \text{tangent to } (\omega) \text{in } A' \Rightarrow BC - \text{polar point } A' \text{ to } (\omega). \\ BC \cap B'C' = \{X\} \end{cases}$$

So, X −pole by AA'.Let $B'C'$ −the rope passing through the pole $X \Rightarrow$

$$(C'B'XQ) = -1 \Rightarrow (A, C'B'XQ) = -1.$$

$$\begin{cases} X - B - A' - C \\ X - C' - Q - B' \\ A(C'B'XQ) = -1 \end{cases} \Rightarrow (A, C'B'XQ) = (ABCXA') = -1 \Rightarrow (BCXA') = -1.$$

Let the fascicle PB, PC, PX, PA' with center P, then $(P, BCXA") = -1, XM \cap PC = \{N'\}$, then $(P, BCXA') = (P, MN'XS) = -1, (P, BCXA') = -1; PM = PB, PS = PA', PX = PX \Rightarrow PN' = PC; PC \cap \omega = \{N\}, N' \in \omega \Rightarrow N = N' \Rightarrow X, M, N$ −collinear,

$(BCXA') = -1 \Rightarrow BN \cap MC \cap PA'$ −are concurrent (from the reciprocical of the theorem Papus)

Reference:

ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

A SIMPLE PROOF FOR HUYGENS' INEQUALITY

By Daniel Sitaru – Romania

HUYGENS' INEQUALITY

$$\textbf{If } 0 < x < \frac{\pi}{2} \textbf{ then } 2\left(\frac{\sin x}{x}\right) + \frac{\tan x}{x} > 3$$

Proof: Let be $f: \left(0, \frac{\pi}{2}\right) \to \mathbb{R}; f(x) = 2\sin x + \tan x - 3x$

$$f'(x) = 2\cos x + \frac{1}{\cos^2 x} - 3, \qquad f''(x) = -2\sin x - \frac{(\cos^2 x)'}{(\cos^2 x)^2}$$

$$f''(x) = -2\sin x - \frac{2(\cos x)' \cdot \cos x}{\cos^4 x}, \quad f''(x) = -2\sin x + \frac{2\sin x}{\cos^3 x}$$

$$f''(x) = -2\sin x\left(1 - \frac{1}{\cos^3 x}\right), \quad f''(x) = \frac{2\sin x\,(1 - \cos^3 x)}{\cos^3 x} > 0$$

$$f' \text{ increasing} \Rightarrow f'(x) > 0 > \lim_{\substack{x\to 0\\ x>0}} f'(x) = 0$$

$$f \text{ increasing} \Rightarrow f(x) > \lim_{\substack{x\to 0\\ x>0}} f(x) = 0 \Rightarrow f(x) > 0$$

$$2\sin x + \tan x - 3x > 0, \qquad 2\sin x + \tan x > 3x,$$

$$2\left(\frac{\sin x}{x}\right) + \frac{\tan x}{x} > 3$$

APPLICATION FOR DANIEL SITARU'S INEQUALITY

By Long Huynh Huu-Vietnam

Solution attempt by Long Huynh Huu (@erugli) for Dan Sitaru's inequality which was posted on Twitter by Nassim Taleb (@nntaleb) [1].
October 18th 2020

1. Schur inequality

A classical application of Schur convexity goes as follows: If a sequence of positive real numbers x, y, z majorises another such sequence a, b, c then

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \geq \frac{1}{a}+\frac{1}{b}+\frac{1}{c}; (1)$$

In this document I want to prove an extension of this result to integrals.

2. Extension to integrals.

Theorem 1. Let $F: I \to \mathbb{R}$ be a convex Lipschitz function on an open interval $I \subset \mathbb{R}$.

Let u, v: $[a, b] \to I$ be monotonically increasing Lipschitz functions on the interval $[a, b]$, such that

$$\int_a^b u(t)\,dt = \int_a^b v(t)\,dt; (2)$$

$$\int_a^x u(t)\,dt \leq \int_a^x v(t)\,dt, (x \in [a, b]); (3)$$

Then,

$$\int_a^b F(u(t))\,dt \geq \int_a^b F\big(v(t)\big)\,dt$$

Proof. Let $n > 0$ be a natural number. Partition $(a, b]$ into n intervals

$I_i = a + (b-a) \cdot \left(\frac{i-1}{n}, \frac{i}{n}\right]$ with $i \in [n]$. We get two increasing sequences

$$u_i = \int_{I_i} u(t)\,dt, (i \in [n])$$

$$v_i = \int_{I_i} v(t)\,dt, (i \in [n])$$

The theorem follows from proving the following limit for u (and the analogous version for v):

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}F\left(n\int_{I_i}u(t)\,dt\right)=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\int_{I_i}F\big(u(t)\big)\,dt=\int_a^b F\big(u(t)\big)\,dt$$

Let $m(I_i)=a+(b-a)\cdot\frac{2i-1}{2n}$ be the midpoint from proving of I_i. Let $L>0$ be the Lipschitz constant for u and let K be the Lipschitz constant for F.

$$\left|n\int_{I_i}u(t)\,dt-u\big(m(I_i)\big)\right|\le n\int_{I_i}\left|u(t)-u\big(m(I_i)\big)\right|dt\le\frac{L}{n};\ \ (4)$$

$$\left|F\left(n\int_{I_i}u(t)\,dt\right)-F\Big(u\big(m(I_i)\big)\Big)\right|\le K\left|n\int_{I_i}u(t)\,dt-u\big(m(I_i)\big)\right|\le\frac{KL}{n};\ \ (5)$$

$$\left|n\int_{I_i}F\big(u(t)\big)\,dt-F\Big(u\big(m(I_i)\big)\Big)\right|\le n\int_{I_i}\left|F\big(u(t)\big)-F\Big(u\big(m(I_i)\big)\Big)\right|dt\le\frac{KL}{n};\ \ (6)$$

Inequalities (4) and (6) are due to Lipschitz continuity of u, and $F\circ u$ respectively.

Inequality (4) implies (5). Inequalities (5) and (6) together imply

$$\left|n\int_{I_i}F\big(u(t)\big)\,dt-nF\left(\int_{I_i}u(t)\,dt\right)\right|\le$$

$$\left|n\int_{I_i}F\big(u(t)\big)\,dt-F\Big(u\big(m(I_i)\big)\Big)\right|+\left|F\Big(u\big(m(I_i)\big)\Big)-nF\left(\int_{I_i}u(t)\,dt\right)\right|\le\frac{2KL}{n};\ (7)$$

Therefore,

$$\frac{1}{n}\sum_{i=1}^{n}F\left(n\int_{I_i}u(t)\,dt\right)=\frac{1}{n}\sum_{i=1}^{n}n\int_{I_i}F\big(u(t)\big)\,dt+o\left(\frac{2KL}{n}\right)=\int_a^b F\big(u(t)\big)\,dt+o\left(\frac{2KL}{n}\right)$$

3 Simplifying the condition

Corollary 1. Let $F: I\to\mathbb{R}$ be a convex Lipschitz function on an open interval $I\subset\mathbb{R}$.

Let $f,g:[a,b]\to I$ be monotonically increasing Lipschitz functions on the interval $[a,b]$, such that

$$\int_a^b f(s)\,ds\ne 0,\int_a^b g(s)\,ds\ne 0,(a<x<b);\ \ \ (8)$$

$\frac{\int_a^x f(s)ds}{\int_a^x g(s)ds}$ is non-decreasing with respect to $x\in(a,b)$; (9)

Then,

$$\int_a^b F\left(\frac{f(t)}{\int_a^b f(s)\,ds}\right)dt\ge\int_a^b F\left(\frac{g(t)}{\int_a^b g(s)\,ds}\right)dt$$

Proof. Set $u(x)=\frac{f(x)}{\int_a^b f(s)ds}$ and $v(x)=\frac{f(x)}{\int_a^b g(s)ds}$. By construction u and v satisfy equation (2).

The second condition (3) requires for $a < x < b$:

$$\frac{\int_a^x f(s)\,ds}{\int_a^b f(s)\,ds} \le \frac{\int_a^x g(s)\,ds}{\int_a^b g(s)\,ds} \Leftrightarrow \frac{\int_a^x f(s)\,ds}{\int_a^x g(s)\,ds} \le \frac{\int_a^b f(s)\,ds}{\int_a^b g(s)\,ds}$$

This inequality holds because the left-hand term is non-decreasing in x, while equality holds for $x = b$. Therefore, Theorem 1 applies.

4. Application to Daniel Sitaru's inequality.

Daniel Sitaru observe that

$$\int_a^b (\log x)^{\log x} dx \cdot \int_a^b (\log x)^{-\log x} dx \ge (b^2 - a^2)\log\sqrt{\frac{b}{a}}, (e \le a \le b); \quad (10)$$

which is equivalent to saying

$$\int_a^b \frac{\int_a^b (\log x)^{\log x} dx}{(\log x)^{\log x}} dx \ge \int_a^b \frac{\int_a^b x\,dx}{x} dx; \quad (11)$$

This follows from Corollary 1 with $F(x) = \frac{1}{x}, f(x) = (\log x)^{\log x}$, and $g(x) = x$. Note that f and g are increasing functions on (e, ∞). Because f and g are strictly positive for $x \ge e$, the non-vanishing conditions (8) are satisfied. We will show the monotonicity condition (9) by taking the derivative.

$$\frac{\partial}{\partial x} \frac{\int_a^b \log(s)^{\log(s)}}{\int_a^x s\,ds} \ge 0$$

$$\Leftrightarrow \frac{\log(x)^{\log(x)} \frac{x^2 - a^2}{2} - x \int_a^b \log(s)^{\log(s)} ds}{\left(\int_a^x s\,ds\right)^2} \ge 0$$

$$\Leftrightarrow \log(x)^{\log(x)} \frac{x^2 - a^2}{2} \ge x \int_a^b \log(s)^{\log(s)} ds\,; (12)$$

Because $\log(s)^{\log(s)}$ is convex on (e, ∞) the mean on $[a, x]$ is bounded by the mean of the values at the endpoints.

$$\frac{1}{x - a} \int_a^x \log(s)^{\log(s)} ds \le \frac{\log(x)^{\log(x)} + \log(a)^{\log(a)}}{2}; (13)$$

Due to monotonicity of $\frac{\log(x)^{\log(x)}}{x}$ on (e, ∞), we further get

$$\frac{\log(x)^{\log(x)} + \frac{a}{a}\log(a)^{\log(a)}}{2} \le \frac{\log(x)^{\log(x)} + \frac{a}{x}\log(a)^{\log(a)}}{2} = \frac{x + a}{2x}\log(x)^{\log(x)}; (14)$$

$$\overset{(13),(14)}{\Longrightarrow} x \int_a^x \log(s)^{\log(s)}\,ds \le \frac{x + a}{2}\log(x)^{\log(x)}$$

Hence we have proven inequality (12) to hold.

References

[1] N. Taleb, https://twiter.comnntaleb/status/1316693195506548736 (2020)

[2] J.M. Steele, The Cauchy-Schwarz master class: an introduction to the art of mathematical inequalities (Cambridge University Press,2004).

A SIMPLE PROOF FOR DOUCET'S INEQUALITY

By Daniel Sitaru-Romania

DOUCET'S INEQUALITY

In ΔABC the following relationship holds:

$$\boldsymbol{s\sqrt{3} \le r + 4R}$$

Proof. $r_a r_b + r_b r_c + r_c r_a = \frac{F}{s-a} \cdot \frac{F}{s-b} + \frac{F}{s-b} \cdot \frac{F}{s-c} + \frac{F}{s-c} \cdot \frac{F}{s-a} =$

$$= F^2 \left(\frac{1}{(s-a)(s-b)} + \frac{1}{(s-b)(s-c)} + \frac{1}{(s-c)(s-a)} \right) =$$

$$= s(s-a)(s-b)(s-c) \left(\frac{1}{(s-a)(s-b)} + \frac{1}{(s-b)(s-c)} + \frac{1}{(s-c)(s-a)} \right) =$$

$$= s(s-c) + s(s-b) + s(s-c) = s(s-c+s-b+s-c) = s^2$$

$$r_a r_b + r_b r_c + r_c r_a = s^2 \quad (1)$$

$$r_a + r_b + r_c = \frac{F}{s-a} + \frac{F}{s-b} + \frac{F}{s-c} = F \left(\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} \right) =$$

$$= \frac{F}{(s-a)(s-b)(s-c)} \big((s-a)(s-c) + (s-c)(s-a) + (s-a)(s-b) \big) =$$

$$= \frac{Fs}{F^2} (3s^2 - s(b+c+c+a+a+b) + ab+bc+ca) =$$

$$= \frac{s}{F}(3s^2 - s \cdot 4s + ab + bc + ca) = \frac{s}{rs}(-s^2 + s^2 + r^2 + 4Rr) = \frac{1}{r}(r^2 + 4Rr) = r + 4R$$

$$r_a + r_b + r_c = r + 4R \quad (2)$$

If $x, y, z \in \mathbb{R}$ then:

$$3(xy + yz + zx) \le (x+y+z)^2 \quad (3)$$

Replace in (3): $x = r_a; y = r_b; z = r_c$

$$3(r_a r_b + r_b r_c + r_c r_a) \le (r_a + r_b + r_c)^2$$

By (1); (2): $3s^2 \leq (r+4R)^2$, $s\sqrt{3} \leq r+4R$

Observation: By Euler's inequality: $r \leq \frac{R}{2}$

$$s\sqrt{3} \leq r+4R \leq \frac{R}{2}+4R=\frac{9R}{2},\ \ 2s\sqrt{3} \leq 9R$$

$$s \leq \frac{9R}{2\sqrt{3}} \Rightarrow s \leq \frac{R\sqrt{3}}{2}$$

which is MITRINOVIC'S INEQUALITY

Reference:

[1] Romanian Mathematical Magazine – www.ssmrmh.ro

MITRINOVIC'S GENERALIZED INEQUALITIES

By D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

If $A_1A_2\dots A_n, n \geq 3$ is a convex polygon, and $M \in Int(A_1A_2\dots A_n)$, with $pr_{A_kA_{k+1}}M = T_k \in [A_kA_{k+1}]$, for any $k \in \{1,2,\dots,n\}, A_{n+1}=A_1$, then

$$\sum_{k=1}^{n}\frac{A_kA_{k+1}}{MT_k} \geq 2n\cdot tan\frac{\pi}{n}$$

Proof: Lemma. *Let $A, B; A \neq B$ be the points in plane and $M \notin AB, T = pr_{AB}M$, then $\frac{AB}{MT} = tan\,u + tan\,v$ where $u = \mu(\angle AMT), v = \mu(\angle TMB)$ are the measures in radians of angles $\angle AMT$ and $\angle TMB$.*

Proof of the Lemma: We have the cases:

$i)$ $T \in (AB)$. We have: $\tan u = \frac{AT}{MT}$ and $\tan v = \frac{BT}{MT}$, so $\tan u + \tan v = \frac{AB}{MT}$.

$ii)$ $T = A$. We have: $\tan u = \frac{AT}{MT} = \frac{AA}{MT} = 0$ and $\tan v = \frac{BT}{MT}$, so $\tan u + \tan v = \frac{AB}{MT}$.

$iii)$ $T = B$. We have: $\tan u = \frac{AB}{MT}$ and $\tan v = \frac{BT}{MT} = \frac{BB}{MT} = 0$, so $\tan u + \tan v = \frac{AB}{MT}$.

From Lemma, we have:

$\frac{A_kA_{k+1}}{MT_k} = \tan u_k + \tan v_k$, $\forall k = \overline{1,n}$, where $u_k = \mu(\angle A_kMT_k), v_k = \mu(\angle T_kMA_{k+1})$,

$\forall k = \overline{1,n}$ and then

$$\sum_{k=1}^{n}\frac{A_kA_{k+1}}{MT_k}=\sum_{k=1}^{n}(\tan u_k+\tan v_k).$$

Since the function $f:\left[0,\frac{\pi}{2}\right)\to[0,\infty), f(x)=\tan x$ is convex on $\left[0,\frac{\pi}{2}\right)$ we can apply Jensen's inequality and we obtain that

$$\sum_{k=1}^{n}\frac{A_kA_{k+1}}{MT_k}=\sum_{k=1}^{n}(\tan u_k+\tan v_k)\geq 2n\cdot\tan\left(\sum_{k=1}^{n}(u_k+v_k)\right)$$

Because $\sum_{k=1}^{n}(u_k+v_k)=2\pi$, we deduce that

$$\boldsymbol{\sum_{k=1}^{n}\frac{A_kA_{k+1}}{MT_k}\geq 2n\cdot\tan\frac{2\pi}{2n}=2n\cdot\tan\frac{\pi}{n}}$$

and we have done.

Observation 1. If $A_1A_2\dots A_n$ is circumscribed on the circle $C(I,r)$ and $M=I$, we have $MT_k=r, k=\overline{1,n}$ and the given inequality becomes

$$\boldsymbol{\frac{1}{r}\sum_{k=1}^{n}A_kA_{k+1}=\frac{2s}{r}\geq 2n\cdot\tan\frac{\pi}{n}\Leftrightarrow s\geq nr\cdot\tan\frac{\pi}{n}};(*)$$

The inequality $(*)$ is generalization of Mitrinovic's inequality $\boldsymbol{s\geq 3r\sqrt{3}}$ $\quad(\boldsymbol{M})$

Observation 2. If $A_1A_2A_3$ is a triangle, then the given inequality becomes

$$\boldsymbol{\frac{A_1A_2}{MT_1}+\frac{A_2A_3}{MT_2}+\frac{A_3A_1}{MT_3}\geq 6\tan\frac{\pi}{3}=6\sqrt{3}};(**)$$

For $M=I$, we obtain (M).

Reference: ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

PROPOSED PROBLEMS
PROBLEMS FOR JUNIORS

J.560 In any ΔABC the following inequality holds:

$$\frac{a}{h_b+h_c}+\frac{b}{h_c+h_a}+\frac{c}{h_a+h_b}\geq\sqrt{3}$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

J.561 If M is an interior point in ΔABC and d_a, d_b, d_c are the distances from point M to the sides BC, CA, respectively AB, then:

$$\frac{a^3}{d_a}+\frac{b^3}{d_b}+\frac{c^3}{d_c}\ge 24F$$

where F is the area of the triangle.

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

J.562 Let $m,n,x,y,z>0$, then in any ΔABC with the area F the following inequality holds:

$$\frac{y+z}{x}(mb+nc)+\frac{z+x}{y}(mc+na)+\frac{x+y}{z}(ma+nb)\ge 8\sqrt[4]{27}\sqrt{mn}\sqrt{F}$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

J.563 Let $m,n,p,t\in\mathbb{R}_+=[0,\infty)$; $m+n, p+t\in\mathbb{R}_+^*=(0,\infty)$ and M an interior point in ΔABC and x,y,z the distances from point M to the apices A,B,C and u,v,w the distances from point M to the sides $[BA],[CA],[AB]$. Prove that:

$$\frac{(mx+xy)^2}{u(pv+tw)}+\frac{(my+nz)^2}{v(pw+tu)}+\frac{(mz+nx)^2}{w(pu+tv)}\ge\frac{12(m+n)^2}{p+t}$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

J.564 If $m,n\in\mathbb{R}_+=[0,\infty), m+n=4, x,y,z\in\mathbb{R}_+^*=[0,\infty)$ and ABC is a triangle with the area F, then:

$$\left(\frac{x^2\cdot a^m}{h_a^n}+\frac{y^2\cdot b^m}{h_b^n}+\frac{z^2\cdot c^m}{h_c^n}\right)\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)\ge\frac{2^{m+2}F^{2-n}}{27}$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

J.565 If $x,y,z\in\mathbb{R}_+^*=(0,\infty)$ and $u\in\mathbb{R}_+=[0,\infty)$, then in any ΔABC the following inequality holds:

$$\frac{y+z+6u}{x+3u}a^2+\frac{z+x+6u}{y+3u}b^2+\frac{x+y+6u}{z+3u}c^2\ge 8\sqrt{3}F$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

J.566 Let $m,n\in\mathbb{R}_+=[0,\infty), m+n=4$ and M is an interior point in ΔABC with the area F and x,y,z the distances from M to the apices A,B,C and u,v,w the distances from M to the sides BC,CA, respectively AB, then:

$$\frac{x^2\cdot a^m}{u(v+w)h_a^n}+\frac{y^2\cdot b^m}{v(w+u)h_b^n}+\frac{z^2c^m}{w(u+v)h_c^n}\ge 2^mF^{m-2}$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

J.567 If $t\ge 0$, then in any ABC triangle with the area F the following inequality holds:

$$\frac{y+z}{x}m_a^{t+1}+\frac{z+x}{y}m_b^{t+1}+\frac{x+y}{z}m_c^{t+1}\ge 2^{t+1}\left(\sqrt{3}\right)^{1-t}\left(\frac{F}{R}\right)^{t+1}$$

$$\forall x,y,z\in\mathbb{R}_+^*=(0,\infty)$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

J.568 In any ΔABC with the area F, the following inequality holds:

$$n_a^2 + n_b^2 + n_c^2 \geq 3\sqrt{3}F$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.569 If ΔDEF is orthic triangle of ΔABC then:

$$S[DEF] \leq \left(\frac{r}{R}\right)^2 \cdot S[ABC]$$

Proposed by Marian Ursărescu – Romania

J.570 Solve for real numbers:

$$2^x + 2^{\frac{10}{x}} + 2^{x+\frac{10}{x}} = 164$$

Proposed by Marian Ursărescu – Romania

J.571 If N – nine point center then:

$$AN + BN + CN \leq 3R$$

Proposed by Marian Ursărescu – Romania

J.572 In acute ΔABC the following relationship holds:

$$\frac{1}{\cos^2 A\,(\cos B + \cos C)^2} + \frac{1}{\cos^2 B\,(\cos C + \cos A)^2} + \frac{1}{\cos^2 C\,(\cos A + \cos B)^2} \geq 12$$

Proposed by Marian Ursărescu – Romania

J.573 If $x, y, z > 0, x + y + z = 3$ then:

$$\frac{1}{\sqrt{x + y^2 + z^2}} + \frac{1}{\sqrt{x^2 + y + z^2}} + \frac{1}{\sqrt{x^2 + y^2 + z}} \leq \sqrt{3}$$

Proposed by Marian Ursărescu – Romania

J.574 If in $\Delta ABC, AA', BB', CC'$ - Gergonne's cevians then:

$$a \cdot AA'^2 + b \cdot BB'^2 + c \cdot CC'^2 \geq 54\sqrt{3}r^3$$

Proposed by Marian Ursărescu – Romania

J.575 If $a, b, c > 0$ then:

$$\frac{a^4+1}{8b^4} + \frac{b^4+1}{8c^4} + \frac{c^4+1}{8a^4} \geq \frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2}$$

Proposed by Marin Chirciu – Romania

J.576 In ΔABC the following relationship holds: $\sum w_a^3 w_b \geq 243r^4$

Proposed by Marin Chirciu – Romania

J.577 In ΔABC the following relationship holds:

$$\frac{m_a^{n+1}}{m_a^n} + \frac{m_b^{n+1}}{m_c^n} + \frac{m_c^{n+1}}{m_a^n} \geq 6r\left(2 - \frac{r}{R}\right), n \in \mathbb{N}$$

Proposed by Marin Chirciu – Romania

J.578 In ΔABC the following relationship holds:

$$6r \leq \sum \frac{a^2}{h_b + h_c} \leq 2(2R - r)$$

Proposed by Marin Chirciu – Romania

J.579 In acute ΔABC the following relationship holds:

$$\sqrt{\frac{3}{2}(1+\cos A\cos B\cos C)}\le\frac{27}{32}\sec\frac{A}{2}\sec\frac{B}{2}\sec\frac{C}{2}$$

Proposed by Marin Chirciu – Romania

J.580 In ΔABC the following relationship holds:

$$\frac{3}{4}\le\sum\frac{a^2}{(b+c)^2}\le\frac{R}{2r}-\frac{1}{4}$$

Proposed by Marin Chirciu – Romania

J.581 In ΔABC the following relationship holds:

$$\sum\sqrt{\frac{1}{1+\left(\frac{a\cot\frac{A}{2}}{r_a}\right)^3}}\ge 1$$

Proposed by Marin Chirciu – Romania

J.582 In ΔABC the following relationship holds:

$$\frac{R}{2r}+\frac{p^2}{p^2+\lambda r(R-2r)}\ge 2, 0\le\lambda\le\frac{27}{2}$$ (Generalization A. Abdullayev)

Proposed by Marin Chirciu – Romania

J.583 If $a,b,c>0$ such that $a+b+c=3$ and $0\le\lambda\le 2$ then:

$$\frac{1}{1+\lambda ab^2}+\frac{1}{1+\lambda bc^2}+\frac{1}{1+\lambda ca^2}\ge\frac{3}{\lambda+1}$$

Proposed by Marin Chirciu – Romania

J.584 If $a,b,c>0$ and $\lambda\ge 1$ then:

$$\frac{a^2}{bc(a^2+\lambda ab+b^2)}+\frac{b^2}{ca(b^2+\lambda bc+c^2)}+\frac{c^2}{ab(c^2+\lambda ca+a^2)}\ge\frac{27}{(\lambda+2)(a+b+c)^2}$$

Proposed by Marin Chirciu – Romania

J.585 If $a,b,c>0$ such that $abc=1$ and $0\le\lambda\le 2$ then:

$$\frac{a^4}{b^4+\lambda c^2}+\frac{b^4}{c^4+\lambda a^2}+\frac{c^4}{a^4+\lambda b^2}\ge\frac{2}{\lambda+1}$$

Proposed by Marin Chirciu – Romania

J.586 In ΔABC the following relationship holds:

$$\sum\frac{r_b+r_c}{b+c}\tan\frac{A}{2}\le\frac{3R}{4r}$$

Proposed by Marin Chirciu – Romania

J.587 In $\Delta ABC, AA', BB', CC'$ - internal bisectors, $\Delta A''B''C''$ - circumcevian triangle of incenter.

Prove that:

$$\frac{[A'B'C']}{[A''B''C'']} \ge \frac{r^2}{R^2}$$

Proposed by Marin Chirciu – Romania

J.588 In ΔABC the following relationship holds:

$$\frac{r_a^{n+1}}{r_a^n} + \frac{r_b^{n+1}}{r_c^n} + \frac{r_c^{n+1}}{r_a^n} \ge p\sqrt{3}, n \in \mathbb{N}$$

Proposed by Marin Chirciu – Romania

J.589 In ΔABC the following relationship holds:

$$\frac{w_a^3}{w_a^2} + \frac{w_b^3}{w_c^2} + \frac{w_c^3}{w_a^2} \ge 2r\left(5 - \frac{r}{R}\right)$$

Proposed by Marin Chirciu – Romania

J.590 In ΔABC the following relationship holds:

$$\sum \frac{a^2}{h_b + h_c} \le \sum \frac{a^2}{r_b + r_c}$$

Proposed by Marin Chirciu – Romania

J.591 In acute ΔABC the following relationship holds:

$$\sqrt{2(1 + \cos A \cos B \cos C)} \le \frac{3}{16} \csc\frac{A}{2} \csc\frac{B}{2} \csc\frac{C}{2}$$

Proposed by Marin Chirciu – Romania

J.592 If $a, b > 0$ then:

$$\frac{(a+b)^3}{8} + \frac{8a^3b^3}{(a+b)^3} \ge ab\sqrt{ab} + \left(\frac{\left(\sqrt{a} - \sqrt{b}\right)^2}{2} + \frac{2ab}{a+b}\right)^3$$

Proposed by Daniel Sitaru– Romania

J.593 In ΔABC the following relationship hold:

$$\prod_{cyc} (m_a^5 - h_a^5 + w_a^5) \le \left(\prod_{cyc} (m_a - h_a + w_a)\right)^5$$

Proposed by Daniel Sitaru– Romania

J.594 Find all $x, y, z > 0$ such that:

$$4 \sin x \cdot \sin y \cdot \sin z \cdot \sin(x + y + z) = 1$$

Proposed by Daniel Sitaru– Romania

J.595

$$\begin{cases} f,g,h:\mathbb{R}\to\mathbb{R} \\ f(x)+g(x)+h(x)=3x+3,\forall x\in\mathbb{R} \\ f^2(x)+g^2(x)+h^2(x)=3x^2+6x+5,\forall\in\mathbb{R} \\ f^3(x)+g^3(x)+h^3(x)=3x^3+9x^2+15x+9,\forall x\in\mathbb{R} \end{cases}$$

Solve for real numbers:

$$f(x)\cdot g(x)\cdot h(x)=0$$

Proposed by Daniel Sitaru– Romania

J.596 If $a,b\ge 0$ then:

$$\sqrt{ab}+\sqrt[7]{\left(\frac{2ab}{a+b}\right)^7-\left(\sqrt{ab}\right)^7+\left(\frac{a+b}{2}\right)^7}\ge\frac{2ab}{a+b}+\frac{a+b}{2}$$

Proposed by Daniel Sitaru– Romania

J.597 Find $x,y,z>0$ such that:

$$\frac{(1+x^2)(1+y^2)}{(1+x)(1+y)}+\frac{(1+y^2)(1+z^2)}{(1+y)(1+z)}+\frac{(1+z^2)(1+x^2)}{(1+z)(1+x)}+24\sqrt{2}=36$$

Proposed by Daniel Sitaru– Romania

J.598 If $x,y,z\in\mathbb{R}$ then:

$$\frac{(x^{12}+x^6+1)(y^{24}+y^{12}+1)(z^{36}+z^{18}+1)}{(x^8+1)(y^{16}+1)(z^{24}+1)}>x^2y^4z^6$$

Proposed by Daniel Sitaru– Romania

J.599 Solve for complex numbers:

$$\begin{cases} \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1 \\ \frac{x}{y}+\frac{y}{z}+\frac{z}{x}+\frac{x}{z}+\frac{y}{x}+\frac{z}{y}+2=0 \end{cases}$$

Proposed by Daniel Sitaru– Romania

J.600 In $\Delta ABC, N-$ ninepoint center, the following relationship holds:

$$\left(\frac{a^2+R^2}{NB}\right)^2+\left(\frac{b^2+R^2}{NC}\right)^2+\left(\frac{c^2+R^2}{NA}\right)^2\ge 192r^2$$

Proposed by Daniel Sitaru– Romania

J.601 If $a,b,c,x,y,z\in\mathbb{R}_+^*$, then:

$$\frac{x^4+y^4}{(ax+by)^2+cyz}+\frac{y^4+z^4}{(ay+bz)^2+czx}+\frac{z^4+x^4}{(az+bx)^2+cxy}\ge\frac{2}{3}\cdot\frac{(x+y+z)^2}{(a+b)^2+c}$$

Proposed by D.M. Bătinețu-Giurgiu – Romania

J.602 If $m \geq 0; x, y, z > 0$, then in ΔABC with the area F the following inequality holds:

$$\sum_{cyc}\left(\frac{x^2+y^2}{z^2}\right)^{m+1}\cdot\frac{a^{4m+3}}{h_a}\geq\frac{2^{5m+4}}{3^m}F^{2m+1}$$

Proposed by D.M. Bătinețu-Giurgiu– Romania

J.603 Let $x, y, z \in \mathbb{R}_+^* = [0, \infty), m, n, p \in \mathbb{R}_+ = [0, \infty), m + n = 2$ and ABC a triangle with the area F, then:

$$\frac{4x+3y+z+2p}{y+3z+p}\cdot\frac{a^m}{h_a^n}+\frac{x+4y+3z+2p}{z+3x+p}\cdot\frac{b^m}{h_b^n}+\frac{3x+y+4z+2p}{x+3y+p}\cdot\frac{c^m}{h_c^m}\geq$$
$$\geq 2^{3-n}\sqrt{3}\cdot F^{1-n}$$

Proposed by D.M. Bătinețu-Giurgiu,Neculai Stanciu – Romania

J.604 Let $m \in \mathbb{N}, M$ an interior point in $ABCD$ tetrahedron, $x_A = MA, x_B = MB, x_C = MC$, $x_D = MD$ and h_a, h_b, h_c, h_d the altitudes heights, then:

$$4m+\left(\frac{x_A}{h_a}\right)^{m+1}+\left(\frac{x_B}{h_b}\right)^{m+1}+\left(\frac{x_C}{h_c}\right)^{m+1}+\left(\frac{x_D}{h_d}\right)^{m+1}\geq 3(m+1)$$

Proposed by D.M. Bătinețu-Giurgiu,Neculai Stanciu– Romania

J.605 If M is an interior point in ABC triangle and $x = MA, y = MB, z = MC$, then:

$$\sum_{cyc}x^2\sin^2 A\geq\frac{16R^2}{3}\sin^2 A\sin^2 B\sin^2 C$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.606 If $x, y, z \in \mathbb{R}_+^* = (0, \infty)$, then in ΔABC with the area F the following inequality holds:

$$\frac{(y+z)m_a}{x(b+c)}+\frac{(z+x)m_b}{y(c+a)}+\frac{(x+y)m_c}{z(a+b)}\geq 4\sqrt[4]{27}\cdot\frac{\sqrt{F}}{R},$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.607 If $x, y, z > 0$, then in ΔABC triangle the following inequality holds:

$$\frac{y+z}{x}h_a+\frac{z+x}{y}h_b+\frac{x+y}{z}h_c\geq 18\sqrt[3]{2}\cdot r\cdot\sqrt[3]{\frac{r}{R}}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.608 Let $m, n, p, t, u \in \mathbb{R}_+^* = (0, \infty)$ and ABC a triangle with the area ABC, then:

$$\sum_{cyc}\frac{(ma+nb)^4}{pa+tb+uc}\geq\frac{8(m+n)^4\cdot s^3}{9(p+t+u)}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.609 If $x, y, z > 0$, then in ΔABC the following inequality holds:

$$\left(\left(\frac{y+z}{x}\right)^2 a^4+2\right)\left(\left(\frac{z+x}{y}\right)^2 b^4+2\right)\left(\left(\frac{x+y}{z}\right)^2 c^4+2\right)\geq 72F^2$$

where F is the triangle's area.

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.610 If $x,y,z>0$; then in any ΔABC with the area F the following inequality holds:

$$\frac{x}{y+z}m_a^2+\frac{y}{z+x}m_b^2+\frac{z}{x+y}m_c^2\geq\frac{4F^2}{R^2}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.611 If $x,y,z\in\mathbb{R}_+^*=(0,\infty)$ then in any ΔABC with the area F the following inequality holds:

$$\frac{y+z}{x\cdot h_b^2}c^2+\frac{z+x}{y\cdot h_c^2}a^2+\frac{x+y}{z\cdot h_c^2}b^2\geq 8$$

Proposed by D.M. Bătinețu-Giurgiu,Neculai Stanciu – Romania

J.612 Let $m,n\in\mathbb{R}_+=[0,\infty), m+n=4$ and $x,y,z\in\mathbb{R}_+^*=(0,\infty)$, then in any ΔABC with the area F the following inequality holds:

$$\frac{xa^m}{(y+z)h_a^n}+\frac{y\cdot b^m}{(z+x)h_b^n}+\frac{z\cdot c^m}{(x+y)\cdot h_c^n}\geq 2^{m-1}F^{m-2}$$

Proposed by D.M. Bătinețu-Giurgiu,Neculai Stanciu– Romania

J.613 Let $x,y,z\in\mathbb{R}_+^*=(0,\infty)$, then in ΔABC with the area F the following inequality holds:

$$\frac{xa^3}{(y+z)h_a}+\frac{yb^3}{(z+x)h_b}+\frac{zc^3}{(x+y)h_c}\geq 4F$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.614 If $t,u,v\geq 0; t+u>0$ the in ΔABC with the area F the following inequality holds:

$$\frac{y+z}{x}(tm_b+um_c)^{v+1}+\frac{z+x}{y}(tm_c+um_a)^{v+1}+\frac{x+y}{z}(tm_a+um_b)^{v+1}\geq$$

$$\geq 2^{v+2}\left(\sqrt{3}\right)^{1-v}\left(\frac{F}{R}\right)^{v+1}, \forall x,y,z>0$$

Proposed by D.M. Bătinețu-Giurgiu– Romania

J.615 In $\Delta ABC, I$ – incentre the following relationship holds:

$$\sum_{cyc}\left(r_a+\frac{m_a w_a}{h_a}+AI\right)\leq\left(\sum_{cyc}(2m_a+h_a)-3r\right)\sqrt{\frac{R}{2r}}$$

Proposed by Bogdan Fuștei – Romania

J.616 In $\Delta ABC, g_a$ – Gergonne's cevian the following relationship holds:

$$m_a\sqrt{2} \geq g_a + \frac{|b-c|}{2}\sqrt{\frac{2h_a-3r}{r}}$$

Proposed by Bogdan Fuștei – Romania

J.617 In $\Delta ABC, I$ – incenter, R_a, R_b, R_c – circumradii in $\Delta BIC, \Delta CIA, \Delta AIB$ then:

$$\sqrt{2(h_a+h_b+h_c)} \geq \frac{R_aR_b+R_bR_c+R_cR_a}{R\sqrt{R}}$$

Proposed by Bogdan Fuștei – Romania

J.618 In $\Delta ABC, I$ – incenter, N_a – Nagel's point, n_a – Nagel's cevian the following relationship holds:

$$\frac{AN_a}{AI}+\frac{BN_a}{BI}+\frac{CN_a}{CI} \leq \sum_{cyc}\frac{n_a}{h_a}\sqrt{\frac{r_a}{m_a}}$$

Proposed by Bogdan Fuștei – Romania

J.619 In $\Delta ABC, n_a$ – Nagel's cevian the following relationship holds:

$$\frac{1}{\sqrt{2}}\sum_{cyc}\frac{n_a}{r_a}+\sum_{cyc}\sqrt{\frac{h_a}{r_a}} \leq \frac{s}{r}$$

Proposed by Bogdan Fuștei – Romania

J.620 In ΔABC the following relationship holds:

$$3\prod_{cyc}\frac{(a+b)w_c}{2c} \geq \sum_{cyc}m_ah_bh_c$$

Proposed by Bogdan Fuștei – Romania

J.621 In $\Delta ABC, I$ – incenter, N_a – Nagel's point, the following relationship holds:

$$\frac{AN_a}{AI}+\frac{BN_a}{BI}+\frac{CN_a}{CI} \leq 2\left(\frac{R}{r}-1\right)\sum_{cyc}\sin\frac{A}{2}$$

Proposed by Bogdan Fuștei – Romania

J.622 In ΔABC the following relationship holds:

$$\sum_{cyc(a,b,c)}\frac{r_a-r}{w_a}\sqrt{\frac{h_a}{r_a}} = \sqrt{\frac{2R}{r}}$$

Proposed by Bogdan Fuștei – Romania

J.623 In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$\frac{n_a}{a}+\frac{n_b}{b}+\frac{n_c}{c} \leq \frac{s}{2r}\left(\frac{R}{r}-1\right)$$

Proposed by Bogdan Fuștei – Romania

J.624 Solve for real numbers:

$$\begin{cases} x,y,z>0 \\ \dfrac{x^2+3}{2x+y+z}+\dfrac{y^2+3}{x+2y+z}+\dfrac{z^2+3}{x+y+2z}=x+y+z \\ xy+yz+zx=3 \end{cases}$$

Proposed by Daniel Sitaru– Romania

J.625 Solve for real numbers:

$$\begin{cases} x,y,z,t>0 \\ xyz+yzt+ztx+txy=1 \\ \dfrac{x^6}{yzt}+\dfrac{y^6}{ztx}+\dfrac{z^6}{txy}+\dfrac{t^6}{xyz}=1 \end{cases}$$

Proposed by Daniel Sitaru– Romania

J.626 Solve for real numbers:

$$\sin^2 x\cdot\cos^2 t+\sin^2 y\cdot\cos^2 x+\sin^2 z\cdot\cos^2 y+\sin^2 t\cdot\cos^2 z=2$$

Proposed by Daniel Sitaru– Romania

J.627 If $a,b,c>0,\frac{ab}{(a+b)^2}+\frac{bc}{(b+c)^2}+\frac{ca}{(c+a)^2}=\frac{3}{4}$ then:

$$16\sum_{cyc}\frac{\sqrt{ab}}{a+b}+\sum_{cyc}\frac{(a+b)^2}{ab}\geq 12+4\sum_{cyc}\frac{a+b}{\sqrt{ab}}$$

Proposed by Daniel Sitaru– Romania

J.628 If $a,b,c\in\mathbb{C},|a|=|b|=|c|=3$ then:

$$\sum_{cyc}|a+3|+3\sum_{cyc}|a^2+1|+\sum_{cyc}|a^3+3|\geq 18$$

Proposed by Daniel Sitaru– Romania

J.629 If $x,y,z>0$ then:

$$\frac{1}{\sqrt{(x+y)(y+z)}}+\frac{1}{\sqrt{(y+z)(z+x)}}+\frac{1}{\sqrt{(z+x)(x+y)}}\leq\frac{3}{2}\sqrt{\frac{3}{xy+yz+zx}}$$

Proposed by Daniel Sitaru– Romania

J.630 Solve for real numbers:

$$\begin{cases} x,y,z>0 \\ \dfrac{2x^7}{y^6+z^6}+\dfrac{2y^7}{z^6+x^6}+\dfrac{2z^7}{x^6+y^6}=3\sqrt{\dfrac{x^7+y^7+z^7}{x^5+y^5+z^5}} \\ \left[\dfrac{x+1}{2}\right]=\dfrac{y+z}{3},[*]-GIF \end{cases}$$

Proposed by Daniel Sitaru– Romania

J.631 If $x, y, z > 0$ then:

$$\frac{(x+y+z)^5}{xy+yz+zx} \geq 81$$

Proposed by Daniel Sitaru– Romania

J.632 If $a, b, x, y, z, t > 0$ then:

$$b^3\left(\frac{2x}{y}+\frac{2z}{t}+\frac{x}{z}+\frac{z}{x}\right)+a^3\left(\frac{2y}{z}+\frac{2t}{x}+\frac{y}{t}+\frac{t}{y}\right) \geq 64a^4b^4(a+b)$$

Proposed by Daniel Sitaru– Romania

J.633 $a, b, c \in \mathbb{C}^*$ - different in pairs, $|a| = |b| = |c|, A(a), B(b), C(c)$. Prove that:

$$\left(\sum_{cyc}\big((a-b)|a-c|+(a-c)|a-b|\big)\right)^2 = \left(\sum_{cyc}|a-b|\right)^2 \cdot \sum_{cyc}|a-b|^2 \Rightarrow$$

$$\Rightarrow AB = BC = CA.$$

Proposed by Marian Ursărescu – Romania

J.634 In ΔABC the following relationship holds:

$$h_a^3h_b + h_b^3h_c + h_c^3h_a \geq \frac{54r^4(5R-r)}{R}$$

Proposed by Marian Ursărescu – Romania

J.635 In ΔABC the following relationship holds:

$$m_a^3m_b + m_b^3m_c + m_c^3m_a \geq 81r^3(2R-r)$$

Proposed by Marian Ursărescu – Romania

J.636 In ΔABC the following relationship holds:

$$a^2b + b^2c + c^2a \leq 9R\sqrt{\frac{9R^4-48r^4}{2}}$$

Proposed by Marian Ursărescu – Romania

J.637 In $\Delta ABC, AA_1, AA_2, BB_1, BB_2, CC_1, CC_2$ – are isogonal in pairs. Prove that:

$$\frac{1}{AA_1}+\frac{1}{AA_2}+\frac{1}{BB_1}+\frac{1}{BB_2}+\frac{1}{CC_1}+\frac{1}{CC_2} \leq \frac{2}{r}$$

Proposed by Marian Ursărescu – Romania

J.638 In ΔABC the following relationship holds:

$$m_a \sin a + m_b \sin B + m_c \sin C \leq \frac{9\sqrt{3}R}{4}$$

Proposed by Marian Ursărescu – Romania

J.639 $a, b, c \in \mathbb{C}^*$ – different in pairs, $|a| = |b| = |c| = 1, A(a), B(b), C(c)$

Prove that:

$$\sum_{cyc} |a + b - 2c| = \sum_{cyc} |a^2 - ab - ac + bc| \Rightarrow AB = BC = CA$$

Proposed by Marian Ursărescu – Romania

J.640 In ΔABC the following relationship holds:

$$\sqrt[3]{n_a m_b h_c} \geq 3r$$

Proposed by Marian Ursărescu – Romania

J.641 $a, b, c \in \mathbb{C}^*$ - different in pairs, $|a| = |b| = |c|, A(a), B(b), C(c)$. Prove that:

$$\sum_{cyc} \left| \frac{(a-b)|a-c| + (a-c)|a-b|}{b+c-2a} \right|^2 = \frac{1}{3}\left(\sum_{cyc} |a-b|\right)^2 \Rightarrow AB = BC = CA$$

Proposed by Marian Ursărescu – Romania

J.642 In acute ΔABC the following relationship holds:

$$\frac{m_a^4}{w_b} + \frac{m_b^4}{w_c} + \frac{m_c^4}{w_a} \geq \frac{(R+r)^4}{r}$$

Proposed by Marian Ursărescu – Romania

J.643 $z_1, z_2, z_3 \in \mathbb{C} - \{0\}$, different in pairs, $A(z_1), B(z_2), C(z_3)$

$$\frac{|z_1 - z_2|}{2 + |z_1 + z_2|} + \frac{|z_2 - z_3|}{2 + |z_2 + z_3|} + \frac{|z_3 - z_1|}{2 + |z_3 + z_1|} = \sqrt{3}$$

Prove that: $AB = BC = CA$.

Proposed by Marian Ursărescu – Romania

J.644 In ΔABC the following relationship holds:

$$\sqrt[3]{m_a r_a w_a} \leq \frac{3R}{2}$$

Proposed by Marian Ursărescu – Romania

J.645 Solve for real numbers:

$$\begin{cases} \left(\dfrac{x^3}{y} + xy + \dfrac{y^3}{x}\right)^2 = \sqrt{27(x^8 + y^8 + x^4y^4)} \\ \dfrac{5x^4 - 10xy + 1}{x^4y^2 - 10y^4 + 5x^2} = \dfrac{y}{x^2} \end{cases}$$

Proposed by Orlando Irahola Ortega-Bolivia

J.646 Solve for real numbers:

$$\begin{cases} \dfrac{x^5}{y} + x^2y^2 + \dfrac{y^5}{x} = \sqrt{3(x^8 + y^8 + x^4y^4)} \\ \dfrac{3xy + y^4}{1 + 3y^2} = \dfrac{x}{\pi} \end{cases}$$

Proposed by Orlando Irahola Ortega-Bolivia

J.647 Solve for real numbers:

$$\frac{x^3}{8} + \frac{10}{x} = \sqrt{\frac{5}{8}x^4 - \frac{25}{4}x^2 + \frac{16}{x^4} + 50}$$

Proposed by Orlando Irahola Ortega-Bolivia

J.648 Solve for real numbers:

$$8\sqrt{x^4+1}+5\sqrt{x^3+1}=7x^2+12$$

Proposed by Orlando Irahola Ortega-Bolivia

J.649 Solve for real numbers:

$$7x^3+70x^2-105x+91=$$

$$=\sqrt{x^7-49x^5+15435x^4+70609x^3-329182x^2-807373x+796544}$$

Proposed by Orlando Irahola Ortega-Bolivia

J.650 If $a,b,c\in\mathbb{R}, a\neq b\neq c$ and

$$\frac{(a+c-b)(a+b-c)}{(b-c)^2}+\frac{(a+b-c)(b+c-a)}{(c-a)^2}+\frac{(b+c-a)(a+c-b)}{(a-b)^2}=0$$

$$\frac{1}{a(a-b)(a-c)}+\frac{1}{b(b-c)(b-a)}+\frac{1}{c(c-a)(c-b)}=\frac{3}{2abc}$$

Prove that: $a^3+b^3+c^3+5abc=(a+b)(b+c)(c+a)$

Proposed by Orlando Irahola Ortega-Bolivia

J.651 Solve for real numbers:

$$1-x=x\left(18x^3+13x^2-12x-2+\frac{1}{x}\right)^{\frac{1}{3}}$$

Proposed by Orlando Irahola Ortega-Bolivia

J.652 Solve for real numbers:

$$\frac{x^2-x+1}{x^2+x+1}=\sqrt{\frac{x^3-1}{x^3+1}}$$

Proposed by Orlando Irahola Ortega-Bolivia

J.653 Solve for real numbers:

$$x^2+4x=\sqrt{40x^2+32x-16}$$

Proposed by Orlando Irahola Ortega-Bolivia

J.654 Solve for real numbers:

$$x^3+6x^2=\sqrt{96x^4+160x^3-240x^2-192x+64}$$

Proposed by Orlando Irahola Ortega-Bolivia

J.655 Solve for real numbers:

$$\frac{x}{27}=\frac{x^8-84\,x^6+1134x^4-2916x^2+729}{x^8-324x^6+10206x^4-61236x^2+59049}$$

Proposed by Orlando Irahola Ortega-Bolivia

J.656 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{m_a}{b+c}\le\sqrt{\frac{(2R-r)^2}{r^2}-\frac{R-2r}{R-r}}+\sum_{cyc}\frac{2r_a}{n_a+s}$$

Proposed by Bogdan Fuștei-Romania

J.657 In ΔABC the following relationship holds:

$$\sqrt{\frac{2R}{r}\left(\frac{(2R-r)^2}{r^2}-\frac{R-2r}{R-r}\right)}\ge\frac{1}{3}\cdot\sum_{cyc}\frac{n_a}{h_a}\cdot\sum_{cyc}\frac{r_b+r_c}{m_a}$$

Proposed by Bogdan Fuștei-Romania

J.658 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{m_a}{b+c}\ge\frac{1}{4}\sum_{cyc}\frac{w_b+w_c}{a}$$

Proposed by Bogdan Fuștei-Romania

J.659 In ΔABC the following relationship holds:

$$\frac{n_a}{h_a}+\frac{n_b}{h_b}+\frac{n_c}{h_c}\ge\sum_{cyc}\frac{\sqrt{4r^2+(n_a-g_a)^2}}{2r}$$

Proposed by Bogdan Fuștei-Romania

J.660 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{n_ag_a}{h_bh_c}\ge\frac{m_a}{r_a}+\frac{m_b}{r_b}+\frac{m_c}{r_c}$$

Proposed by Bogdan Fuștei-Romania

J.661 In ΔABC the following relationship holds:

$$\prod_{cyc}\left(\frac{am_a^2}{2r^2}-\frac{n_a^2}{h_a^2}\right)\ge\left(\sum_{cyc}\frac{n_a}{h_a}+2\sum_{cyc}\frac{r_a}{n_a+s}-1\right)^3$$

Proposed by Bogdan Fuștei-Romania

J.662 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\sqrt{4r^2+(n_a-g_a)^2}}{n_a}\le 2$$

Proposed by Bogdan Fuștei-Romania

J.663 In ΔABC the following relationship holds:

$$\sqrt{\frac{(2R-r)^2}{r^2}-\frac{R-2r}{R-r}}\ge\frac{n_a}{h_a}+\frac{n_b}{h_b}+\frac{n_c}{h_c}$$

Proposed by Bogdan Fuștei-Romania

J.664 In ΔABC the following relationship holds:

$$\prod_{cyc}\left(\frac{r_a}{a}-\frac{n_a}{2h_a}\right)\le\frac{r_ar_br_c}{(n_a+s)(n_b+s)(n_c+s)}$$

Proposed by Bogdan Fuștei-Romania

J.665 In ΔABC the following relationship holds:

$$\frac{1}{2r^2}\sum_{cyc} am_a^2 \ge \frac{1}{r}\sum_{cyc} n_a + 2\sum_{cyc}\frac{2r_a+h_a}{n_a+s} + \sum_{cyc}\frac{n_a^2}{h_a^2} - 3$$

Proposed by Bogdan Fuștei-Romania

J.666 In ΔABC the following relationship holds:

$$\sqrt{\frac{(2R-r)^2}{r^2} - \frac{R-2r}{R-r}} \ge \sum_{cyc}\frac{2m_a-g_a}{h_a}$$

Proposed by Bogdan Fuștei-Romania

J.667 In ΔABC the following relationship holds:

$$\frac{(a+b)(b+c)(c+a)}{abc} \ge \frac{8}{3}\cdot\frac{m_ah_bh_c+m_bh_ah_c+m_ch_ah_b}{w_aw_bw_c}$$

Proposed by Bogdan Fuștei-Romania

J.668 In ΔABC the following relationship holds:

$$\frac{s}{r} \le \sqrt{\frac{(2R-r)^2}{r^2} - \frac{R-2r}{R-r}} + \sum_{cyc}\frac{2n_a}{n_a+s}$$

Proposed by Bogdan Fuștei-Romania

J.669 In ΔABC the following relationship holds:

$$81\sqrt{3}\prod_{cyc}(2b^2+2c^2+a^2) > 4096s^3r_ar_br_c$$

Proposed by Daniel Sitaru– Romania

J.670 In ΔABC the following relationship holds:

$$am_a+bm_b+cm_c+6F < 3s^2-r^2-4Rr$$

Proposed by Daniel Sitaru– Romania

J.671 In ΔABC the following relationship holds:

$$72\sum_{cyc}\left(\frac{1}{b+c}+\frac{2}{c+a}\right)\left(\frac{1}{c+a}+\frac{2}{a+b}\right)\left(\frac{1}{a+b}+\frac{2}{b+c}\right) \le \left(\frac{s}{rR}\right)^3$$

Proposed by Daniel Sitaru– Romania

J.672

$$A=\frac{1}{n}\sum_{i=1}^{n} a_i\,, Q=\sqrt{\frac{1}{n}\sum_{i=1}^{n} a_i^2}\,, G=\sqrt[n]{\prod_{i=1}^{n} a_i}\,, H=n\left(\sum_{i=1}^{n}\frac{1}{a_i}\right)^{-1}, a_i>0, i\in\overline{1,n}$$

Prove that:

$$nA^2 \geq Q^2 + (n-1)\left(\frac{G^n}{H}\right)^{\frac{2}{n-1}}, n \in \mathbb{N}, n \geq 2$$

Proposed by Seyran Ibrahimov-Azerbaijan

J.673 If $a, b > 0, n \in \mathbb{N}, n \geq 2$ then:

$$\sqrt[n]{\left(\prod_{k=2}^{n} \sqrt[k]{\frac{a^k + b^k}{2}}\right)^{n-1}} \leq \frac{a^n + b^n}{a + b}$$

Proposed by Seyran Ibrahimov-Azerbaijan

J.674 If $x_i > 0, i \in \overline{1,n}$ then:

$$e^{\frac{1}{n}\sum_{i=1}^{n} x_i} + \underbrace{min}_{1 \leq k \leq n}\left(\frac{1}{k}\sum_{i=1}^{k} e^{x_i} - e^{\frac{1}{k}\sum_{i=1}^{k} x_i}\right) \leq \frac{1}{n}\sum_{i=1}^{n} e^{x_i}$$

Proposed by Seyran Ibrahimov-Azerbaijan

J.675 $p, q > 1, \frac{1}{p} + \frac{1}{q} = 1, x_i, y_i > 0, i \in \overline{1,n}$. Prove that:

$$\sqrt[p]{p} \cdot \sqrt[q]{q}\left(\sum_{i=1}^{n} \sqrt{x_i y_i}\right)^2 \leq \left(\sum_{i=1}^{n} x_i\right)^p + \left(\sum_{i=1}^{n} y_i\right)^q$$

Proposed by Seyran Ibrahimov-Azerbaijan

J.676 If $x_i > 0, i \in \overline{1,n}, n \in \mathbb{N}, n \geq 3$ then:

$$(n+1)^{n(n+1)} \cdot \prod_{i=1}^{n} x_i \cdot \left(\sum_{i=1}^{n} x_i\right)^{n^2} \leq n^{n^2} \cdot \left(\prod_{i=1}^{n} x_i + \sum_{i=1}^{n} x_i\right)^{n+1}$$

Proposed by Seyran Ibrahimov-Azerbaijan

J.677 If $x, y, z, t > 0, n \in \mathbb{N}, n \geq 2$ then:

$$\sum_{cyc} \frac{x}{(y+z+t)^n} \geq \frac{4^n}{3^n(x+y+z+t)^{n-1}}$$

Proposed by Seyran Ibrahimov-Azerbaijan

J.678 If $x_i, a_i, i \in \overline{1,n}, n \in \mathbb{N}, n \geq 2, p > 1, 0 \leq (n-1)q_i \leq M - 1$ then:

$$\frac{1}{M} \cdot \left(\sum_{i=1}^{n} x_i\right)^p \left(\sum_{i=1}^{n} a_i\right)^{-1} \leq \left(\sum_{i=1}^{n} \frac{x_i^{\frac{p}{p-1}}}{\left(q_i \sum_{\substack{i=1 \\ i \neq j}}^{n} a_j + a_i\right)^{\frac{1}{p-1}}}\right)^{p-1}$$

Proposed by Seyran Ibrahimov-Azerbaijan

J.679 If $a, b, c, d > 0, p = a + b + c, q = ab + bc + ca, r = abc$ then:

$$p^2q^2 + 18^3 + 3^7 \cdot r^2 \geq \sqrt{338364} \cdot pqr$$

Proposed by Seyran Ibrahimov-Azerbaijan

J.680 If $a, b, c > 0$ then:

$$a + b + c \geq 3\sqrt[3]{abc} + \frac{1}{4}\sum_{cyc}\left(\sqrt{\frac{\sqrt[3]{a^2} + \sqrt[3]{b^2}}{2}} - \sqrt[6]{ab}\right) \cdot \sum_{cyc}\left(\sqrt{\frac{\sqrt[3]{a^4} + \sqrt[3]{b^4}}{2}} - \sqrt[3]{ab}\right)$$

Proposed by Seyran Ibrahimov-Azerbaijan

J.681 If $x, y, z > 0$ then:

$$x + y + z \geq \frac{4}{3}\left(\sum_{cyc}\frac{x}{\sqrt{2x + y + z}}\right)^2$$

Proposed by Seyran Ibrahimov-Azerbaijan

J.682 If $a, b \geq 0$ then:

$$2(a + b) \geq \frac{1}{3}\sqrt{3(a^2 + ab + b^2)} + 3\sqrt{ab}$$

Proposed by Seyran Ibrahimov-Azerbaijan

J.683 If $t \geq 3$ then in ΔABC holds:

$$\frac{a^t}{b + c - a} + \frac{b^t}{c + a - b} + \frac{c^t}{a + b - c} \geq \frac{2^{t-1}}{3^{\frac{t-3}{2}}}(4Rr + r^2)^{\frac{t-1}{2}}$$

Proposed by Seyran Ibrahimov-Azerbaijan

J.684 If $a, b \geq 0$ then:

$$a + b \geq \left(\sqrt{a} + \sqrt{b}\right) \cdot \sqrt[3]{\frac{\sqrt[3]{a^2} + \sqrt[3]{b^2}}{2}} \geq 2\sqrt{ab}$$

Proposed by Seyran Ibrahimov-Azerbaijan

J.685 If $a, b, c, d > 0$ then:

$$a + b + c + d \geq \frac{1}{9}\sum_{cyc}\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right)^2 \geq 4(abcd)^{\frac{1}{4}}$$

Proposed by Seyran Ibrahimov-Azerbaijan

J.686 In ΔABC the following relationship holds:

$$a^2 + b^2 + c^2 \geq \frac{1}{2}\sum_{cyc}\left(\sqrt{\frac{a^4 + b^4}{2}} + ab\right) \geq \frac{1}{2}\sum_{cyc}\left(\sqrt{\frac{a^4 + b^4}{2}} + \frac{2a^2b^2}{a^2 + b^2}\right) \geq$$

$$\geq \frac{1}{2}\sum_{cyc}\left(\frac{a^2+b^2}{2}+ab\right)\geq \frac{1}{2}\sum_{cyc}\left(\frac{a^2+b^2}{2}+\frac{2a^2b^2}{a^2+b^2}\right)\geq \sum_{cyc} ab \geq 4\sqrt{3}F$$

Proposed by Seyran Ibrahimov-Azerbaijan

J.687

$$A=\frac{1}{n}\sum_{i=1}^{n}a_i\,, Q=\sqrt{\frac{1}{n}\sum_{i=1}^{n}a_i^2}\,, G=\sqrt[n]{\prod_{i=1}^{n}a_i}\,, H=n\left(\sum_{i=1}^{n}\frac{1}{a_i}\right)^{-1}, a_i>0$$

Prove that: $Q(A+G+H)\leq 3A^2, n\in\mathbb{N}-\{0,1\}$

Proposed by Seyran Ibrahimov-Azerbaijan

J.688 Solve for real numbers:

$$[x]^2+x^2=2x[x],\ \ [*]-GIF$$

Proposed by Jalil Hajimir-Canada

J.689 Solve for real numbers:

$$[x]^{\sin\pi x}=2,\qquad [*]-GIF$$

Proposed by Jalil Hajimir-Canada

J.690 Solve for real numbers:

$$x[e^x]+[x]e^x=x,\ \ [*]-GIF$$

Proposed by Jalil Hajimir-Canada

J.691 If $x,y,z\in\mathbb{R}, (x^2+y^2)(y^2+z^2)(z^2+x^2)\neq 0$ then:

$$\frac{|x|}{\sqrt{x^2+y^2}}+\frac{|y|}{\sqrt{y^2+z^2}}+\frac{|z|}{\sqrt{z^2+x^2}}\leq\frac{2\sqrt{3}}{2}$$

Proposed by Jalil Hajimir-Canada

J.692 Prove that for any c_a,c_b,c_c – cevians in ΔABC holds:

$$\frac{\sqrt{c_a}+\sqrt{c_b}+\sqrt{c_c}}{c_a+c_b+c_c}\leq\sqrt{\frac{3}{2F\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}}$$

Proposed by Radu Diaconu – Romania

J.693 In acute $\Delta ABC, AD\perp BC, D\in(BC), r_1, r_2$ – inradii in $\Delta ABD, \Delta ACD$.

Prove that:

$$\sum_{cyc}\mu(A)\cos A\leq\frac{\pi}{2}\cdot\frac{a+AD}{r_1+r_2+s}$$

Proposed by Radu Diaconu – Romania

J.694 In acute ΔABC the following relationship holds:

$$\left(\sum_{cyc}\frac{\sin 2A}{\sqrt{\sin 2B+\sin 2C}}\right)\left(\sum_{cyc}\frac{h_a}{\sqrt{h_b+h_c}}\right)\geq 9\cdot\sqrt[4]{\frac{27}{4}}\cdot\frac{r\sqrt{r}}{R}$$

Proposed by Radu Diaconu – Romania

J.695 $\Delta ABC, \Delta ABD$ are such that: $\Delta ABC \cap \Delta ABD = [AB], m(\sphericalangle BDA) = 90^\circ$.

Prove that: $DC^2 \geq 4F$

Proposed by Radu Diaconu – Romania

J.696 In $\Delta ABC, O$ – circumcentre, $m(\sphericalangle A) = 90^\circ$,

R_1, R_2 – circumradii of $\Delta AOB, \Delta AOC$. Prove that:

$$\min\left(\frac{\sqrt{R_1}}{m_b m_c}, \frac{\sqrt{R_2}}{m_c m_a}, \frac{\sqrt{\frac{R}{2}}}{m_a m_b}\right) < \sqrt{\frac{7}{r}} \cdot \frac{R}{12r^2}$$

Proposed by Radu Diaconu – Romania

J.697 If in $\Delta ABC, 2a = b + c, g_a$ – Gergonne's cevian then:

$$\frac{\max(g_a, g_b, g_c)}{r_a} \geq \frac{2r}{R}$$

Proposed by Radu Diaconu – Romania

J.698 In $\Delta ABC, m(\sphericalangle A) = 90^\circ, O$ – circumcenter, I – incenter, holds:

$$1 < \frac{[ABO]}{[BIC]} < \frac{5}{4}$$

Proposed by Radu Diaconu – Romania

J.699 In $\Delta ABC, I$ – incenter, R_a, R_b, R_c – circumradii of $\Delta BIC, \Delta CIA, \Delta AIB$. Prove that:

$$R_a + R_b + R_c \leq \sqrt{3} \cdot \max(a, b, c)$$

Proposed by Radu Diaconu – Romania

J.700 If in $\Delta ABC, m(\sphericalangle A) = 90^\circ$ then:

$$\max(\csc^3 B, \csc^3 C) > \frac{4}{\left(1 + \frac{r}{R}\right)^2}$$

Proposed by Radu Diaconu – Romania

J.701 If in $\Delta ABC, m(\sphericalangle A) = 90^\circ$ then:

$$\frac{3\sqrt{3}}{\sqrt{4R + 2r}} + \sum_{cyc} \frac{1}{\sqrt{a}} \geq 2\sqrt{2} \cdot \sum_{cyc} \frac{1}{\sqrt{4R + 2r - b}}$$

Proposed by Radu Diaconu – Romania

J.702 In $\Delta ABC, K$ – Lemoine's point, the following relationship holds:

$$\left(\sum_{cyc} \frac{AK}{w_a}\right)\left(\sum_{cyc} \frac{1}{\varphi + \sin\frac{A}{2}}\right) > \frac{12r(a + b + c)}{a^2 + b^2 + c^2}, \varphi \leq 1$$

Proposed by Radu Diaconu – Romania

J.703 In ΔABC the following relationship holds:

$$32(a+b+c)r^2 \le \prod_{cyc} \frac{ah_a+bh_b}{r_c} \le \frac{16}{\sqrt{3}}(a^2+b^2+c^2)r$$

Proposed by Ertan Yildirim-Turkey

J.704 In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{h_a+h_b}{r_c} = 6$$

Proposed by Ertan Yildirim-Turkey

J.705 In ΔABC the following relationship holds:

$$24r \le \frac{(a+b)^2}{r_a+r_b} + \frac{(b+c)^2}{r_b+r_c} + \frac{(c+a)^2}{r_c+r_a} \le 12R$$

Proposed by Ertan Yildirim-Turkey

J.706 In ΔABC the following relationship holds:

$$\frac{sabc}{r_a\, r_b r_c R} + \frac{R}{s}\sum_{cyc} \frac{a}{r_a} + \sum_{cyc} \frac{ab}{r_a r_b} \le 4 + \frac{8R}{3r}$$

Proposed by Ertan Yildirim-Turkey

J.707 In ΔABC the following relationship holds:

$$\frac{cosA}{r_b+r_c} + \frac{cosB}{r_c+r_b} + \frac{cosC}{r_a+r_b} \le \frac{1}{4r}$$

Proposed by Ertan Yildirim-Turkey

J.708 In ΔABC the following relationship holds:

$$\frac{r_b+r_c}{b+c}\sqrt{\frac{1-cosA}{1+cosA}} + \frac{r_c+r_b}{c+a}\sqrt{\frac{1-cosB}{1+cosB}} + \frac{r_c+r_a}{c+a}\sqrt{\frac{1-cosC}{1+cosC}} \ge \frac{3}{2}$$

Proposed by Ertan Yildirim-Turkey

J.709 In ΔABC the following relationship holds:

$$2 \le \frac{a^2}{w_b^2+w_c^2} + \frac{b^2}{w_c^2+w_a^2} + \frac{c^2}{w_a^2+w_b^2} \le \frac{R}{r}$$

Proposed by Ertan Yildirim-Turkey

J.710 In ΔABC the following relationship holds:

$$1 < \frac{m_a}{b+c} + \frac{m_b}{c+a} + \frac{m_c}{a+b} \le \frac{a+b+c}{8r}$$

Proposed by Ertan Yildirim-Turkey

J.711 In ΔABC the following relationship holds:

$$\frac{r_a(r_b+r_c)}{a}+\frac{r_b(r_c+r_a)}{b}+\frac{r_c(r_a+r_b)}{c}=\frac{3(a+b+c)}{2}$$

Proposed by Ertan Yildirim-Turkey

J.712 In ΔABC the following relationship holds:

$$\frac{b+c-a}{h_a}+\frac{c+a-b}{h_b}+\frac{a+b-c}{h_c}=\frac{4(r_a+r_b+r_c)}{a+b+c}$$

Proposed by Ertan Yildirim-Turkey

J.713 In acute ΔABC the following relationship holds:

$$\frac{cos(A-B)}{a+b}+\frac{cos(B-C)}{b+c}+\frac{cos(C-A)}{c+a}\le\frac{a+b+c}{24r^2}$$

Proposed by Ertan Yildirim-Turkey

J.714 In ΔABC the following relationship holds:

$$\sum_{cyc} m_a m_b\left(\frac{1}{sin^2A}+\frac{1}{sin^2B}\right)\le\frac{R^2}{2r^2}(a^2+b^2+c^2)$$

Proposed by Ertan Yildirim-Turkey

J.715 In ΔABC the following relationship holds:

$$72r^2\le\sum_{cyc}\frac{w_a^2+w_b^2}{sinA\cdot sinB}\le\frac{9R^3}{r}$$

Proposed by Ertan Yildirim-Turkey

J.716 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{sinB+sinC}{w_a}\le\frac{\sqrt{3}}{r}$$

Proposed by Ertan Yildirim-Turkey

J.717 In ΔABC the following relationship holds:

$$4R\le\frac{1}{a+b+c}\sum_{cyc} bc\left(tan\frac{A}{2}+cot\frac{A}{2}\right)\le\frac{2R^2}{r}$$

Proposed by Ertan Yildirim-Turkey

J.718 In ΔABC the following relationship holds:

$$\frac{a}{r_b r_c}+\frac{b}{r_c r_a}+\frac{c}{r_a r_b}=\frac{2(2R-r)}{F}$$

Proposed by Ertan Yildirim-Turkey

J.719 In ΔABC the following relationship holds:

$$\frac{h_a(h_b+h_c)}{(b+c)m_a}+\frac{h_b(h_c+h_a)}{(c+a)m_b}+\frac{h_c(h_a+h_b)}{(a+b)m_c}\le sinA+sinB+sinC$$

Proposed by Ertan Yildirim-Turkey

J.720 In $\Delta ABC, I$ −incenter, R_a, R_b, R_c −circumradii of $\Delta BIC, \Delta CIA, \Delta AIB$. Prove that:

$$\left(\frac{s}{R}\right)^2\le\left(\frac{r_a}{R_a}\right)^2+\left(\frac{r_b}{R_b}\right)^2+\left(\frac{r_c}{R_c}\right)^2\le 5+\frac{3r}{R}+\left(\frac{r}{R}\right)^2$$

Proposed by Ertan Yildirim-Turkey

J.721 In ΔABC the following relationship holds:

$$\frac{a+b}{cosA+cosB}+\frac{b+c}{cosB+cosC}+\frac{c+a}{cosC+cosA}=\frac{(a+b+c)R}{r}$$

Proposed by Ertan Yildirim-Turkey

J.722 In ΔABC

$$3\le\frac{m_a}{r_a}+\frac{m_b}{r_b}+\frac{m_c}{r_c}\le\frac{\sqrt{p^2-4r^2-7r}}{r}$$

Proposed by Marin Chirciu – Romania

J.723 If $a,b,c>0$ such that $ab+bc+ca=3$ and $0\le\lambda\le1$ then:

$$\frac{1}{a^2+\lambda}+\frac{1}{b^2+\lambda}+\frac{1}{c^2+\lambda}\ge\frac{3}{\lambda+1}$$

Proposed by Marin Chirciu – Romania

J.724 If $a,b,c\in\mathbb{R}^*$ then:

$$\frac{\sqrt{a^4+b^4}}{a^2-ab+b^2}+\frac{\sqrt{b^4+c^4}}{b^2-bc+c^2}+\frac{\sqrt{c^4+a^4}}{c^2-ca+a^2}\le3\sqrt{2}$$

Proposed by Marin Chirciu – Romania

J.725 In ΔABC

$$9r\le\sum\frac{w_bw_c}{w_a}\le\frac{9R^2}{4r}$$

Proposed by Marin Chirciu – Romania

J.726 In ΔABC

$$\frac{1}{2r}\le\sum\frac{\cot B+\cot C}{b+c}\le\frac{R}{4r^2}$$

Proposed by Marin Chirciu – Romania

J.727 In ΔABC

$$9r\le\sum\frac{m_bm_c}{m_a}\le R\left(\frac{2R}{r}+\frac{1}{2}\right)$$

Proposed by Marin Chirciu – Romania

J.728 In acute ΔABC

$$\sum \frac{\tan B + \tan C}{b + c} \geq \frac{3}{R}$$

Proposed by Marin Chirciu – Romania

J.729 In ΔABC

$$9r \leq \sum \frac{h_b h_c}{h_a} \leq \frac{9R}{2}$$

Proposed by Marin Chirciu – Romania

J.730 In acute ΔABC

$$\sum \frac{\tan B + \tan C}{b + c} \geq 3\left(\frac{2r}{R}\right)^2 \sum \frac{\cot B + \cot C}{b + c}$$

Proposed by Marin Chirciu – Romania

J.731 In ΔABC

$$4R + r \leq \sum \frac{r_b r_c}{r_a} \leq \frac{(2R - r)^2}{r}$$

Proposed by Marin Chirciu – Romania

J.732 In ΔABC

$$\frac{3}{2R} \leq \sum \frac{\sin B + \sin C}{b + c} \leq \frac{3}{4r}$$

Proposed by Marin Chirciu – Romania

J.733 If $a, b, c, d, e \geq 1$ then: $4a + 4b + 3c + 2d + e \leq 10 + ab(1 + c + cd + cde)$

Proposed by Daniel Sitaru– Romania

J.734 If $a, b, c > 0, (4a - b - c)(4b - c - a)(4c - a - b) = 64$ then:

$$a^3b^3c^3 \geq (a^2 + bc)(b^2 + ca)(c^2 + ab)$$

Proposed by Daniel Sitaru– Romania

J.735 $A(1,0,0), B(0,2,0), C(1,1,1), D(3,-1,2)$. Find the area of circumsphere of tetrahedron $ABCD$.

Proposed by Daniel Sitaru– Romania

J.736 In $\Delta ABC, T$ – Toricelli's point the following relationship holds:

$$TA^2 + TB^2 + TC^2 \geq 12r^2$$

Proposed by Daniel Sitaru– Romania

J.737 Solve for natural numbers:

$$\cos\frac{x}{32} + \cos\frac{2x}{32} + \cos\frac{3x}{32} + \cdots + \cos\frac{31x}{32} = 1$$

$$\cos^4\frac{x}{64}+\cos^4\frac{2x}{64}+\cos^4\frac{3x}{64}+\cdots+\cos^4\frac{63x}{64}=12$$

Proposed by Daniel Sitaru– Romania

J.738 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{(h_b^{n+1}+h_c^{n+1})^2}{h_b^n+h_c^n}\le\frac{27}{2}R^2, n\in\mathbb{N}$$

Proposed by Marin Chirciu-Romania

J.739 If $a,b,c,d>0$ then

$$4(1-a+a^2)(1-b+b^2)(1-c+c^2)(1-d+d^2)\ge(ab+cd)^2$$

Proposed by Marin Chirciu-Romania

J.740 In ΔABC the following relationship holds:

$$\sum_{cyc}sin^3\frac{A}{2}sin\frac{B}{2}\ge\frac{r}{4R}\left(\frac{5}{4}+\frac{r}{2R}\right)$$

Proposed by Marin Chirciu-Romania

J.741 In ΔABC the following relationship holds:

$$\sum_{cyc}cos^3\frac{A}{2}cos\frac{B}{2}\ge\frac{s}{4R}\left(\frac{3\sqrt{3}}{4}+\frac{s}{2R}\right)$$

Proposed by Marin Chirciu-Romania

J.742 In ΔABC the following relationship holds:

$$\frac{2}{R}\le\sum_{cyc}\frac{cotB+cotC}{a}\le\frac{1}{r}$$

Proposed by Marin Chirciu-Romania

J.743 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{(w_b^{n+1}+w_c^{n+1})^2}{w_b^n+w_c^n}\le\frac{27}{2}R^2, n\in\mathbb{N}$$

Proposed by Marin Chirciu-Romania

J.744 In acute ΔABC the following relationship holds:

$$\sum_{cyc}\frac{a}{b^2+c^2-a^2}\ge\frac{1}{2s}\left(\frac{2R^2}{r^2}+\frac{R}{r}-1\right)$$

Proposed by Marin Chirciu-Romania

J.745 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{acos(B-C)}{b+c}\le\frac{3R}{4r}$$

Proposed by Marin Chirciu-Romania

J.746 In ΔABC the following relationship holds:

$$\frac{3\lambda}{2}+\prod_{cyc}\frac{r_a}{w_a}\geq 1+\lambda\sum_{cyc}\frac{a}{b+c},\lambda\leq\frac{1}{2}$$

Proposed by Marin Chirciu-Romania

J.747 If $x,y,z>0$ such that $x+y+z=1$ and $n\leq 43,2$ then

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+n(xy+yz+zx)\geq 9+\frac{n}{3}$$

Proposed by Marin Chirciu-Romania

J.748 In $\Delta ABC, R_a, R_b, R_c$ −circumradii of $\Delta BIC, \Delta CIA, \Delta AIB, I$ −incenter.

$$\sum_{cyc}\frac{w_b+w_c}{w_a}R_a^2\geq 2r(5R+2r)$$

Proposed by Marin Chirciu-Romania

J.749 If $a,b>0,x,y>0$ and $n\in\mathbb{N}^*$ then

$$\frac{a^{2n+1}}{ax+by}+\frac{b^{2n+1}}{bx+ay}\geq\frac{a^{2n}+b^{2n}}{x+y}$$

Proposed by Marin Chirciu-Romania

J.750 In ΔABC the following relationship holds:

$$\sum_{cyc}\left(\frac{b+c-a}{a}\right)^2+\lambda\frac{r}{R}\geq 3+\frac{1}{2}\lambda,\lambda\leq 10$$

Proposed by Marin Chirciu-Romania

J.751 In ΔABC the following relationship holds:

$$R^2\geq 4r^2+\frac{1}{8}\left(1+\frac{2r}{R}\right)\sum_{cyc}(b-c)^2$$

Proposed by Marin Chirciu-Romania

J.752 If $x,y,z>0,x+y+z\geq 3,n\in\mathbb{N},n\geq 2$ and $\lambda\geq 0$ then:

$$\frac{x^n}{y^2+\lambda z^2}+\frac{y^n}{z^2+\lambda x^2}+\frac{z^n}{x^2+\lambda y^2}+\frac{x^2+y^2+z^2}{\lambda+1}\geq\frac{6}{\lambda+1}$$

Proposed by Marin Chirciu-Romania

J.753 In triangle ABC, n_a – Nagel's cevian, the following relationship holds:

$$n_a^2+n_b^2+n_c^2\overset{(1)}{\geq} m_a^2+m_c^2+m_c^2+r(R-2r)\overset{(2)}{\geq} m_a^2+m_b^2+m_c^2$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

J.754 If $x_i > 0, \forall i = 1,2,\dots,2021$ then:

$$\frac{1}{x_1}+\frac{2}{x_1+x_2}+\cdots+\frac{2021}{x_1+x_2+\cdots+x_{2021}}<4\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_{2021}}\right)$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

J.755 In any triangle ABC the following relationship holds:

$$n_a^2+n_b^2+n_c^2+\frac{2r}{R}(R-2r)\ge m_a^2+m_b^2+m_c^2$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

J.756 In any triangle ABC the following relationship holds:

$$h_bh_a^4+h_ch_b^4+h_ah_c^4<as_aw_ag_ah_a+bs_bw_bg_bh_b+cs_cw_cg_ch_c\le\frac{3\sqrt{3}}{16}\cdot R^5\cdot\left(\frac{h_a}{r}\right)^3$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

J.757 Find all positive real numbers α such that:

$$\frac{a^3}{b^3}+\frac{b^3}{c^3}+\frac{c^3}{a^3}\ge\frac{\alpha(a^3+b^3+c^3)}{abc},\forall a,b,c>0$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

J.758 In any triangle ABC the following relationship holds:

$$m_a^2+m_b^2+m_c^2\ge w_a^2+w_b^2+w_c^2+r(R-2r)+\frac{r}{4R}(R^2-4r^2)$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

J.759 If a,b,c are positive real numbers such that:

1. $\frac{a^5}{b}+\frac{b^5}{c}+\frac{c^5}{a}\ge(a^2+b^2+c^2)\left(\frac{a+b+c}{3}\right)^2$

2. $\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}\ge ab+bc+ca+\frac{2020}{2021}\left[\frac{(a^2-b^2)(a-b)}{b}+\frac{(b^2-c^2)(b-c)}{c}+\frac{(c^2-a^2)(c-a)}{a}\right]$

Proposed by Nguyen Van Canh – BenTre – Vietnam

J.760 If a,b,c are positive real numbers such that $a^2+b^2+c^2=3$, then:

1. $\frac{a^4+b^4+c^4}{ab+bc+ca}+\frac{3abc}{a+b+c}\ge 2$

2. $\frac{(b+c)^2+(a+c)^2}{ab+bc+ca}\ge\frac{a+b}{3}$

Proposed by Nguyen Van Canh – BenTre – Vietnam

J.761 In any triangle ABC the following relationship holds:

$$\frac{m_a}{m_b+m_c}+\frac{m_b}{m_a+m_c}+\frac{m_c}{m_a+m_c}+\sqrt{\frac{2m_am_bm_c}{(m_a+m_b)(m_b+m_c)(m_c+m_a)}}\ge 2$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

J.762 Solve for real numbers:

$$x+\left[x+\frac{1}{3}\right]+\left[x+\frac{2}{3}\right]=10(x-[x]),\ \ [*]-GIF$$

Proposed by Rajeev Rastogi-India

J.763 If $x, y, z \in \mathbb{R}, x + y + z = 0$ then:

$$\Omega = \sum_{cyc} \frac{2x^2 + 3x + 5}{2x^2 + yz}$$

Proposed by Rajeev Rastogi-India

J.764 p –is prime number fixed. Find all ordered pairs of positive integers such that:

$$(x + y)(x^3 + 7y) = p^4$$

Proposed by Rajeev Rastogi-India

J.765 Find number of positive real solutions of the equation:

$$7\{x\}^2 + 7\{x\}[x] - 10[x] = 0, [*] - GIF, \{*\} = * - [*]$$

Proposed by Rajeev Rastogi-India

J.766 If $0 < a, b, c < 1$ then:

$$\frac{1}{a(1-b)} + \frac{1}{b(1-c)} + \frac{1}{c(1-a)} \geq \frac{9}{1 - abc}$$

Proposed by Rajeev Rastogi-India

J.767 In ΔABC the following relationship holds:

$$\left(4 - \frac{2m_a m_b}{m_a^2 + m_b^2}\right)^2 \leq \left(\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2}\right)\left(\frac{a^2}{c^2} + \frac{b^2}{a^2} + \frac{c^2}{b^2}\right)$$

Proposed by Adil Abdullayev-Azerbaijan

J.768 In ΔABC the following relationship holds:

$$\frac{(r_a + r_b + r_c)^3}{4 r_a r_b r_c} \geq \frac{3R}{r} + \sum_{cyc} \left(\frac{r_a}{r_b + r_c}\right)^2$$

Proposed by Adil Abdullayev-Azerbaijan

J.769 In ΔABC the following relationship holds:

$$\frac{r_a m_a^2 + r_b m_b^2 + r_c m_c^2}{r_a + r_b + r_c} = ab + bc + ca$$

Proposed by Adil Abdullayev-Azerbaijan

J.770 In ΔABC the following relationship holds:

$$4\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right) \leq \frac{h_a}{r_a} + \frac{h_b}{r_b} + \frac{h_c}{r_c} + 3$$

Proposed by Adil Abdullayev-Azerbaijan

J.771 In ΔABC the following relationship holds:

$$\frac{r_b r_c}{w_a^2} + \frac{r_c r_a}{w_b^2} + \frac{r_a r_b}{w_c^2} + \frac{16abc}{(a+b)(b+c)(c+a)} \geq 5$$

Proposed by Adil Abdullayev-Azerbaijan

J.772 In ΔABC the following relationship holds:

$$2\left(\frac{m_a^2}{r_br_c}+\frac{m_b^2}{r_cr_a}+\frac{m_c^2}{r_ar_b}\right)+\frac{8abc}{(a+b)(b+c)(c+a)}\geq 7$$

Proposed by Adil Abdullayev-Azerbaijan

J.773 In acute ΔABC the following relationship holds:

$$\cos(A-B)\cos(B-C)\cos(C-A)\leq\frac{64abc}{9(a+b)(b+c)(c+a)-8abc}$$

Proposed by Adil Abdullayev-Azerbaijan

J.774 In ΔABC the following relationship holds:

$$\frac{h_a^2}{w_a^2}+\frac{h_b^2}{w_b^2}+\frac{h_c^2}{w_c^2}+3\leq 4\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\right)$$

Proposed by Adil Abdullayev-Azerbaijan

J.775 If $x,y,z>0$ then:

$$\frac{(x+y)(y+z)(z+x)}{8xyz}\geq\left(1+\frac{x^2+y^2+z^2-xy-yz-zx}{x^2+y^2+z^2+3xy+3yz+3zx}\right)^2$$

Proposed by Adil Abdullayev-Azerbaijan

J.776 In ΔABC the following relationship holds:

$$\prod_{cyc}\frac{\cos^2\frac{A}{2}+\cos^2\frac{B}{2}}{2\cos^2\frac{C}{2}}\leq\frac{R}{2r}$$

Proposed by Adil Abdullayev-Azerbaijan

J.777 In $\Delta ABC, n_a$ – Nagel's cevian, T – Toricelli point, the following relationship holds:

$$n_a+n_b+n_c+2\sum_{cyc}\frac{h_ar_a}{n_a+s}\geq\frac{3\sqrt{3}}{2}(AT+BT+CT)$$

Proposed by Bogdan Fuștei – Romania

J.778 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{a}{\sqrt{h_a-2r}}\geq 2\sqrt{2R-r}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)$$

Proposed by Bogdan Fuștei – Romania

J.779 In $\Delta ABC, T$ – Toricelli's point, the following relationship holds:

$$\frac{m_a+m_b+m_c+2(w_a+w_b+w_c)}{3s}\sum_{cyc}\cos\frac{A}{2}\leq\sum_{cyc}\frac{w_b+w_c}{BT+TC}$$

Proposed by Bogdan Fuștei – Romania

J.780 In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$\frac{3s}{4r} \geq \sum_{cyc} \frac{n_a}{a} \sqrt{\frac{2r_a}{h_a}}$$

Proposed by Bogdan Fuștei – Romania

J.781 In ΔABC, the following relationship holds:

$$1 + \sum_{cyc} \frac{m_a}{m_b + m_c - m_a} \leq \frac{2R(m_a + m_b + m_c)}{9r^2}$$

Proposed by Bogdan Fuștei – Romania

J.782 In ΔABC, the following relationship holds:

$$\frac{m_a}{w_a} + \frac{m_b}{w_b} + \frac{m_c}{w_c} \geq \sum_{cyc} \frac{m_b + m_c - m_a}{\sqrt{w_b w_c}}$$

Proposed by Bogdan Fuștei – Romania

J.783 If $x, y, z, t > 0$ then:

$$\frac{x}{2y} + \frac{y}{2z} + \frac{z}{2t} + \frac{t}{2x} + \frac{16xyzt}{(x+y)(y+z)(z+t)(t+x)} \geq 3$$

Proposed by Marin Chirciu – Romania

J.784 In ΔABC:

$$\sum \sin^3 A \sin B \geq \frac{rp^2}{2R^3} \geq \frac{27}{2}\left(\frac{R}{r}\right)^3$$

Proposed by Marin Chirciu – Romania

J.785 In ΔABC:

$$9r \leq \sum \sqrt{\frac{w_b^2 + w_c^2}{2}} \leq \frac{9R}{2}$$

Proposed by Marin Chirciu – Romania

J.786 In ΔABC:

$$p^4 \geq p^2(12R^2 + 4Rr - 2r^2) - r(4R + r)^3$$

Proposed by Marin Chirciu – Romania

J.787 In ΔABC:

$$\frac{2R - 3r}{2Rr} \leq \sum \frac{r_a}{bc} \tan^2 \frac{A}{2} \leq \frac{2R^2 - 4Rr + r^2}{2Rr^2}$$

Proposed by Marin Chirciu – Romania

J.788 In ΔABC:

$$12\left(\frac{2r}{R}\right)^2 \le \sum m_a m_b\left(\frac{1}{r_b}+\frac{1}{r_c}\right)^2 \le \frac{R^2}{r^2}\left(\frac{2R}{r}-1\right)$$

Proposed by Marin Chirciu – Romania

J.789 If $x,y,z>0$ and $x+y+z=3$ then:

$$\frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3}\ge x^3+y^3+z^3$$

Proposed by Marin Chirciu – Romania

J.790 In $\Delta ABC, A_1, B_1, C_1$ are contact points with incircle. Prove that:

$$3\left(\frac{R}{r}\right)^2 \le \left(\frac{AB}{A_1B_1}\right)^2+\left(\frac{BC}{B_1C_1}\right)^2+\left(\frac{CA}{C_1A_1}\right)^2 \le \frac{2R}{r}\left(\frac{2R}{r}-1\right)$$

Proposed by Marin Chirciu – Romania

J.791 If $a,b,c>0$ and $a+b+c=abc$ and $n\in\mathbb{N}$ then:

$$\sum a^n(bc-1)\ge 6\left(\sqrt{3}\right)^n$$

Proposed by Marin Chirciu – Romania

J.792 In $\Delta ABC, I$ – incenter, R_a, R_b, R_c – circumradii of $\Delta BIC, \Delta CIA, \Delta AIB$. Prove that:

$$\sum \frac{R_a^2}{r_a(\sin B+\sin C)}\le \frac{3R^3}{r(a+b+c)}$$

Proposed by Marin Chirciu – Romania

J.793 In $\Delta ABC, I$ – incenter, R_a, R_b, R_c – circumradii of $\Delta BIC, \Delta CIA, \Delta AIB$. Prove that:

$$\sum \frac{R_a^2}{r_a^2-r^2}\ge \frac{3}{2}$$

Proposed by Marin Chirciu – Romania

J.794 In ΔABC:

$$\sum \tan^3\frac{A}{2}\tan\frac{B}{2}\ge \frac{r(4R+r)}{p^2}\ge \frac{2r}{3R}$$

Proposed by Marin Chirciu – Romania

J.795 In acute triangle ABC the following relationship holds:

$$\left(1+\frac{\cos^2 A}{\cos B}\right)\left(1+\frac{\cos^2 B}{\cos C}\right)\left(1+\frac{\cos^2 C}{\cos A}\right)\ge \frac{p^2}{2R^2}$$

Proposed by Alex Szoros – Romania

J.796 In ΔABC if $abc=1$, then:

$$\frac{2}{R}\le\frac{a^3}{r_a}+\frac{b^3}{r_b}+\frac{c^3}{r_c}\le\frac{1}{r}$$

Proposed by Alex Szoros – Romania

J.797 Prove that the following inequality is true in every triangle:

$$S\le\frac{abc(b+c)l_a}{(a+b)(a+c)(b+c-a)}$$

Proposed by Alex Szoros – Romania

J.798 In ΔABC the following relationship holds:

$$4(R+r)^3\ge\left(\frac{a^2}{r_a}+\frac{2bc}{\sqrt{r_br_c}}\right)\left(\frac{b^2}{r_b}+\frac{2ca}{\sqrt{r_cr_a}}\right)\left(\frac{c^2}{r_c}+\frac{2ab}{\sqrt{r_ar_b}}\right)\ge 27R^2r$$

Proposed by Alex Szoros – Romania

J.799 In ΔABC the following relationship holds:

$$\left(\frac{R}{r}\right)^3\ge\frac{(a^2+b^2)(b^2+c^2)(c^2+a^2)}{(m_al_a+rr_a)(m_bl_b+rr_b)(m_cr_c+rr_c)}\ge 8$$

Proposed by Alex Szoros – Romania

J.800 Let $a,b,x,y,z>0$. Prove that:

$$\sum\frac{\left(\frac{x}{ay+bz}\right)^3}{\left(\frac{x}{ay+bz}\right)^2+\frac{xy}{(ay+bz)(az+bx)}+\left(\frac{y}{az+bx}\right)^2}\ge\frac{1}{a+b}$$

Proposed by Alex Szoros – Romania

J.801 If $x,y,z>0,\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=3$ then:

$$\frac{y^3z}{x^6(y^3+z)}+\frac{z^3x}{y^6(z^3+x)}+\frac{x^3y}{z^6(x^3+y)}\ge\frac{3}{2}$$

Proposed by Rajeev Rastogi – India

J.802 Find the number of values $n\in\{10,11,\dots,2020\}$ such that:

$$n^4+6n^3+25n^2+12\ \vdots\ 5$$

Proposed by Rajeev Rastogi – India

J.803 Find the number of ways of selecting 12 squares of size 1×1 from a chessboard of size 5×5 such that no two chosen squares have a side in common.

Proposed by Rajeev Rastogi – India

J.804 If $a,b,c,d>0,a+b+c+d=4$ then:

$$2+\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}\ge ab+ac+ad+bc+bd+cd$$

Proposed by Rajeev Rastogi – India

J.805 Find $x, y \in \mathbb{Z}$ such that:

$$x|y \text{ and } x^2 + y^2|y^4 + 2080$$

Proposed by Mehmet Șahin –Turkey

J.806 Solve for integers:

$$\sqrt[3]{(x+y)^2} + \sqrt[3]{(y+z)^2} + \sqrt[3]{(z+x)^2} = x + y + z$$

Proposed by Mehmet Șahin –Turkey

J.807 If $a, b, c > 0, 2021(ab + bc + ca) > a + b + c$ then:

$$2021(a + b + c) > 3$$

Proposed by Lucian Tuțescu – Romania

J.808 Solve for integers:

$$\sqrt[3]{a^2 + b^2} + \sqrt[3]{2ab} = a + b$$

Proposed by Mehmet Șahin –Turkey

J.809 Solve for integers:

$$x^2\sqrt{yz} + y^2\sqrt{zx} + z^2\sqrt{xy} = 3xyz$$

Proposed by Mehmet Șahin –Turkey

J.810 In $\Delta ABC, n_a$ – Nagel's cevian the following relationship holds:

$$\frac{h_a h_b + h_b h_c + h_c h_a}{(h_a - 2r)(h_b - 2r)(h_c - 2r)} \le \frac{n_a n_b + n_b n_c + n_c n_a}{r^3}$$

Proposed by Bogdan Fuștei – Romania

J.811 In $\Delta ABC, n_a$ – Nagel's cevian, g_a – Gergonne's cevian, the following relationship holds:

$$\sum_{cyc} \frac{h_a}{g_a + s - a} \ge \frac{g_a + g_b + g_c}{2r} - \frac{1}{2}\sum_{cyc} \frac{a}{AI}$$

Proposed by Bogdan Fuștei – Romania

J.812 In $\Delta ABC, n_a$ – Nagel's cevian, g_a – Gergonne's cevian, the following relationship holds:

$$\sum_{cyc} \frac{n_a^2 + g_a^2}{r_b + r_c} \ge \frac{a^2 + b^2 + c^2}{2R}$$

Proposed by Bogdan Fuștei – Romania

J.813 In $\Delta ABC, n_a$ – Nagel's cevian, g_a – Gergonne's cevian, the following relationship holds:

$$\sum_{cyc} \frac{\sqrt{n_a r_a g_a}}{a} \ge \frac{h_a + h_b + h_c}{2\sqrt{r}}$$

Proposed by Bogdan Fuștei – Romania

J.814 In $\Delta ABC, n_a$ – Nagel's cevian the following relationship holds:

$$\cot\frac{A}{2} \le \frac{n_b}{h_b} + \frac{n_c}{h_c} - \frac{|b-c|+2r}{2\sqrt{2}r} + 2\left(\frac{r_b}{n_b+s} + \frac{r_c}{n_c+s} - \frac{r_a}{n_a+s}\right)$$

Proposed by Bogdan Fuștei – Romania

J.815 In $\Delta ABC, n_a$ – Nagel's cevian the following relationship holds:

$$s = \frac{n_a+n_b+n_c}{3} + \frac{2}{3}\sum_{cyc}\frac{r_a h_a}{s+n_a}$$

Proposed by Bogdan Fuștei – Romania

J.816 In ΔABC the following relationship holds:

$$\frac{3}{4} + \sum_{cyc}\frac{m_a}{h_b+h_c} \ge \frac{g_a+g_b+g_c}{4r}$$

Proposed by Bogdan Fuștei – Romania

J.817 In ΔABC the following relationship holds:

$$m_a \ge h_a + \frac{2(m_b-m_c)^2}{3a}$$

Proposed by Bogdan Fuștei – Romania

J.818 In ΔABC the following relationship holds:

$$2\sum_{cyc}\frac{g_a^2}{h_a^2} - 1 \le \frac{r_a^2+r_b^2+r_c^2}{s^2} + \sum_{cyc}\frac{2\sqrt{3}a}{m_a+w_b+w_c}$$

Proposed by Bogdan Fuștei – Romania

J.819 In ΔABC the following relationship holds:

$$\frac{3R}{2r} \ge \sqrt[3]{\frac{(m_a+n_a+h_a)(m_b+n_b+h_b)(m_c+n_c+h_c)}{h_a h_b h_c}}$$

Proposed by Bogdan Fuștei – Romania

J.820 If $0 < x < \frac{\pi}{2}$ then:

$$\frac{8\sin^6 x}{1+\cot x} + \frac{8\cos^6 x}{1+\tan x} \ge \sin^3(2x)$$

Proposed by Daniel Sitaru– Romania

J.821 If $0 \le x < \frac{\pi}{16}$ then:

$$\cos^{1216} x \ge \cos 8x \cdot \cos^9(6x) \cdot \cos^{34}(4x) \cdot \cos^{71}(2x)$$

Proposed by Daniel Sitaru– Romania

J.822 a, b, c, d – sides, r – inradii, s – semiperimeter, F – area in a bicentric quadrilateral. Prove that:

$$a^4 + b^4 + c^4 + d^4 \geq 8F^2\left(1 - \sqrt{\frac{r}{s}}\right)$$

Proposed by Daniel Sitaru– Romania

J.823 If e, f – diagonals, R – circumradii, r – inradii, s – semiperimeter in a bicentric quadrilateral then:

$$2R \cdot \sqrt[3]{ef}\left(\sqrt[3]{e} + \sqrt[3]{f}\right) \leq s\left(r + \sqrt{r + 4R^2}\right)$$

Proposed by Daniel Sitaru– Romania

J.824 If a, b, c, d – sides, R – circumradii, r – inradii, s – semiperimeter in a bicentric quadrilateral then:

$$3\sum_{cyc} a^2 - 2\sum_{cyc} ab \leq 4(2R^2 - rs)$$

Proposed by Daniel Sitaru– Romania

J.825 Solve for real numbers:

$$\begin{cases} x, y, z, t > 0 \\ \dfrac{(2-x)(2-y)(2-z)(2-t)}{(x+y)(y+z)(z+t)(t+x)} = \dfrac{1}{16} \\ x + y + z + t = 4 \end{cases}$$

Proposed by Daniel Sitaru– Romania

J.826 In ΔABC the following relationship holds:

$$\left(\sqrt[4]{s-a} + \sqrt[4]{s-b} + \sqrt[4]{s-c}\right)^2 \leq \frac{3(3R - 6r + h_a + h_b + h_c)}{\sqrt{s}}$$

Proposed by Daniel Sitaru– Romania

J.827 Find $x, y > 0$ such that:

$$27xy + 27(1 - x - y)(x + y + xy) = 10$$

Proposed by Daniel Sitaru– Romania

J.828 a, b, c, d – sides, s – semiperimeter, r – inradii in a bicentric quadrilateral.

Prove that:

$$\sum_{cyc} \frac{1}{a^4 + b^4 + c^4 + r^2 s^2} \leq \frac{1}{16r^4}$$

Proposed by Daniel Sitaru– Romania

J.829 If $x, y, z \in \mathbb{C}, x + y + z = 0$ then:

$$|x| + |y| + |z| \leq |x - z| + |z - y| + |y - x|$$

Proposed by Daniel Sitaru– Romania

J.830 In ΔABC the following relationship holds:

$$\sum_{cyc}\left(\sqrt{\frac{a^4+b^4}{2}}+\frac{2a^2b^2}{a^2+b^2}\right)\geq 8\sqrt{3}F$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

J.831 Prove that in any triangle ABC is true the following inequality

$$\frac{484}{7+\sin^2 A}+\frac{1936}{10+\sin^2 B}+\frac{2025}{11+\sin^2 C}\geq 400$$

Proposed by Neculai Stanciu – Romania

J.832 Let the triangle ABC with the area F, usual notations and

$M\in(BC), N\in(CA), P\in(AB)$. Prove that:

$$a^3(BN+CP)+b^3(CP+AM)+c^3(AM+BN)\geq 16\sqrt{3}\cdot F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

J.833 If $x_1,x_2,\dots,x_n>0, x_1+x_2+\dots+x_n=1$ then:

$$x_1\cdot x_2\cdot\ldots\cdot x_n\geq\sqrt[n]{x_1^{\frac{1}{x_1}}\cdot x_2^{\frac{1}{x_2}}\cdot\ldots\cdot x_n^{\frac{1}{x_n}}}, n\in\mathbb{N}, n\geq 2$$

Proposed by Marius Drăgan, Neculai Stanciu – Romania

J.834 If $t>0$ then in $\Delta ABC, \Delta A'B'C'$ holds:

$$\sum_{cyc}(aa')^t\geq 3^{1-t}\cdot 4^t\cdot\left(rr'(r+4R)(r'+R')\right)^{\frac{t}{2}}$$

Proposed by Cristian Miu-Romania

J.835 $ABCD$ – cyclic quadrilateral, O – circumcentre, $OE\perp DC, OF\perp CB, OH\perp AB, OG\perp AD$. Prove that:

$$\frac{AB+DC}{BC-AD}+\frac{OH+OE}{OF-OG}=0$$

Proposed by Amerul Hassan-Myanmar

J.836 Solve for real numbers:

$$\begin{cases}\sqrt{x+y}-\sqrt{x-y}+\sqrt{x^2-y^2}=5\\ 2x+3\sqrt{x^2-y^2}=19\end{cases}$$

Proposed by Denisa Lepădatu-Romania

J.837 Prove that, if the real numbers x, y, z, λ satisfy the relation

$$x^2+y^2+z^2-xy-xz-yz+3\lambda(\lambda-x+z)=0$$

x, y and z form an arithmetic progression.

Proposed by Denisa Lepădatu-Romania

J.838 Solve for real numbers:

$$\frac{\sqrt{z-1}+x}{\sqrt{x-1}-x}=\frac{\sqrt[3]{x-1}+x}{\sqrt[3]{x-1}-x}$$

Proposed by Denisa Lepădatu-Romania

J.839 Let $b\geq c\geq a\geq 0$ and $a^3+\frac{1}{b}+\frac{1}{c}=2$. Prove that:

$$\frac{1}{a+c}+\frac{2}{c+b}+\frac{3b+2a+c}{2c+ab}\geq\frac{8}{a+b+c}$$

Proposed by Minh Nhat Nguyen-Vietnam

J.840 Find the solution: $2^{2020}+3^{2020}+6^{2020}=n^{2021}$ $(n\in\mathbb{N})$

Proposed by Ilir Demiri-Azerbaijan

PROBLEMS FOR SENIORS

S.248 If $A\in M_n(\mathbb{R}), n\geq 2, A$ – symmetric, invertible then:

$$\det(A^2+A^{-2}+2A+2A^{-1}+3I_n)\geq 9^n$$

Proposed by Marian Ursărescu – Romania

S.249 Find $m, n\in\mathbb{N}^*$ such that x^2-x+3 divide $(x+2)^m-(x^2+2)^n, x\in\mathbb{R}$.

Proposed by Marian Ursărescu – Romania

S.250 If $A\in M_n(\mathbb{R}), A=A^T, \det A\neq 0$ then:

$$\det[4(A^2+A^{-2})+25(A+A^{-1})+42I_n]>1$$

Proposed by Marian Ursărescu – Romania

S.251 $a_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}$

Find:

$$\Omega = \lim_{n\to\infty} \sqrt{n} \prod_{k=1}^{n} \left(1 - \frac{1}{a_{k+1}\sqrt{k+1}}\right)$$

Proposed by Marian Ursărescu – Romania

S.252 If $A, B \in M_2(\mathbb{C}), \det(A+B) = 1$ then:

$$\det(A \cdot \det B + B \cdot \det A) = \det(AB)$$

Proposed by Marian Ursărescu – Romania

S.253 If $A, B \in M_2(\mathbb{R}), AB = BA, \det A = \alpha > 0, \det(A + i\alpha B) = 0$ then:

Find:

$$\Omega = \det(A^2 - \alpha AB + \alpha^2 B^2)$$

Proposed by Marian Ursărescu – Romania

S.254

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{R}[X], n \geq 2$$

If $a_0, a_1, \ldots, a_n > 0$ then: $P\left(1 + \frac{1}{n}\right) \geq P(1) + \frac{1}{n}P'(1)$

Proposed by Marian Ursărescu – Romania

S.255 $A, B \in M_2(\mathbb{R}), Tr\ ((AB)^2) = Tr\ (A^2B^2), n \in \mathbb{N}, n \geq \mathbb{N}, n \geq 2$. Find:

$$\Omega = Tr\ [(AB - BA)^n]$$

Proposed by Marian Ursărescu – Romania

S.256

$$\Omega_n = \int_0^1 \frac{(x-1)^{2n} + (x-1)^n + 1}{x^2 + x + 1} dx, n \in \mathbb{N}, n \geq 2$$

Find n such that $\Omega_n \in \mathbb{Q}$

Proposed by Marian Ursărescu – Romania

S.257 Find $A, B \in M_2(\mathbb{R})$ such that:

$$\det A < 0, \det(A - B) > 0, \det(A + B) < 0, \det(A + 2B) > 0$$

Proposed by Marian Ursărescu – Romania

S.258 If $a, b \in \mathbb{R}, A, B \in M_n(\mathbb{R}), AB = BA$ then:

$$\det(I_n + 2(a^2 + b^2)(A^2 + B^2) + 2(a+b)(A+B) + 8abAB) \geq 0$$

Proposed by Marian Ursărescu – Romania

S.259 $a,b,c>0,\frac{a}{1+b}+\frac{b}{1+c}+\frac{c}{1+a}=1,\lambda\geq 1$. Find $\max P$.

$$P=(a^2+\lambda bc)(b^2+\lambda ca)(c^2+\lambda ab)$$

Proposed by Marin Chirciu– Romania

S.260 If $a,b,c>0$ such that $a^2+b^2+c^2=3$ and $0\leq\lambda\leq 2$ then:

$$\frac{1}{1+\lambda ab^2}+\frac{1}{1+\lambda bc^2}+\frac{1}{1+\lambda ca^2}\geq\frac{3}{\lambda+1}$$

Proposed by Marin Chirciu – Romania

S.261

$$\begin{cases}\tan^{-1}\left(\frac{x+y}{1-xy}\right)=\sqrt[3]{\tan^{-1}z}\\ \tan^{-1}\left(\frac{y+z}{1-yz}\right)=\sqrt[3]{\tan^{-1}x}\\ \tan^{-1}\left(\frac{z+x}{1-zx}\right)=\sqrt[3]{\tan^{-1}y}\end{cases}$$

Find: $\Omega=x+y+z,x,y,z\in\mathbb{R}$

Proposed by Daniel Sitaru– Romania

S.262

$$f\in C^1([a,b]),f(0)=0,f\left(\frac{\pi}{2}\right)=96,f'(x)=f'\left(\frac{\pi}{2}-x\right),\forall x\in[a,b]$$

Find:

$$\Omega=\int_0^{\frac{\pi}{2}}x\left(\frac{\pi}{2}-x\right)f(x)dx$$

Proposed by Daniel Sitaru– Romania

S.263 If $f:\mathbb{R}\to(0,\infty),f$ - continuous, $a>0,f(x)=f(-x),\forall x\in\mathbb{R}$ then:

$$\int_{\frac{1}{a}}^{a}\frac{x+\log x}{xf\left(x-\frac{1}{x}\right)}dx=\frac{1}{2}\int_{\frac{1+\sqrt{1+4a^2}}{2a}}^{\frac{a+\sqrt{4+a^2}}{2}}\frac{dx}{f(x)}$$

Proposed by Daniel Sitaru– Romania

S.264 If $1<a<b\leq e$ then:

$$125^a\cdot(4a+b)^{a+4b}\leq 125^b\cdot(a+4b)^{4a+b}$$

Proposed by Daniel Sitaru– Romania

S.265 If $0 < a \le b < 1$ then:

$$\sin\left(\frac{3a+b+2}{4}\right)\sin\left(\frac{a+3b+6}{4}\right) \le \sin\left(\frac{a+3b+2}{4}\right)\sin\left(\frac{3a+b+6}{4}\right)$$

Proposed by Daniel Sitaru,– Romania

S.266 If $x > 1, p, q, r \in \mathbb{N}$ then:

$$\frac{(x+1)^{2(p+q+r)}(x^2-1)^3}{(x^{2p+2}-1)(x^{2q+2}-1)(x^{2r+2}-1)} \le \frac{(2p)!\,(2q)!\,(2r)!}{p!\,q!\,r!}$$

Proposed by Daniel Sitaru– Romania

S.267 Prove that:

$$\begin{vmatrix} \sin x \sin y & \sin y & 3\sin x & 3 \\ \sin x & 1 & \sin x \cos y & \cos y \\ 2\sin y & \sin y \cos x & 6 & 3\cos x \\ 2 & \cos x & 2\cos y & \cos x \cos y \end{vmatrix} \ne 0, \forall x, y \in \mathbb{R}$$

Proposed by Daniel Sitaru– Romania

S.268

$$\Omega = \lim_{n\to\infty}\left(\left(\int_{\frac{2}{n}}^{\frac{3}{n}} \frac{\tan^{-1}(x+1)}{x(1+x^2)}\right)\left(\int_{2n}^{3n} \frac{\tan^{-1}(x+1)}{x}dx\right)\left(\int_{\frac{1}{n}}^{n} \frac{\tan^{-1}(2x)-\tan^{-1}(3x)}{x}dx\right)\right)$$

Proposed by Daniel Sitaru– Romania

S.269 Find:

$$\Omega = \lim_{x\to\infty}\left(x\int_x^{x+\frac{2}{x}}\left(t\cdot sin^{-1}\left(\frac{1}{t}\right)\right)dt\right)$$

Proposed by Vasile Mircea Popa-Romania

S.270 If $a, b, c, d \ge 0, a+b+c+d = 2$ then:

$$\frac{2}{9} \le \frac{a}{1+a^3} + \frac{b}{1+b^3} + \frac{c}{1+c^3} + \frac{d}{1+d^3} \le \frac{16}{9}$$

Proposed by Vasile Mircea Popa-Romania

S.271 Find:

$$\Omega = \lim_{x\to 1}\sum_{k=1}^{n}\left[\sum_{i=1}^{k} i^2(i+1)^2\right]^{-1}$$

Proposed by Vasile Mircea Popa-Romania

S.272 Find:

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\left(2^{\frac{k^2}{n^3}}-1\right)$$

Proposed by Vasile Mircea Popa-Romania

S.273 We consider the equation: $x^4-8x^3+12x^2-8x+1=0$

Show that the equation has two real roots, of which the largest is:

$$a=2+\sqrt[4]{2}+\sqrt[4]{4}+\sqrt[4]{8}$$

Proposed by Vasile Mircea Popa-Romania

S.274 In any acute or right triangle ABC the following relationship holds:

$$0<\sqrt{cosA}+\sqrt{cosB}+\sqrt{cosC}-cosA-cosB-cosC\le\frac{3}{2}(\sqrt{2}-1)$$

Proposed by Vasile Mircea Popa-Romania

S.275 Find:

$$\Omega=\lim_{n\to\infty}\left(n-\sum_{k=1}^{n}\sqrt[4]{1+\frac{k^3}{n^4}}\right)$$

Proposed by Vasile Mircea Popa-Romania

S.276 If $0<a\le b, f:(0,\infty)\to(0,\infty), f-$ continuous then:

$$\int_a^b f^6(x)\,dx\cdot\left(\int_a^b\frac{1}{f(x)}dx\right)^3\ge(b-a)\left(\int_a^b f(x)\,dx\right)^3$$

Proposed by Daniel Sitaru – Romania

S.277 If $f:[0,1]\to\mathbb{R}, f-$ continuous, $n\in\mathbb{N}, n\ge1$ then:

$$e^{2n}+2n\int_0^1 f^2(e^x)\,dx\ge1+4n\int_1^e x^{n-1}f(x)dx$$

Proposed by Daniel Sitaru– Romania

S.278 In any triangle ABC the following relationship holds:

$$\begin{vmatrix}(b+c)^2 & a^2 & a^2\\ b^2 & (c+a)^2 & b^2\\ c^2 & c^2 & (a+b)^2\end{vmatrix}\ge93312r^6$$

Proposed by D.M. Bătinețu-Giurgiu– Romania

S.279 If $0 < a, b, c < 1$ then:

$$\frac{1}{1-a^3b^2}+\frac{1}{1-b^3c^2}+\frac{1}{1-c^3a^2} \ge \frac{1}{1-a^3bc}+\frac{1}{1-ab^3c}+\frac{1}{1-abc^3}$$

Proposed by Daniel Sitaru– Romania

S.280 Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=0}^{n}\sum_{i=0}^{k}\sum_{j=0}^{i}(-1)^j \cdot \binom{i}{j} \cdot \frac{3^{i-j}}{4^i}\right)$$

Proposed by Daniel Sitaru– Romania

S.281 If $0 < a \le b < \frac{\pi}{8080}$ then:

$$\int_a^b \frac{\tan(2021x)\cdot\tan(2022x)\cdot\tan(2023x)}{8\tan^3 x}dx > 10^9$$

Proposed by Daniel Sitaru– Romania

S.282 If $x \in \mathbb{R}$ then:

$$\cos^4(\cos x)+\cos^2(\cos x)-2 \le \sin^6(\cos x) \le \cos^4(\cos x)+\cos^2(\cos x)+1$$

Proposed by Daniel Sitaru– Romania

S.283 If $0 < a \le b, f: [a,b] \to (0,\infty), f$ – continuous, then:

$$(b-a)\left(\int_a^b f^3(x)\,dx\right)\left(\int_a^b \frac{dx}{f^2(x)}\right) \ge \left(\int_a^b \sqrt[3]{f^5(x)}\,dx\right)\left(\int_a^b \frac{dx}{\sqrt[3]{f(x)}}\right)^2$$

Proposed by Daniel Sitaru– Romania

S.284 Find a closed form:

$$\Omega = \left(\int_{\frac{\pi}{6}}^{1} \frac{x\log x\,dx}{36x^4+\pi^2}\right)\left(\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{x\log x\,dx}{324x^4+\pi^4}\right)\left(\int_{1}^{\frac{\pi}{3}} \frac{x\log x\,dx}{9x^4+\pi^2}\right)$$

Proposed by Daniel Sitaru– Romania

S.285 In ΔABC the following relationship holds:

$$a^2\mu(A)+b^2\mu(B)+c^2\mu(C) \ge 6\pi Rr$$

Proposed by Marian Ursărescu – Romania

S.286

$$x_0=\frac{1}{2}, x_1=1, 15^{x_{n+2}}=12^{x_{n+1}}+9^{x_n}$$

Prove that the sequence =$(x_n)_{n\in\mathbb{N}}$ is increasing, bounded and find:

$$\Omega = \lim_{n\to\infty} x_n$$

Proposed by Marian Ursărescu – Romania

S.287 Find:

$$\Omega = \lim_{n\to\infty} \sum_{k=1}^{n} (\arg(2k+i))^2, i^2 = -1$$

Proposed by Marian Ursărescu – Romania

S.288 $A \in M_4(\mathbb{R}), \det A = -1$. Prove that:

$$\det(A^2 + I_4) \geq (Tr\, A^*)^2$$

Proposed by Marian Ursărescu – Romania

S.289 Find:

$$\Omega(p) = \lim_{n\to\infty} \frac{\sqrt[p]{n} - \sqrt[p]{n-1} + \sqrt[p]{n-2} - \sqrt[p]{n-3} + \cdots + (-1)^{n-1}}{\sqrt[np]{n!}}$$

Proposed by Marian Ursărescu – Romania

S.290 $P(x) = ax^4 + bx^3 + 2ax^2 + cx + a, a, b, c \in \mathbb{R}, a \neq 0$.

If P has all roots real numbers then: $|b - c| \geq 4$

Proposed by Marian Ursărescu – Romania

S.291 $P(x) = x^6 + ax^5 + bx^4 + cx^3 + bx^2 + dx + 1, a, b, c, d \in \mathbb{R}, a \neq 0$.

If P has all roots real numbers then: $|a - c + d| \geq 8$.

Proposed by Marian Ursărescu – Romania

S.292 Find:

$$\Omega = \lim_{n\to\infty} \frac{1}{n} \left(\sum_{1\leq i<j\leq n} i \cdot \sin j \right) \left(\sum_{1\leq i<j\leq n} j \cdot \sin i \right) \left(\sum_{1\leq i<j\leq n} (i \cdot j + \sin i \cdot \sin j)^2 \right)^{-1}$$

Proposed by Daniel Sitaru– Romania

S.293 Find without any software:

$$\Omega(n,x) = \int \frac{\csc(2x)}{(1+\tan^3 x)^n} dx, n \in \mathbb{N}$$

Proposed by Daniel Sitaru – Romania

S.294 Find a closed form:

$$\Omega = \lim_{n\to\infty} \left(\log n - \frac{1}{\pi} \sum_{k=1}^{n} \int_{-\frac{1}{k}}^{\frac{1}{k}} (x^8 + x^4 + 1) \cos^{-1}(kx)\, dx \right)$$

Proposed by Daniel Sitaru – Romania

S.295 If $\alpha \in \mathbb{R}, x, y, z, t > 0$ then:

$$\sum_{cyc} \frac{6x + 3(y+z)\sin 2\alpha}{y+z+t} \geq 8(1 + \sin 2\alpha)$$

Proposed by Daniel Sitaru – Romania

S.296 If $a, b \in \mathbb{R}, a \leq b$ then:

$$512(a^{10} + b^{10}) \leq (a+b)^{10} + 9216(b-a)^2 b^8$$

Proposed by Daniel Sitaru – Romania

S.297 If $a, b, c > 0$ then:

$$\left(1 + \frac{e}{e^{a+b+c}}\right) \prod_{cyc} (1 + e^a)^a \leq \left(1 + \sqrt[4]{e}\right)\left(1 + \frac{e}{e^{a+b+c}}\right)^{a+b+c}$$

Proposed by Daniel Sitaru – Romania

S.298 Solve for real numbers:

$$4\sin^2 x + \int_0^{\sin x} \sin^{-1}(2t^2 - 1)\, dt = 0$$

Proposed by Daniel Sitaru – Romania

S.299 If $a, b > 0$ then: $(4ab)^{\sqrt{ab}} \cdot e^{30(a^2+b^2)} \leq (ae^{30a} + be^{30b})^{a+b}$

Proposed by Daniel Sitaru – Romania

S.300 If $0 < x + y + z < \frac{\pi^2}{4}$ then:

$$\frac{x \cdot \cos(\sqrt{z}) + y \cdot \cos(\sqrt{x}) + z \cdot \cos(\sqrt{y})}{\cos\left(\sqrt{\frac{xy+yz+zx}{x+y+z}}\right)} \geq x + y + z$$

Proposed by Daniel Sitaru – Romania

S.301 If $a, b, c, d > 0$ then:

$$\log(1 + a^4)\log(1 + b^4)\log(1 + c^4)\log(1 + d^4) \leq (1 + \log(abcd))^4$$

Proposed by Daniel Sitaru – Romania

S.302 Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n-1}(n-k)\int_{\frac{k}{n}}^{\frac{k+1}{n}} \frac{\log(1+x)}{(1-x)(1+x^2)}dx\right)$$

Proposed by Daniel Sitaru – Romania

S.303 Prove without any software:

$$\int_0^1 \frac{x^2}{e^{2x^2}}dx + \frac{1}{2}\int_0^1 \frac{1}{e^{2x^2}}dx > \frac{1}{4e^2}$$

Proposed by Daniel Sitaru – Romania

S.304 Evaluate:

$$\int_{-a}^{a} \frac{dx}{1 + x^n + \sqrt{1 + x^{2n}}}$$

Proposed by Jalil Hajimir-Canada

S.305 If $x, y, z > 0$ determine the minimum value of:

$$\Omega = \frac{5x}{y+z} + \frac{4y}{z+x} + \frac{3z}{x+y}$$

Proposed by Jalil Hajimir-Canada

S.306 Find the maximum value of:

$$\Omega = \left(\sqrt{x} + \sqrt{y}\right)\left(\frac{1}{\sqrt{3x+y}} + \frac{1}{\sqrt{x+3y}}\right), x, y > 0$$

Proposed by Jalil Hajimir-Canada

S.307 Find without any software:

$$\Omega = \int_0^{\pi} \frac{sin^3 x}{9 - cos^2 x} dx$$

Proposed by Jalil Hajimir-Canada

S.308 Solve:

$$\left[\frac{\left(e^x + 2\sqrt{x} + 3\sqrt{x+1}\right)\left(e^{2x} + 13x + 9\right)}{\left(e^x + \sqrt{x} + \sqrt{x+1}\right)^3}\right] \leq 4, \qquad [*] - GIF$$

Proposed by Jalil Hajimir-Canada

S.309 Prove:

$$\int_0^{[n]} [x] e^{\frac{x-2}{2}} \sqrt{log(x+1)} dx > \binom{[n]}{2}, \quad [*] - GIF$$

Proposed by Jalil Hajimir-Canada

S.310 Find the maximum value of:

$$f(x, y, z) = \sum_{cyc} \frac{\sqrt{xy}}{x + y + 2z}, x, y, z > 0$$

Proposed by Jalil Hajimir-Canada

S.311 Let x, y and z be positive real numbers. Determine the maximum value of:

$$f(x,y,z)=\sqrt{\frac{17x}{8x+9y}}+\sqrt{\frac{17y}{8y+9z}}+\sqrt{\frac{17z}{8z+9x}}$$

Proposed by Jalil Hajimir-Canada

S.312 Prove without any software:

$$1<\int_0^{2\pi}\frac{e^{sinx}dx}{5\sqrt{2+sinx}}<1.1$$

Proposed by Jalil Hajimir-Canada

S.313 In ΔABC the following relationship holds:

$$\frac{\mu(A)\cdot r_a+\mu(B)\cdot r_b+\mu(C)\cdot r_c}{2s}\geq\frac{4\pi\cdot\sin 2A\cdot\sin 2B\cdot\sin 2C}{9}$$

Proposed by Radu Diaconu – Romania

S.314 Prove without softs:

$$\sqrt{\left(\int_0^1 e^{\frac{x^2}{12}}dx\right)\cdot\int_0^1\left(\frac{1}{1+2^{2x-1}}\right)^2dx}>\frac{1}{2}$$

Proposed by Radu Diaconu – Romania

S.315 In acute ΔABC the following relationship holds:

$$\cos\frac{\mu^2(A)}{4}+\cot\frac{\mu^2(B)}{4}+\sec\frac{\mu^2(C)}{4}>2+\frac{r}{2R}$$

Proposed by Radu Diaconu – Romania

S.316 In $\Delta ABC, m(\sphericalangle A)=90°, I$ – incenter, I_a, I_b, I_c – excenters, holds:

$$\frac{9r}{4R}<\left(\sum_{cyc}\frac{II_a}{a}\right)\left(\sum_{cyc}\frac{\mu^2(A)}{\pi^2}\right)<\frac{3}{2}+\frac{R}{r}$$

Proposed by Radu Diaconu – Romania

S.317 In $\Delta ABC, m(\sphericalangle A)=90°$ if and only if:

$$\begin{vmatrix}\sqrt{2R} & -\sqrt{b} & \sqrt{2r} & \sqrt{c}\\ \sqrt{b} & -\sqrt{2R} & \sqrt{c} & \sqrt{2r}\\ \sqrt{2r} & \sqrt{c} & -\sqrt{2R} & \sqrt{b}\\ \sqrt{c} & \sqrt{2r} & -\sqrt{b} & \sqrt{2R}\end{vmatrix}=0$$

Proposed by Radu Diaconu – Romania

S.318 In acute ΔABC the following relationship holds:

$$\sum_{cyc}\frac{a\cdot\mu(A)\cdot\sin A\cdot\tan A}{h_a}\geq\frac{8\pi}{81}\cdot\left(\frac{s}{R}\right)^3$$

Proposed by Radu Diaconu – Romania

S.319 In ΔABC the following relationship holds:

$$\min\left(\sqrt{\mu(B)\mu(C)}\cdot\sec\frac{A}{2},\sqrt{\mu(C)\mu(A)}\cdot\sec\frac{B}{2},\sqrt{\mu(A)\mu(B)}\cdot\sec\frac{C}{2}\right)\leq\frac{2s\pi}{27r}$$

Proposed by Radu Diaconu – Romania

S.320 In ΔABC the following relationship holds:

$$\left(ar_a\mu(A)\right)^n+\left(br_b\mu(B)\right)^n+\left(cr_c\mu(C)\right)^n\geq\frac{(2\pi F)^n}{3^{n-1}},n\in\mathbb{N},n\geq 2$$

Proposed by Radu Diaconu – Romania

S.321 Let ΔDEF be the orthic triangle of acute $\Delta ABC, H-$ orthocenter. Prove that:

$$\sum\frac{AH\cdot r_a}{EF}\leq\frac{R}{2r}\sum\frac{AH\cdot h_a}{EF}$$

Proposed by Marin Chirciu – Romania

S.322 In ΔABC

$$\left(\frac{2r}{R}\right)^{\frac{3}{2}}\leq\frac{3}{\sqrt{5+\sum\sec^2\frac{A}{2}}}\leq 1$$

Proposed by Marin Chirciu – Romania

S.323 If $x,y,z>0$ such that $x+y+z=xy+yz+zx$ and $\lambda\geq 2$ then:

$$\frac{1}{x+\lambda}+\frac{1}{y+\lambda}+\frac{1}{z+\lambda}\leq\frac{3}{\lambda+1}$$

Proposed by Marin Chirciu – Romania

S.324 In ΔABC

$$\sum\frac{\cos B+\cos C}{a}\leq\frac{p}{9r}\sum\frac{\sin B+\sin C}{a}$$

Proposed by Marin Chirciu – Romania

S.325 In ΔABC

$$\frac{9}{4p}\left(\frac{2r}{R}\right)\leq\sum\frac{\cos B+\cos C}{b+c}\leq\frac{9}{4p}$$

Proposed by Marin Chirciu – Romania

S.326 If $a,b,c>0$ such that $a^2+b^2+c^2=3$ and $\lambda\leq 4$ then:

$$\frac{a^4+b^4+c^4}{ab+bc+ca}+\frac{\lambda abc}{a+b+c}\geq 1+\frac{\lambda}{3}$$

Proposed by Marin Chirciu – Romania

S.327 If $x,y>0$ such that $x+y=2$ and $n\in\mathbb{N}$ find the maximum value of

$$P=(xy)^n(n+1-nxy)$$

Proposed by Marin Chirciu – Romania

S.328 In acute ΔABC

$$\sqrt{2(1+\cos A\cos B\cos C)} \geq 12\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

Proposed by Marin Chirciu – Romania

S.329 $0 < a \leq b, f, f': [a,b] \to (0,\infty), f$ – nonconstant, derivable. Prove that:

$$6\int_a^b \frac{f'(x)}{\sqrt{1+f^2(x)}}dx + f^3(b) - f^3(a) \geq 6(b-a)$$

Proposed by Daniel Sitaru – Romania

S.330 If $a,b,c > 0, a^2+b^2+c^2 = 3$ then:

$$\sqrt[3]{\left(1+\frac{3}{ab+bc+ca}\right)^{(a+b+c)^2}} \leq \left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right)$$

Proposed by Daniel Sitaru – Romania

S.331 If $0 < a \leq b$ then:

$$\left(\int_a^b \sqrt{1+\cos^2 x}\,dx - \sin b + \sin a\right)\left(\int_a^b \sqrt{1+\cos^2 x}\,dx + \sin b - \sin a\right) \geq (b-a)^2$$

Proposed by Daniel Sitaru – Romania

S.332 If $a,b,c > 0, abc = 8$ then:

$$\log(1+a)^{\log(1+b)^{\log(1+c)}} \leq \log^3 3$$

Proposed by Daniel Sitaru – Romania

S.333 If $0 < a \leq b$ then:

$$\int_a^b \arctan x \cdot \arctan\left(\frac{1}{x}\right)dx \leq \frac{\pi}{4}\ln\left(\frac{1+b^2}{1+a^2}\right)$$

Proposed by Daniel Sitaru – Romania

S.334 If $0 < a \leq b < 2$ then:

$$\int_a^b \frac{\sqrt{x}+\sqrt{2-x}}{\sec x + \sin x \cdot \tan x}dx \leq 2\arctan\left(\frac{\sin b - \sin a}{1+\sin a \sin b}\right)$$

Proposed by Daniel Sitaru – Romania

S.335 Solve for real numbers:

$$\begin{cases} x,y,z > 0 \\ \tan(1024x)+\tan(1024y)+\tan(1024z) = 0 \\ x+y+z = \pi \end{cases}$$

Proposed by Daniel Sitaru – Romania

S.336 Find without any software:

$$\Omega = \int\left(\frac{\sinh^2 x}{\sinh(2x)-2x} + \frac{x\cdot\sinh x}{\sinh x - x\cdot\cosh x}\right)dx$$

Proposed by Daniel Sitaru – Romania

S.337 Find without any software:

$$\Omega = \int \left(x+\frac{5}{x}\right)\left(1-\frac{5}{x^2}\right)\sin\left(\ln\left(x+\frac{5}{x}\right)\right)dx$$

Proposed by Daniel Sitaru – Romania

S.338 If $0 < a \le b < \pi$ then:

$$48\left|\cos b - 2\cos\frac{a+b}{2} + \cos a\right| \le (b-a)^3$$

Proposed by Daniel Sitaru – Romania

S.339 Find:

$$\Omega = \lim_{n\to\infty} \sum_{1\le i<j\le n} \frac{\cos\left(\frac{j-i}{n}\right) - \cos\left(\frac{j+i}{n}\right)}{\sqrt{i^2+j^2+n^4}}$$

Proposed by Daniel Sitaru – Romania

S.340 Find:

$$\Omega = \lim_{n\to\infty}\left((n-1)!\left(\sqrt{2+\sqrt[3]{3+\cdots+\sqrt[n+1]{n-1}}} - \sqrt{2+\sqrt[3]{3+\cdots+\sqrt[n]{n}}}\right)\right)$$

Proposed by Daniel Sitaru – Romania

S.341 In ΔABC the following relationship holds:

$$R^n \ge (2r)^n + \frac{1}{8R}\left(R^{n-1} + R^{n-1}\cdot 2r + \cdots + (2r)^n\right)\sum_{cyc}(b-c)^2, n\in\mathbb{N}, n\ge 2$$

Proposed by Marin Chirciu-Romania

S.342 If $a,b,c>0$ and $\lambda \ge \sqrt{3}-1$ then

$$\frac{a^2}{(a+\lambda b)(a+\lambda c)} + \frac{b^2}{(b+\lambda a)(b+\lambda c)} + \frac{c^2}{(c+\lambda a)(c+\lambda b)} \ge \frac{3}{(\lambda+1)^2}$$

Proposed by Marin Chirciu-Romania

S.343 If $a,b,c>0, a+b+c=3$ then

$$(a^2+b^2+c^2)(ab+bc+ca)^2 \le 27$$

Proposed by Marin Chirciu-Romania

S.344 In $\Delta ABC, R_a, R_b, R_c$ –circumradii of $\Delta BIC, \Delta CIA, \Delta AIB, I$ –incenter.

$$\sum_{cyc}\frac{s_b+s_c}{s_a}R_a^2 \ge 2r(5R+2r)$$

Proposed by Marin Chirciu-Romania

S.345 If $x_1, x_2, x_3, \dots, x_{2021}$ are positive real numbers such that:

$$\frac{1}{x_1+1}+\frac{1}{x_2+1}+\cdots+\frac{1}{x_{2021}+1}=\frac{1}{2020}$$

Prove that: $x_1 x_2 \dots x_{2021} \le \frac{1}{2020^{2021}}$

Proposed by Nguyen Van Canh – BenTre – Vietnam

S.346 Find all positive real numbers α such that:

$$\tan(2020\alpha x) \ge x, \forall x \in \left[0, \frac{\pi}{2}\right)$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

S.347 In any triangle ABC the following relationship hold:

$$3 \le \frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c} \le \frac{\sqrt{p^2+2Rr+5r^2}}{2r}$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

S.348 If a, b, c are positive real numbers such that $a+b+c \le 3$ then:

$$a^3\left(\frac{b+c}{2}\right)^4+b^3\left(\frac{a+c}{2}\right)^4+c^3\left(\frac{a+b}{2}\right)^4 \le \frac{1}{3}(a^2+b^2+c^2)^2$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

S.349 Find all positive real numbers α such that:

$$e^x-\alpha x \ge 0, \forall x \ge 0$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

S.350 In any triangle ABC holds:

$$\sqrt{bcm_a}+\sqrt{acm_b}+\sqrt{abm_c} \le 9R\sqrt{\frac{R}{2}}$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

S.351 In any triangle ABC holds:

$$3 \le \sqrt{\frac{m_a}{w_a}}+\sqrt{\frac{m_b}{w_b}}+\sqrt{\frac{m_c}{w_c}} \le \sqrt{\frac{4R}{r}+1}$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

S.352 Find all positive real numbers α, β such that:

$$\max\{3x^2-(\alpha x+\beta)\} \le \frac{1}{3}, \forall x \in [0;1]$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

S.353 In any triangle ABC (acute) the following relationship holds:

$$\sum \frac{a}{b^2+c^2-a^2} + \sqrt[3]{\frac{8abc}{(a+b-c)(b+c-a)(c+a-b)}} \ge \frac{p+4Rr+4r^2}{2(Rr+r^2)}$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

S.354 In any triangle ABC the following relationship holds:

$$1.\ \left(\frac{\sqrt{h_a h_b}+\sqrt{h_b h_c}+\sqrt{h_c h_a}}{r_a+r_b+r_c}\right)\left(\frac{\sqrt{w_a w_b}+\sqrt{w_b w_c}+\sqrt{w_c w_a}}{4R+r}\right) \le \frac{R}{2r}$$

$$2.\ \min\left\{\sqrt{3}a\sin^3\frac{A}{2}; \sqrt{3}b\sin^4\frac{B}{2}; \sqrt{3}c\sin^5\frac{C}{2}\right\} < 27R$$

$$3.\ \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{1}{2}\cdot\sqrt[2020]{\frac{ab+bc+ca}{a^2+b^2+c^2}} \ge 2$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

S.355 $f: \mathbb{R} \to (0,\infty), f$ –derivable, $f'(x) = \frac{sin(\pi x)}{1+\frac{1}{x^2}}, f\left(\frac{1}{3}\right) = 0$. Prove that:

$$\frac{\sqrt{3}}{1440} < \int_{\frac{1}{3}}^{\frac{1}{2}} f(x)dx < \frac{1}{360}$$

Proposed by Rajeev Rastogi-India

S.356 Prove without softs:

$$\frac{\sqrt{5}}{2e^4}\left(1+\frac{1}{\sqrt{2}e^5}\right) < \int_{-2}^{3} \frac{1}{e^{x^2}\sqrt{1+x^2}}dx < 5$$

Proposed by Rajeev Rastogi-India

S.357 $f, g: \left(0, \frac{\pi}{2}\right) \to (0,\infty), f, g$ –continuous,

$$\int_0^{\frac{\pi}{2}} \left(\frac{4sinx}{f(x)} - 1\right) f^2(x)dx + \int_0^{\frac{\pi}{2}} \left(\frac{4cosx}{g(x)} - 1\right) g^2(x)dx = 2\pi$$

Find: $\Omega = \lim_{x\to 0}\left[\frac{f^2(x)}{xg^2(x)}\right], [*] - GIF$

Proposed by Rajeev Rastogi-India

S.358 $f: \mathbb{R} \to \mathbb{R}, f(x) = \begin{cases} \frac{1}{log2019}(2019^{x^{2020}-1} + (x-1)^{2019}cos\frac{1}{x-1}, x \neq 1 \\ 1, x = 1 \end{cases}$. Find:

$$\Omega = \lim_{x\to 0}\left(\sum_{k=1}^{10} \frac{f(kx+1)-1}{x}\right)$$

Proposed by Rajeev Rastogi-India

S.359 Exists $f: \mathbb{R} \to \mathbb{R}, f$ –bijective such that $f(x) + f(-x) = 2020, \forall x \in \mathbb{R}$? Find in that case:

$$\Omega = \int_{1-x}^{2019+x} f^{-1}(t)dt$$

Proposed by Rajeev Rastogi-India

S.360 Find without softs:

$$\Omega = \left(\sum_{k=1}^{4} tan^2 \frac{k\pi}{5}\right)\left(\sum_{k=1}^{3} cos \frac{2k\pi}{7}\right)^{-1} + \prod_{k=1}^{4} tan \frac{k\pi}{5}\left(\prod_{k=1}^{3} cos \frac{2k\pi}{7}\right)^{-1}$$

Proposed by Rajeev Rastogi-India

S.361 Find:

$$\Omega = \underbrace{min}_{x\in\mathbb{R}}\left(\underbrace{max}_{y\in\mathbb{R}}\left(\left|x + siny + \frac{2}{3 + siny}\right|\right)\right)$$

Proposed by Rajeev Rastogi-India

S.362 If $n \in \mathbb{N}, n \geq 2$ then:

$$\sqrt[n]{\frac{n^2 + n + \sqrt[n]{n}}{n}} + \sqrt[n]{\frac{n^2 + n - \sqrt[n]{n}}{n}} < 4$$

Proposed by Rajeev Rastogi-India

S.363 In ΔABC the following relationship holds:

$$\left(\frac{am_a + bm_b + cm_c}{a + b + c}\right)\left(\frac{m_a + m_b + m_c}{m_a^2 + m_b^2 + m_c^2}\right) \leq 2\sqrt{3}$$

Proposed by Rajeev Rastogi-India

S.364 $\begin{cases} [x](x^2 + 2020) = x([x]^2 + 2020), x \in \mathbb{R} - \mathbb{Z} \\ 3[y^3] + 3[y^2] + 2[y] = y - [y] - 2, y \in \mathbb{R} \end{cases}$, $[*] - GIF$. Find:

$$\Omega = |[x] + [y]|$$

Proposed by Rajeev Rastogi-India

S.365 Find all positive $n \in \mathbb{Z}$ such that: $\sqrt{\frac{4n-1}{n+5}} \in \mathbb{Q}$. ***Proposed by Rajeev Rastogi-India***

S.366 In $\Delta ABC, g_a$ – Gergonne's cevian, the following relationship holds:

$$\frac{2r}{R} + \frac{8r_a r_b r_c}{g_a g_b g_c} \geq 9$$

Proposed by Adil Abdullayev-Azerbaijan

S.367 In $\Delta ABC, n_a$ – Nagel's cevian the following relationship holds:

$$n_a n_b n_c(a^2 + b^2 + c^2) \geq 9R^2 h_a h_b h_c$$

Proposed by Adil Abdullayev-Azerbaijan

S.368 In ΔABC the following relationship holds:

$$\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}\geq\sqrt{10-\frac{w_aw_bw_c}{r_ar_br_c}}$$

Proposed by Adil Abdullayev-Azerbaijan

S.369 In ΔABC the following relationship holds:

$$9+\frac{8r_ar_br_c}{w_aw_bw_c}\geq 17\cdot\sqrt[3]{\frac{2s^2}{27Rr}}$$

Proposed by Adil Abdullayev-Azerbaijan

S.370 Prove that in any triangle:

$$\sqrt{27r^2+k}\leq p\leq\sqrt{\frac{27R^2}{4}-k}$$

$$k=\frac{r^2(R-2r)}{R-r}$$

Proposed by Adil Abdullayev-Azerbaijan

S.371 In acute $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$\frac{n_a\cos A}{s_a}+\frac{n_b\cos B}{s_b}+\frac{n_c\cos C}{s_c}\geq\frac{3}{2}$$

Proposed by Adil Abdullayev-Azerbaijan

S.372 In ΔABC the following relationship holds:

$$\left(\frac{R}{2r}\right)^2+\left(\frac{w_aw_bw_c}{r_ar_br_c}\right)^2\geq 2$$

Proposed by Adil Abdullayev-Azerbaijan

S.373 Find:

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\ln(n+k)-\ln n}{n+k}$$

Proposed by Adil Abdullayev-Azerbaijan

S.374 In ΔABC the following relationship holds:

$$\frac{n_a^2}{m_a^2}\geq 1+\frac{3a(a+2b+2c)(b-c)^2}{(a^2+b^2+c^2)^2}$$

Proposed by Adil Abdullayev-Azerbaijan

S.375 In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$\frac{a}{n_a}+\frac{b}{n_b}+\frac{c}{n_c}\ge\frac{\sqrt{2r}}{s}\cdot\sum_{cyc}\sqrt{r_a-r}$$

Proposed by Bogdan Fuștei – Romania

S.376 In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$\left(\sum_{cyc}\frac{a}{AI}\right)^2\ge\sqrt[3]{\frac{n_a^2n_b^2n_c^2}{r_a^2r_b^2r_c^2}}+2\sqrt[3]{\frac{2r}{R}}$$

Proposed by Bogdan Fuștei – Romania

S.377 In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$2\left(\frac{R}{r}-1\right)\ge\sqrt[3]{\frac{n_a^2n_b^2n_c^2}{r_ar_br_ch_ah_bh_c}}+\sqrt[3]{\frac{R}{2r}}$$

Proposed by Bogdan Fuștei – Romania

S.378 In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$\sum_{cyc}\frac{a}{\sqrt{h_a-2r}}\ge\sqrt{R}\cdot\sum_{cyc}\frac{n_a}{h_a}+\frac{r_a+r_b+r_c}{m_a+m_b+m_c}\sum_{cyc}\sqrt{2(r_a-r)}$$

Proposed by Bogdan Fuștei – Romania

S.379 In $\Delta ABC, n_a$ – Nagel's cevian, g_a – Gergonne's cevian, the following relationship holds:

$$3\ge\sum_{cyc}\sqrt{\frac{2r}{r_b+r_c}}+\sum_{cyc}\frac{m_a+\sqrt{g_a}\left(\sqrt{n_a}-\sqrt{g_a}\right)}{s\sqrt{2}}$$

Proposed by Bogdan Fuștei – Romania

S.380 In $\Delta ABC, T$ – Toricelli point, the following relationship holds:

$$(m_a^2+m_b^2+m_c^2)\left(\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right)\ge\frac{9}{2}(AT+BT+CT)$$

Proposed by Bogdan Fuștei – Romania

S.381 In $\Delta ABC, T$ – Toricelli point, the following relationship holds:

$$\sum_{cyc}\frac{a}{m_a+w_b+w_c}\ge\frac{AT+BT+CT}{s}$$

Proposed by Bogdan Fuștei – Romania

S.382 In $\Delta ABC, T$ – Toricelli point, the following relationship holds:

$$\sum_{cyc}\frac{b^2+c^2}{TB+TC}\ge\sqrt{3}\left[(m_a+m_b+m_c)\sqrt{3}-s\right]$$

Proposed by Bogdan Fuștei – Romania

S.383 In acute $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$(a+b+c)\left(\frac{1}{s_a}+\frac{1}{s_b}+\frac{1}{s_c}\right) \geq \sqrt{2}\left(\sum_{cyc}\frac{n_a}{w_a}+2\left(\sqrt{\frac{R}{r}}+\sqrt{\frac{r}{R}}\right)\right)$$

Proposed by Bogdan Fuștei – Romania

S.384 Prove that:

$$\sum_{cyc} yz\sqrt{y^2+yz+z^2} \geq \prod_{cyc}\sqrt{y^2+yz+z^2}\,; x,y,z>0$$

Proposed by Bogdan Fuștei – Romania

S.385 In $\Delta ABC, n_a$ – Nagel's cevian, g_a – Gergonne's cevian, the following relationship holds:

$$\prod_{cyc}\frac{n_a^2+g_a^2}{h_b^2+h_c^2} \geq \frac{1}{48}\sum_{cyc}\left(\frac{r_b+r_c}{m_a}\right)^4$$

Proposed by Bogdan Fuștei – Romania

S.386 In $\Delta ABC, T$ – Toricelli point, the following relationship holds:

$$3\sqrt{2}s \geq \sqrt{AT+BT+CT}\cdot\sum_{cyc}\sqrt{m_a+w_b+w_c}$$

Proposed by Bogdan Fuștei – Romania

S.387 In $\Delta ABC, n_a$ – Nagel's cevian, g_a – Gergonne's cevian, the following relationship holds:

$$2(m_a+m_b+m_c)-s \geq \sum_{cyc}\frac{n_a^2+h_a(r_b+r_c+2r)}{n_a+g_a+s-a}$$

Proposed by Bogdan Fuștei – Romania

S.388 If $x,y,z>0$ such that $xyz=1$ and $\lambda>0$ then:

$$(x+y+z)\left(\frac{x}{x+\lambda}+\frac{y}{y+\lambda}+\frac{z}{z+\lambda}\right) \geq \frac{9\lambda}{\lambda^3+1}$$

Proposed by Marin Chirciu – Romania

S.389 If $a,b,c,d>0$ such that $a+b+c+d\leq 4$ and $n\in\mathbb{N}^*$ then:

$$\frac{1}{a^n}+\frac{1}{b^n}+\frac{1}{c^n}+\frac{1}{d^n} \geq 4$$

Proposed by Marin Chirciu – Romania

S.390 In ΔABC:

$$\sum\frac{(b^{n+1}+c^{n+1})^2}{b^{2n}+c^{2n}} \leq 18R^2, n\in\mathbb{N}$$

Proposed by Marin Chirciu – Romania

S.391 If $x, y, z > 0$ such that $\frac{1}{x^n} + \frac{1}{y^n} + \frac{1}{z^n} = 3$ and $\lambda \geq 0, \mu \geq 0, n, k \in \mathbb{R}$ then:

$$\lambda \sum x^n + \mu \left(\sum x^k + \sum \frac{1}{x^k} \right) \geq 3(\lambda + 2\mu)$$

Proposed by Marin Chirciu – Romania

S.392 If $a, b, c > 0$ and $a + b = c = 3$ and $\lambda \geq 0$ then:

$$\frac{a^6}{a^2 + \lambda b} + \frac{b^6}{b^2 + \lambda c} + \frac{c^6}{c^2 + \lambda a} \geq \frac{3}{\lambda + 1}$$

Proposed by Marin Chirciu – Romania

S.393 If $a, b, c > 0$ such that $a + b + c = 2$ and $\lambda \leq 2$ then:

$$\sum \frac{b^2 + \lambda bc + c^2}{b + c} \geq \lambda + 2$$

Proposed by Marin Chirciu – Romania

S.394 If $a, b, c > 0$ such that $abc = 1$ and $0 \leq \lambda \leq 2$ then:

$$\frac{a^2}{b^2 + \lambda c} + \frac{b^2}{c^2 + \lambda a} + \frac{c^2}{a^2 + \lambda b} \geq \frac{3}{\lambda + 1}$$

Proposed by Marin Chirciu – Romania

S.395 In ΔABC the following relationship holds:

$$\lambda + \frac{1}{3} \sum r_a \geq \sqrt[3]{(\lambda + m_a)(\lambda + m_b)(\lambda + m_c)} \geq \lambda + \sqrt[3]{r_a r_b r_c}, \lambda \geq 0$$

Proposed by Alex Szoros – Romania

S.396 Prove that the following inequality is true in every triangle:

$$\min\left\{\frac{R}{r}, \frac{r_a}{r_b} + \frac{r_b}{r_a}\right\} \geq \frac{a}{b} + \frac{b}{a}$$

Proposed by Alex Szoros – Romania

S.397 Let $x, y, z > 0$. Prove that:

$$\frac{(x+y)(y+z)(z+x)}{8xyz} \geq \frac{1}{2} + \frac{1}{3}\left(\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}\right)$$

Proposed by Alex Szoros – Romania

S.398 Find the greatest real number n such that the inequality

$$a^5 + b^5 + c^5 + nabc \leq 1 + nabc(ab + bc + ca)$$

holds for any positive numbers a, b, c with $a + b + c = 1$.

Proposed by Alex Szoros – Romania

S.399 In ΔABC the following relationship holds:

$$\frac{R}{r} + 1 \geq \sum \left(\frac{b}{c} + \frac{c}{b}\right) \cos A \geq \frac{6r}{R}$$

Proposed by Alex Szoros – Romania

S.400 In ΔABC the following relationship holds:

$$\left(\frac{R}{r}\right)^{\lambda} \geq 1+\frac{\lambda(h_a^2+h_b^2+h_c^2)}{h_ah_b+h_bh_c+h_ch_a}, \lambda \geq 1$$

Proposed by Alex Szoros – Romania

S.401 In acute triangle ABC the following relationship holds

$$\left(\frac{R}{r}\right)^3-6\left(\frac{R}{r}\right)^2+12\frac{R}{r} \geq 8\sum \frac{a}{h_a(\tan B+\tan C)}$$

Proposed by Alex Szoros – Romania

S.402 Evaluate:

$$\int_0^4 \left(\sqrt[3]{\frac{1}{2}\left(x+\sqrt{x^2+4}\right)}+\sqrt[3]{\frac{1}{2}\left(x-\sqrt{x^2+4}\right)}\right)dx$$

Proposed by Alex Szoros – Romania

S.403 In ΔABC the following relationship holds: $\frac{R}{r} \geq \frac{am_a}{S} \geq 2+\frac{(b-c)^2}{2S} \geq \frac{b}{c}+\frac{c}{b}$

Proposed by Alex Szoros – Romania

S.404 In ΔABC the following relationship holds:

$$3(a^2+b^2+c^2) \geq \sum \frac{(m_a+l_a)^6}{3m_a^4+10m_a^2l_a^2+3l_a^4} \geq (a+b+c)^2$$

Proposed by Alex Szoros – Romania

S.405 In ΔABC the following relationship holds:

$$\left(\frac{R^2}{r^2}+\frac{5R}{2r}\right)\left(\frac{b+c}{a}\right)\sin\frac{A}{2} \geq (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Proposed by Alex Szoros – Romania

S.406 $f: \mathbb{R}^+ \to \mathbb{R}$, differentiable, $f(1)=0, f'(1)=1$,

$$f(xy)=\frac{f(x)}{y}+\frac{f(y)}{x}, \forall x,y>0, a_1=1, a_2=\frac{1}{4}, a_{n+1}=\frac{(n-1)a_n}{n-a_n}, n \geq 2$$

Find:

$$\Omega=\left[\sum_{n=1}^{100}\left(\frac{9a_na_{n+1}}{(2a_n+1)(2a_{n+1}+1)}\right)+\lim_{x\to\infty} e^{f(x)}\right], [*]-GIF$$

Proposed by Rajeev Rastogi – India

S.407 Prove without softs:

$$\int_0^{\frac{\pi}{2}} e^{-k\sin x}\,dx < \frac{\pi}{2k}(1-e^{-k}), k>0$$

Proposed by Rajeev Rastogi – India

S.408 Let $b_1, b_2, \dots$ be a sequence of real numbers such that for each $n \geq 1$

$$b_{n+1}^2 \geq \frac{b_1^2}{1^3} + \frac{b_2^2}{2^3} + \cdots + \frac{b_n^2}{n^3}$$

If N be the least positive integer satisfying the inequality

$$\sum_{n=1}^{N} \frac{b_{n+1}}{b_1 + b_2 + \cdots + b_n} \geq \frac{2021}{1015}$$

then find the value of $(N - 200)$

Proposed by Rajeev Rastogi – India

S.409

$$f = x^8 - 9x^7 + 31x^7 + a_0x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5 \in \mathbb{R}[X]$$

If the equation $f(x) = 0$ has all roots $x_i \in \mathbb{R}$ then $x_i \in [2,4], i \in \overline{1,8}$

Proposed by Rajeev Rastogi – India

S.410 Find:

$$\Omega = \lim_{n\to\infty} \left(\sqrt[n]{\int_0^1 (1 - x^2)^n dx} + \sqrt[n]{\int_0^1 (1 - x^3)^n \, dx} \right)$$

Proposed by Rajeev Rastogi – India

S.411 Find the product of all non-real roots of the equation:

$$54x^4 - 36x^3 + 18x^2 - 6x + 1 = 0$$

Proposed by Rajeev Rastogi – India

S.412 Find without softs:

$$\int \frac{2x\cos^2 x - (x-1)^2 e^x \cos^2 x - (1 + \sin x)\sqrt{1 + x^2}(1 + e^x)^2}{\sqrt{1 + x^2}(1 + e^x)\cos^2 x} dx$$

Proposed by Rajeev Rastogi – India

S.413 Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that:

$$5f(f(x) + y) = 2f(x + y) + 3(f(x) + y), \forall x, y \in \mathbb{R}$$

Proposed by Rajeev Rastogi – India

S.414 Find without softs:

$$\Omega = \int \frac{2\sin x \, e^{x+\sin x} + 2(1 + \sin 2x)\cos 2x - (\sin x + \cos x)^4 \cos x}{(1 + \sin 2x)e^{\sin x}} dx$$

Proposed by Rajeev Rastogi – India

S.415 $f:[0,1]\to[0,\infty), f$ – twice derivable, $f'(x)>0, f''(x)>0, \forall x\in[0,1]$

Prove that:

$$\int_0^1 f^3(x)\,dx+\frac{4}{27}\ge\left(\int_0^1 f(x)\,dx\right)^2$$

Proposed by Rajeev Rastogi – India

S.416

$$\Omega=\lim_{n\to\infty}\left(\int_0^1 x^n\sqrt{1-x^2}\,dx\right)\left(\int_0^1 x^{n-2}\sqrt{1-x^2}\,dx\right)^{-1}$$

Proposed by Rajeev Rastogi – India

S.417 Find:

$$\Omega=\lim_{x\to\infty}\left(\frac{1}{x^2}\lim_{n\to\infty}\left(\int_0^{n[x]}(nx-[nx])^n dx+\int_0^{n[x+1]}(nx-[nx])^n dx\right)\right),[*]-GIF$$

Proposed by Rajeev Rastogi – India

S.418 Find without softs:

$$\omega_1=\int_0^2\left(\sqrt[3]{x+\sqrt{1+x^2}}+\sqrt[3]{x-\sqrt{1+x^2}}\right)dx,\omega_2=\int_1^{e^2}\left(\frac{1}{2}+\sqrt{\frac{1}{4}+\ln x}\right)dx$$

$$\omega_3=\int_1^2\frac{e^{x^2-[x]}}{e^{x-[x]}}dx,\Omega=\omega_1+\omega_2+\omega_3$$

Proposed by Rajeev Rastogi – India

S.419 Find the sum of all possible values of $\alpha\in\mathbb{Z}$ for which the equation

$$3x^3-(3+3\alpha)x^2-(35\alpha-3)x+3-32\alpha=0$$

has three positive integral roots.

Proposed by Rajeev Rastogi – India

S.420 In ΔABC the following relationship holds:

$$\sqrt[3]{\frac{(b+c-a)(c+a-b)(a+b-c)}{a^2b^2c^2(a^2+b^2)(b^2+c^2)(c^2+a^2)}}\ge\frac{1}{4sR^2}$$

Proposed by Rajeev Rastogi – India

S.421 In acute $\Delta ABC, D, E, F$ – midpoints of $(BC),(CA),(AB), X\in(BC)$,

$Y\in(CA), Z\in(CA), XE\perp AC, YD\perp BC, ZF\perp AB$. Prove that:

$$\frac{YD}{h_a}+\frac{XE}{h_b}+\frac{ZF}{h_c}\ge 12\left(\frac{r}{R}\right)^2$$

Proposed by Mehmet Șahin –Turkey

S.422 In acute $\Delta ABC, G$ – centroid, $GD \perp BC, GE \perp CA, GF \perp AB$

$D \in (BC), E \in (CA), F \in (AB)$. Prove that:

$$\left(\frac{BC}{EF}\right)^2 + \left(\frac{CA}{FD}\right)^2 + \left(\frac{AB}{DE}\right)^2 \leq \frac{3}{2}\left(\frac{R}{r}\right)^3$$

Proposed by Mehmet Şahin –Turkey

S.423 In $\Delta ABC, g_a$ – Gergonne's cevian, the following relationship holds:

$$\sum_{cyc} \frac{h_a}{g_a + s - a} \leq \frac{1}{2}\left(\frac{g_a + g_b + g_c}{r} - \left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)\sqrt{4 - \frac{2r}{R}}\right)$$

Proposed by Bogdan Fuștei – Romania

S.424 In acute ΔABC the following relationship holds:

$$\frac{m_a h_a}{w_a} \leq \frac{n_a^2 + g_a^2 + 2rr_a}{4R} \leq m_a$$

Proposed by Bogdan Fuștei – Romania

S.425 In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$\sqrt{4 - \frac{2r}{R}} \cdot \sum_{cyc}\left(\frac{n_a}{h_a} + \frac{2r_a}{s - n_a}\right) \leq 1 + \frac{4R}{r}$$

Proposed by Bogdan Fuștei – Romania

S.426 In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{n_a g_a (n_a + g_a)}{bc(b + c)} \geq \sum_{cyc} \cos^3 \frac{A}{2}$$

Proposed by Bogdan Fuștei – Romania

S.427 In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$\sqrt{4 - \frac{2r}{R}} \cdot \sum_{cyc}\left(\frac{n_a}{r_a} + \frac{2h_a}{s - n_a}\right) \leq 1 + \frac{4R}{r}$$

Proposed by Bogdan Fuștei – Romania

S.428 In $\Delta ABC, n\_a$ – Nagel's cevian, g_a – Gergonne's cevian, holds:

$$\sum_{cyc}\left(\frac{n_a}{r_a} + \frac{2h_a}{2m_a + s - g_a}\right) \geq \sum_{cyc}\sqrt{\frac{r_b + r_c}{r_a - r}}$$

Proposed by Bogdan Fuștei – Romania

S.429 In $\Delta ABC, n_a$ – Nagel's cevian, g_a – Gergonne's cevian, the following relationship holds:

$$\sum_{cyc} \frac{r_b + r_c}{a} \geq \sum_{cyc}\left(\frac{2m_a - g_a}{h_a} + \frac{2r_a}{s + n_a}\right)$$

Proposed by Bogdan Fuștei – Romania

S.430 In $\Delta ABC, n_a$ – Nagel's cevian, I – incenter, the following relationship holds:

$$\frac{n_a}{h_a} \geq \frac{1}{\sqrt{2}}\left(\frac{m_a}{h_a} + \frac{|b-c|}{2r}\sqrt{1-\frac{AI^2}{w_a^2}}\right)$$

Proposed by Bogdan Fuștei – Romania

S.431 In ΔABC the following relationship holds:

$$\frac{R}{2r} \geq \sqrt{1+\frac{3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2(b^2-c^2)^2}{(a+b+c)^4}}$$

Proposed by Bogdan Fuștei – Romania

S.432 If $0 < a \leq b$ then:

$$(b-a)\cdot\int_a^b\left(\sqrt{e}\right)^{x^2}\cdot\tanh x\,dx \geq (\cosh b - \cosh a)\cdot\int_a^b\left(\sqrt{e}\right)^{-x^2}\cdot\cosh x\,dx$$

Proposed by Daniel Sitaru – Romania

S.433 If $a, b \in \mathbb{R}$ then:

$$2(\sin b - \sin a)^2 \leq 2(\cos b - \cos a)(\cos b - \cos a + 2b - 2a) + 3(b-a)^2$$

Proposed by Daniel Sitaru – Romania

S.434 $n \in \mathbb{N}, A_n(F_n, F_{n+1}), B_n(F_{n+2}, F_{n+3}), C_n(F_{n+4}, F_{n+5}), F_n$ – Fibonacci numbers.

Find:

$$\Omega = \lim_{n\to\infty}\left(area[A_nB_nC_n] + \frac{n}{2n+1}\right)^n$$

Proposed by Daniel Sitaru – Romania

S.435 If $A_1A_2 \ldots A_8$ – regular octagon then:

$$(A_1A_5 + A_3A_5)(A_1A_7 + A_3A_7) = \left(2+\sqrt{2}\right)\cdot A_2A_5\cdot A_2A_7$$

Proposed by Daniel Sitaru – Romania

S.436 a, b, c, d – sides, e, f – diagonals, $\mu(B) = \frac{\pi}{3} + \mu(D), \mu(C) = \frac{\pi}{3} + \mu(A)$ in a convexe quadrilateral (not parallelogram). Prove that:

$$a^2b^2 + c^2d^2 \geq \left(1+\frac{e^2}{f^2}\right)abcd$$

Proposed by Daniel Sitaru – Romania

S.437 Find:

$$\Omega=\lim_{n\to\infty}\left(\sqrt[3]{n+1}\cdot\sum_{k=1}^{n+1}\tan^{-1}\left(\frac{1}{k^2+k+1}\right)-\sqrt[3]{n}\cdot\sum_{k=1}^{n}\tan^{-1}\left(\frac{1}{k^2+k+1}\right)\right)$$

Proposed by Daniel Sitaru – Romania

S.438 Find:

$$\Omega=\lim_{n\to\infty}\sum_{k=3}^{n}\frac{k}{k!+(k-1)!+(k-2)!}$$

Proposed by Daniel Sitaru – Romania

S.439 Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=2}^{n}\frac{k^2}{\sqrt[k]{(k!)^2}}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

S.440 Find:

$$\Omega=\lim_{n\to\infty}\left(\left(\frac{\pi^2}{6}-\sum_{k=1}^{n}\frac{1}{k^2}\right)e^{H_n}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

S.441 If $x,y,z\in(0,e)$ such that $x+y+z=1$ then prove that:

$$(xyz)^3\geq x^{\frac{1}{x}}y^{\frac{1}{y}}z^{\frac{1}{z}}$$

Proposed by Marius Drăgan, Neculai Stanciu – Romania

S.442 If $x,y,z>0, x+y+z=1$ then:

$$(x+1)^3(y+1)^3(z+1)^3\leq(x+1)^{\frac{1}{x}}(y+1)^{\frac{1}{y}}(z+1)^{\frac{1}{z}}$$

Proposed by Marius Drăgan, Neculai Stanciu – Romania

S.443 If $a,b\geq 0, a+b=1$ then:

$$(2^{4b}-1)a^{4b}+(2^{4a}-1)b^{4a}\geq 2a^{2b}b^{2a}$$

Proposed by Seyran Ibrahimov-Azerbaijan

S.444 Find $a\in\mathbb{R}$ such that:

$$\lim_{n\to\infty}\frac{1^a+2^a+3^a+\cdots+n^a}{(n+1)^{a-1}\big((na+1)+(na+2)+\cdots+(na+n)\big)}=\frac{1}{60}$$

Proposed by Mohammad Hamed Nasery-Afghanistan

S.445 Find:

$$\Omega = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{\sin^{-1}\left(\frac{28e^{1+\log k}}{k+n^k}\right)}{\log\left(1+\frac{n}{(k+n^k)\sqrt[n]{n!}}\right)}\right)^{n^2}$$

Proposed by Ruxandra Daniela Tonilă-Romania

S.446 Prove or disprove the equation $9X^2 - 10XY + 9Y^2 = Z\uparrow^{2020}$ has infinitely many integer solutions X, Y, Z such that $\gcd(X, Y, Z) = 1$, where, $Z\uparrow^2 = Z^Z, Z\uparrow^3 = Z^{Z^Z}$,

$Z\uparrow^4 = Z^{Z^{Z^Z}}$, defined analogously for $Z\uparrow^{2020}$

Proposed by Safal Das Biswas-India

S.447 Solve for real numbers: $x^{2x} = 2x$.

Proposed by Ghulam Shah Naseri-Afghanistan

S.448 $\forall\Delta ABC | p_a, p_b, p_c \to$ Spieker cevians,

$$\sum_{cyc} p_a^2 (2s+a)^2 + 8r(2R-r)s^2 \ge \left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2 b + 2\sum_{cyc} ab^2\right)$$

Proposed by Soumava Chakraborty-India

S.449 Given two sequences of positive real numbers $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty}(a_{n+1} - a_n) = e, b_n = \sqrt[n]{\frac{n!}{\sum_{k=1}^{n}\frac{k^k}{n^n}}}$$

Evaluate

$$\lim_{n\to\infty}\left(\frac{a_{n+1}b_{n+1}}{n+1} - \frac{a_n b_n}{n}\right)$$

Proposed by Ty Halpen-USA

S.500 In ΔABC holds:

$$a > b, \cos A + \cos B + \cos C = \frac{5}{4} \Rightarrow a + g_a + OH > b + h_b + 2\sqrt{2}r$$

Proposed by Radu Diaconu-Romania

S.501 For $x, y, z > 0$ prove that:

$$\left(\sum_{cyc} xy\right)^2 \sum_{cyc} x + 3\left(\sum_{cyc} xy\right)\left(\prod_{cyc} x\right) \geq 4\left(\prod_{cyc} x\right)\left(\sum_{cyc} x\right)^2$$

Proposed by Nikos Ntorvas-Greece

S.502 Prove that if

$$S_n = 4\sin(x)\left(\sin(3x) + \sin(7x) + \sin(11x) + \sin(15x) + \cdots + \sin\left((2n-1)x\right)\right)$$

then $S_n = \tan(x)\sin\left((n+1)(2x+\pi)\right) + \cos\left((n+1)(2x+\pi)\right) + 1$

Proposed by Sergio Esteban-Argentina

S.503 Evaluate:

$$\int \frac{x \sin^n x \cos nx}{\cos^n x - \sin nx + \varphi} dx$$

φ: Golden ratio, $n \in \mathbb{N}$

Proposed by Arslan Ahmed-Yemen

S.504 Let $a_k > \frac{k}{k+1}$ $(k = 1,2, \dots, n)$ be real numbers such that:

$$\prod_{k=1}^{n} \frac{a_k^{k+1} + k}{(k+1)a_k - k} = (n+1)!$$

Prove that:

$$\sum_{k=1}^{n} k a_k \geq \frac{n(n+1)}{2}$$

Proposed by Kunihiko Chikaya-Japan

S.505 If $x > 0$ then: $x^2 - 3x + 1 \geq \log x^x - x^{x+1}$

Proposed Lazaros Zachariadis-Greece

S.506 Find:

$$\Omega(\alpha) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=-1090\alpha}^{1010\alpha} \frac{1}{1+3^i}, \alpha \in \mathbb{N} - \{0\}$$

Proposed by Madan Mastermind-India

S.507 Prove without softs:

$$\int_0^{\frac{\pi}{2}} (\sin x)^{\cos x}(\cos x)^{\sin x}dx > \frac{27\pi - 40}{48}$$

Proposed by Precious Itsuokor-Nigeria

S.508 Given two positive integers $p \geq 0, q \geq 2$ and a real number $a > 0$, find Ω

$$\Omega = \lim_{n\to\infty}\sum_{k=1}^{n}\left\{\left(a + \sqrt{a + \frac{k^{q-1}}{n^q}}\right)^{\frac{1}{2p}} - (a + \sqrt{a})^{\frac{1}{2p}}\right\}$$

Proposed by Farid Khelili-Algerie

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

UNDERGRADUATE PROBLEMS

U.142 If $0 < a \leq b$ then:

$$3ab\int_a^b\int_a^b\int_a^b (x+y+z)(x^{-1}+y^{-1}+z^{-1})dx\,dy\,dz \leq (2a+b)(a+2b)(b-a)^3$$

Proposed by Daniel Sitaru – Romania

U.143 Find a closed form:

$$\Omega = \prod_{n=1}^{\infty}\log\left(2+\frac{1}{n}\right)\log\left(2-\frac{1}{n+1}\right)$$

Proposed by Daniel Sitaru – Romania

U.144 Find without any software:

$$\Omega = \int_0^{\infty}\frac{x^2 \cdot tan^{-1}x}{1+x^4}dx$$

Proposed by Vasile Mircea Popa-Romania

U.145 Find:

$$\Omega=\int_0^{\infty}\left(\frac{x^2\cdot tan^{-1}x}{x^4-x^2+1}\right)dx$$

Proposed by Vasile Mircea Popa-Romania

U.146 Find:

$$\Omega=\lim_{\substack{\epsilon\to 0\\ \epsilon>0}}\int_{\epsilon}^{1}\frac{\sqrt{x}logx}{x^3+x\sqrt{x}+1}dx$$

Proposed by Vasile Mircea Popa-Romania

U.147 Find:

$$\Omega=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\int_0^{\infty}\frac{x^{k-1}}{(1+x^2)(1+x^k)^2}dx-log\sqrt{n}\right)$$

Proposed by Vasile Mircea Popa-Romania

U.148 If $0\le x<\frac{\pi}{2}$ then:

$$9+4\sum_{cyc}\sin x\cdot Si(x)\ge 9\cdot\sqrt[9]{\prod_{cyc}\cos^4 x}$$

Proposed by Daniel Sitaru – Romania

U.149 Find a closed form:

$$\Omega=3-\frac{17}{18}+\frac{43}{75}-\frac{81}{196}+\frac{131}{405}-\cdots$$

Proposed by Orlando Irahola Ortega-Bolivia

U.150 If $0<a\le b$ then:

$$32\int_a^b\int_a^b\int_a^b(x+y+z)^3\,dx\,dy\,dz\ge 27(b^2-a^2)^3+108(b-a)(b^2-a^2)(b^3-a^3)$$

Proposed by Daniel Sitaru – Romania

U.151 If $0<a\le b, f,f':(0,\infty)\to(0,\infty), f$ – derivable, f' - continuous, then:

$$18\int_a^b\int_a^b\int_a^b\frac{f(x)f'(y)f'(z)dxdydz}{\big(f(y)+2f(z)\big)\big(3f^2(x)+2f^2(y)+f^2(z)\big)}\le(b-a)\log^4\left(\frac{f(b)}{f(a)}\right)$$

Proposed by Daniel Sitaru – Romania

U.152 If $0<a\le b, f:(0,\infty)\to(0,\infty), f$ – continuous then:

$$\int_a^b\int_a^b\int_a^b \frac{2dxdydz}{f(x)f(y)+f^2(z)} \le (b-a)\left(\int_a^b \frac{dx}{f(x)}\right)^2$$

Proposed by Daniel Sitaru – Romania

U.153 Find without any software:

$$\Omega = \left(\int_0^{10}\int_0^{10}[x+y]dxdty\right)\left(\int_0^{10}\int_0^{10}\{x+y\}dxdy\right), [*]-GIF, \{*\} = * - [*]$$

Proposed by Jalil Hajimir-Canada

U.154 Solve the following integral equation:

$$y(t) + 2\int_0^t y(x)sin(t-y)dx = 3e^{-t}$$

Proposed by Jalil Hajimir-Canada

U.155 Find the general solution:

$$\frac{dy}{dx} = y(2^x - logy)$$

Proposed by Jalil Hajimir-Canada

U.156 Find the general form of solutions:

$$\frac{d^2U}{dx^2} + y\frac{d^2U}{dx^2} = sinx + ycosx$$

Proposed by Jalil Hajimir-Canada

U.157 Find without any software:

$$\int_0^\infty \frac{2^{\frac{x}{3}}}{1+2^x}dx$$

Proposed by Jalil Hajimir-Canada

U.158 Find without any software:

$$\Omega = \int_0^\infty t^5 5^{-t} sin5t\ dt$$

Proposed by Jalil Hajimir-Canada

U.159

$$\Omega(x) = 1 + x\sum_{n=1}^{\infty} 2^{-n}\tanh(2^{-n}x), x > 0$$

Prove that:

$$e^{\Omega(a)}+e^{\Omega(b)}+e^{\Omega(c)}<3e\cdot\cosh\left(\frac{a+b+c}{3}\right),a,b,c>0$$

Proposed by Daniel Sitaru – Romania

U.160 If $0<a\le b$ then:

$$\int_a^b\int_a^b\int_a^b\sqrt{x^2+y^2+z^2}dx\,dy\,dz\le\frac{\sqrt{3}}{3}(b-a)^2(a^2+ab+b^2)\ln\left(\sqrt{\frac{b}{a}}\right)$$

Proposed by Daniel Sitaru – Romania

U.161 Find the minimum value of $n\in\mathbb{N}$ for which there exists positive integers x,y such that:

$$lcm(x,y)=\int_0^\infty 2x^{2n+1}\cdot e^{-x^2}dx,\ \ gcd(x,y)=2020$$

Proposed by Rajeev Rastogi-India

U.162 A positive integer is said to be "Rrian" integer, if it cannot be written as the difference of two square numbers. For example 1,2,4 and 6 are the first four "Rrian" integers. If R_n denotes the n^{th} "Rrian" integer, then find number of distinct non negative integers in tha sequence

$$\left[\frac{1^2}{R_{2020}}\right],\left[\frac{2^2}{R_{2020}}\right],\left[\frac{3^2}{R_{2020}}\right],\dots,\left[\frac{R_{2020}^2}{R_{2020}}\right],[*]-GIF.$$

Proposed by Rajeev Rastogi-India

U.163 If $m,n\in\mathbb{N}-\{0\},gcd(m,n)=1,\varphi$ –Euler's totient function, $\tau(n)$ –number of positive divisors of n then: $\sqrt{\varphi(mn)\cdot\tau(m)\cdot\tau(n)}\ge\frac{2mn}{m+n}$

Proposed by Rajeev Rastogi-India

U.164 Let $\mathcal{A}_k$ counts the total number of digits in $\mathcal{M}^{k^\alpha}$ for all $\mathcal{M}\ge 2$ and $\alpha\ge 1$. For example, for $\alpha,\mathcal{M}=1,3,3^6=729$ so $\mathcal{A}_6=3$. Then find the limit of

$$\lim_{n\to\infty}\frac{1}{n^{\alpha+1}}\sum_{k=1}^{n}\mathcal{A}_k$$

Proposed by Narendra Bhandari – Bajura – Nepal

U.165 Show the last two digits of the followings

$$9^9,9^{9^9},9^{9^{9^9}},9^{9^{9^{9^9}}},9^{9^{9^{9^{9}}}}$$

is always 89.

In general prove that the last two digits of $9\uparrow\uparrow n=\underbrace{9^{9^{\cdot^{\cdot^{9}}}}}_{n\ge 2}$ is 89.

Proposed by Narendra Bhandari – Bajura – Nepal

U.166 Prove that (Variant of classical result).

$$\prod_{n=1}^{\infty}\left(1-\frac{1}{\phi^n}\right)^{\frac{\mu(n)-\phi(n)+\phi\lambda(n)}{n}}=\frac{\sqrt{e^3}}{\sqrt{e^{\vartheta_3(0,\phi^{-1})}}}$$

where ϕ is Golden ratio, $\mu(n)$ is Mobius function, $\phi(n)$ is Euler's totient function, $\lambda(n)$ is Liouville's function, $\vartheta_a(x,q)$ is Jacobi theta function and e is Euler's number.

Proposed by Narendra Bhandari – Bajura – Nepal

U.167 Let the sequence $(Q_k)_{k\geq 1}=n_k, \forall n\geq 1$ with $P(Q_k)=\prod_{l=1}^{k}Q_l, S(Q_k)=\sum_{l=1}^{k}Q_l$ and $\sum_{1\leq n_1\leq n_2\leq,\dots,\leq n_k}\frac{1}{P(Q_k)S(Q_k)}=F(k)$ then prove or disprove that the following equality

$$\sum_{k=1}^{\infty}\sum_{j=0}^{k-1}\cos^{2k}\left(\frac{j\pi}{k}\right)\sum_{i=1}^{k}\sum_{q=1}^{k}\frac{(-1)^{i-1}}{(H_i)^{-1}}\binom{k}{i}\frac{\left(F(k)\right)^{-1}}{(k-q+1)^{k+1}}=2\sqrt[4]{e}+\sqrt{e}I_0(2^{-1})-3$$

where $I_0(x)$ is modified Bessel function of the first kind.

Proposed by Narendra Bhandari – Bajura – Nepal

U.168 Prove

$$\int_0^{\frac{\pi}{2}}\frac{x\sin 4x}{3+\cos 4x}dx=\frac{\pi}{4}\ln\left(\frac{2+\sqrt{2}}{4}\right)$$

Proposed by Narendra Bhandari – Bajura – Nepal

U.169 For $n\geq 0$. Prove or disprove

$$\lim_{k\to\infty}\lim_{n\to\infty}\prod_{m=1}^{k}\left(1+\int_0^{\frac{1}{4}\pi}\left(\frac{1-\tan x}{1+\tan x}\right)^n dx\right)^{\frac{n}{m}}\frac{1}{\sqrt{k}}=e^{\frac{\gamma}{2}}$$

where γ is Euler – Mascheroni constant and e is Euler's number.

Proposed by Narendra Bhandari – Bajura – Nepal

U.170

$$\left(\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j}(ijk)\right)^{-1}\frac{(-1)^{n+1}}{48}\right)=\frac{65}{108}-\frac{\zeta(2)}{6}-\frac{4\log(2)}{9}$$

Also prove the following closed form

$$\sum_{p=0}^{\infty}\left(\lim_{n\to\infty}\frac{1}{n^{2p}}\left(\sum_{k_p=0}^{n}\sum_{k_{p-1}}^{k_p}\dots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}\prod_{j=1}^{p}k_j\right)\right)=\sqrt{e}$$

Proposed by Narendra Bhandari – Bajura – Nepal

U.171 Prove/disprove

$$\psi\left(\frac{19}{20}\right)-\psi_2\left(\frac{1}{20}\right)=$$

$$\frac{\pi^3}{\sqrt{2}}\frac{\left(4+2\sqrt{3}\phi+\sqrt{5-\frac{\sqrt{5}}{\phi}}\right)^{-3}}{(1+\sqrt{3}-\phi(\sqrt{3}-1))\sqrt{\phi^2+1}}\frac{256\cot^2 3^0}{\left(9-\sqrt{5}-2\sqrt{6+3\phi}\right)}\left(1-\frac{8}{4+\sqrt{10-2\sqrt{5}}+2\sqrt{3}\phi}\right)$$

Notation: $\psi_n(x)$ is polygamma function and ϕ is Golden ratio.

Proposed by Narendra Bhandari – Bajura – Nepal

U.172 Generalization of Jay Jay Oweifa's proposal

If a, b and c are non zero real numbers, then prove that:

$$\lim_{n\to\infty}\left(\sqrt[a]{bn}\sqrt[n]{\frac{\sqrt[n]{n}}{n!}}\pm\sqrt[c]{\frac{n}{\sqrt[n]{n!}}}\right)=\sqrt[a]{be}\pm\sqrt[c]{e}$$

Proposed by Narendra Bhandari – Bajura – Nepal

U.173 Evaluate the sum in closed form

$$\sum_{n=0}^{\infty}\frac{F_{2n}}{(3n+1)^2}\binom{2n}{n}^{-1}$$

where F_n is nth Fibonacci number.

Proposed by Narendra Bhandari – Bajura – Nepal

U.174 If $f_n(x)=Li_{2n}\left(\frac{(1-x)^{2n}}{(1+x)^{2n}}\right)$ and $g(x)$ is constant function such that $g(x)=1$ for all $x\in[0,1]$, then prove that

$$\lim_{n\to\infty}\prod_{m=1}^{\infty}\prod_{k=1}^{\infty}\left(g(x)+16n\int_0^1 xf_n(x)\,dx\right)^{\frac{n}{m^3k^2}}\frac{1}{e^{m^{-3}H_m^{(2)}}}=\frac{\sqrt{e^{9\zeta(5)}}}{\sqrt[3]{e^{\pi^2\zeta(3)}}}=\sqrt[6]{\frac{e^{27\zeta(5)}}{e^{2\pi^2\zeta(3)}}}$$

where $\zeta(.)$ is Riemann zeta function, $Li_n(x)$ is Polylogarithmic function and H_n is nth Harmonic numer.

Proposed by Narendra Bhandari – Bajura – Nepal

U.175 Prove that:

$$\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\Gamma\left(n+\frac{1}{2}\right)}{n!}\right)^2\sum_{n=1}^{\infty}\frac{1}{n+1}\left(\frac{n!}{\Gamma\left(n+\frac{3}{2}\right)}\right)^2=(\ln 16^{\pi}-8G)\left(8G-\frac{14\zeta(3)}{\pi}-\frac{4}{\pi}\right)$$

where G is Catalan's constant and $\zeta(.)$ is Riemann zeta function.

Proposed by Narendra Bhandari – Bajura – Nepal

U.176 For $N \geq 2$ natural number, prove or disprove

$$\sum_{k=2}^{N}\sum_{n=2}^{k}\frac{1}{n^k} < \gamma, \frac{\pi}{4} - \sum_{k=2}^{N}\sum_{n=2}^{k}\frac{1}{k^n} > \frac{\gamma}{10}$$

where γ is Euler-Mascheroni constant.

Proposed by Narendra Bhandari – Bajura – Nepal

U.177 Variant Version of Prof. Dan Sitaru's proposal

Prove that:

$$\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\Gamma\left(n+\frac{1}{2}\right)}{n!}\right)^2\sum_{n}^{\infty}\frac{1}{n}\left(\frac{n!}{\Gamma\left(n+\frac{3}{2}\right)}\right)^2 = (\ln 16^{\pi} - 8G)\left(\frac{16}{\pi} - 4\right)$$

Proposed by Narendra Bhandari – Bajura – Nepal

U.178 Prove that:

$$\pi = \sum_{k\geq 0}\frac{2^{k+1}}{2k+1}\binom{2n}{n}^{-1} = \frac{2}{3}\left(\sum_{k\geq 0}\frac{2^k}{2k+3}\binom{2n}{n}^{-1} + 4\right) = \frac{6}{23}\left(\sum_{k\geq 0}\frac{2^k}{2k+5}\binom{2n}{n}^{-1} + \frac{104}{9}\right)$$
$$= \frac{10}{91}\left(\sum_{k\geq 0}\frac{2^k}{2k+7}\binom{2n}{n}^{-1} + \frac{2116}{75}\right) = \frac{70}{1451}\left(\sum_{k\geq 0}\frac{2^k}{2k+9}\binom{2n}{n}^{-1} + \frac{238912}{3675}\right)$$
$$= \frac{126}{5797}\left(\sum_{k\geq 0}\frac{2^k}{2k+11}\binom{2n}{n}^{-1} + \frac{2863204}{19845}\right)$$

Proposed by Narendra Bhandari – Bajura – Nepal

U.179 Prove

$$\int_0^{\frac{\pi}{4}} y^2 \ln\left(\frac{\cos y}{\ln 2}\right)dy = \frac{3\pi}{256}\zeta(3) + \frac{\pi^2}{32}G - \frac{\ln(\ln 4)}{192}\pi^3 - \frac{1536}{6144}\beta(4)$$

where G is Catalan's Constant, $\beta(s)$ is Dirichlet beta function, $\zeta(.)$ is Riemann zeta function.

Proposed by Narendra Bhandari – Bajura – Nepal

U.180 If $ab = 24$ for any positive real numbers a, b and positive integer k, then prove:

$$F(a,b) = \frac{1}{b^2}\left|\int_0^{\frac{1}{a}\pi}\ln\left(\tan^k\left(\frac{x}{b}\right)\right)dx\right| < \frac{k}{e\cdot G}$$

where $|.|$ is absolute value, e is Euler's number and G is Catalan's constant.

Proposed by Narendra Bhandari – Bajura – Nepal

U.181 For all $a > 0$

$$F(a) = \int_0^{\infty}\frac{\sin x}{x+a}dx = \int_0^{\infty}\frac{e^{-ax}}{x^2+1}dx = Ci(a)\sin(a) - Si(a)\cos(a) + \frac{\cos(a)}{2}\pi$$

Proposed by Narendra Bhandari – Bajura – Nepal

U.182 If $0 < a \le b$ then:

$$(b-a)^2\int_a^b \frac{x^2dx}{1+x^2} + (b-a)\int_a^b\int_a^b \frac{y^2dxdy}{(1+x^2)(1+y^2)} +$$

$$+\int_a^b\int_a^b\int_a^b \frac{z^2dxdydz}{(1+x^2)(1+y^2)(1+z^2)} + \log^3\left(\sqrt{\frac{b}{a}}\right) \le (b-a)^3$$

Proposed by Daniel Sitaru – Romania

U.183 Find a closed form:

$$\Omega = \sum_{n=0}^{\infty}\frac{(-1)^n F_{2n}}{9^n}, F_n - \text{Fibonacci numbers.}$$

Proposed by Daniel Sitaru – Romania

U.184 Let the recurrence relation:

$$R(n-3)+R(n-2)+R(n-1)+R(n) = (-1)^{\frac{1}{2}n(n+1)}$$

$$R(0) = -1, R(1) = 0, R(2) = 1$$

Then prove that

$$\int_{-\infty}^{\infty}\left(\sum_{n=1}^{\infty} R(n)x^n\right)\frac{dx}{x} = \pi$$

Proposed by Srinivasa Raghava-AIRMC-India

U.185 Prove that:

$$\int_0^{\frac{\pi}{2}} \sin\left(\frac{x}{2}\right)\tanh^{-1}(\sin(2x))\,dx$$

$$= \log\left(\left(2\sqrt{2-\sqrt{2}}+2\sqrt{2}-1\right)^{\sqrt{2+\sqrt{2}}}\left(1+2\sqrt{2}-2\sqrt{2+\sqrt{2}}\right)^{\sqrt{2-\sqrt{2}}}\right)$$

Proposed by Narendra Bhandari-Nepal

U.186 If

$$\Omega(s) = \int_0^{\infty}\frac{e^{-st}}{t}\left(\int_0^{\infty}\int_0^{\infty} e^{-(m+n)}m^{t-1}n^{t-2}dmdn\right)^{-1} dt$$

Show

$$\Omega(s) = \frac{1}{2} - \frac{1}{\pi}\tan^{-1}\left(\frac{s}{\pi}\right); m, n, , s > 0$$

Proposed by Akerele Olofin-Nigeria

U.187 Prove:

$$\sum_{n=1}^{10}\cos\left(\frac{n\pi}{10}\right)\left(\psi\left(\frac{11-n}{20}\right)-\psi\left(\frac{21-n}{20}\right)\right)=\frac{9(\phi-1)\pi}{2}-5\sqrt{\phi+2}\ln\left(2-\sqrt{\phi+2}\right)$$

ϕ – Golden Ratio

Proposed by Asmat Qatea-Afghanistan

U.188 Nice find:

$$\int_0^{\infty}\frac{e^{-2x}\tanh^2(x)}{x}dx$$

Proposed by Ajetunmobi Abdulqoyyum-Nigeria

U.189 Prove that:

$\sum_{k=0}^{n} k!\times\binom{n}{k}=e\times n!\,\Gamma(1+n,1)$, $e=2.7182\ldots$ Euler constant

$$\Gamma(x,y)=\int_y^{\infty}e^{x-1}e^{-t}\,dt$$

Proposed by Ghazaly Abiodun-Nigeria

U.190 Prove that:

$$\int_0^{\infty}e^{-x}\prod_{k=1}^{\infty}(1-e^{-6xk})(1+e^{-6xk+x})\,(1+e^{-6xk+5x})dx=\frac{\pi}{\sqrt{2}}\frac{\sinh\left(\frac{2\pi\sqrt{2}}{3}\right)}{\left(\cosh\left(\frac{2\pi\sqrt{2}}{3}\right)-\cos\left(\frac{2\pi}{3}\right)\right)}$$

Proposed by Syed Shahabudeen-India

U.191 Find a closed form:

$$\Omega=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\cos x}{\sqrt[7]{\sin^7(2x)\cdot\cos x}}dx$$

Proposed by Mohamed Arahman Jama-Somalia

U.192 Find without softs:

$$\Omega=\int_0^1\frac{\log^2(1+x)-\log x\log(1+x)-\log(1-x)\log(1+x)}{1+x^2}dx$$

Proposed by Precious Itsuokor-Nigeria

U.193 Find:

$$\lim_{x\to 0}\frac{\left(\int_0^{x-\sin(\tan^{-1}x)}\tan\theta\,d\theta\right)\left(\int_{x-\sin x}^{x-\sin(\tan^{-1}x)}\sin\varphi\,d\varphi\right)}{x^3e^{-\frac{1}{x}}\sin x}$$

Proposed by Qusay Yousef-Algerie

U.194 Find:

$$\Omega=\int_{-\pi}^{\frac{3\pi}{2}}\frac{-x}{(\tan x)^n+1}dx+\int_{\frac{\pi}{2}}^{\pi}\frac{\frac{\pi}{2}(\cos x)^n-x\,(\sin x)^n}{(\sin x)^n+(\cos x)^n}dx$$

$$\forall n\in\{2,4,6,\dots etc\}$$

Proposed by Qusay Yousef-Algerie

U.195 Prove that:

$$\sum_{n=1}^{\infty}\frac{H_n\overline{H_n}}{n^3}=$$

$$=\frac{1}{6}\log^3(2)\zeta(2)-\frac{7}{8}\log^2(2)\zeta(3)+4\log(2)\zeta(4)-\frac{193}{64}\zeta(5)-\frac{1}{60}\log^5(2)+$$

$$+\frac{3}{8}\zeta(2)\zeta(3)+2Li_5\left(\frac{1}{2}\right)$$

where $\overline{H_n}=1-\frac{1}{2}+\cdots+\frac{(-1)^{n-1}}{n}$

Proposed by Cornel Ioan Vălean-Romania

U.196 Evaluate:

$$\lim_{n\to\infty}\left(n\int_0^1(\cos x-\sin x)^n dx\right)^{\int_0^1\frac{nx}{nx^3+1}dx}$$

Proposed by Mokhtar Khassani-Algerie

U.197 If $\lim_{n\to\infty}n^2\left(\int_0^1\sqrt[n]{1+x^n}\,dx-1\right)=L$ and

$$\lim_{n\to\infty}n\left(n^2\left(\int_0^1\sqrt[n]{1+x^n}dx-1\right)-L\right)=M \text{ then}$$

$$\lim_{n\to\infty}\left(n\left(n^2\left(\int_0^1\sqrt[n]{1+x^n}dx-1\right)-L\right)-M\right)=?$$

Proposed by Vicky Chaudary-India

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

ROMANIAN MATHEMATICAL MAGAZINE-R.M.M.-AUTUMN 2022

PROBLEMS FOR JUNIORS

JP.376. Let ΔABC be an acute triangle. Prove that:

$$\sqrt{\frac{\sin A}{\sin B \cdot \sin C}}+\sqrt{\frac{\sin B}{\sin C \cdot \sin A}}+\sqrt{\frac{\sin C}{\sin A \cdot \sin B}} \geq \sqrt[4]{108}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

JP.377. Let a, b, c be positive real numbers such that $ab+bc+ca \leq a+b+c$. Prove that:

$$\frac{a}{b^2c^2}+\frac{b}{c^2a^2}+\frac{c}{a^2b^2} \geq 3$$

Proposed by George Apostolopoulos-Messolonghi-Greece

JP.378. Determine all triplets (x, y, z) of positive integers which satisfy the following two equations:

$$xy+z^2=31, \quad x+yz^2=53$$

Proposed by George Apostolopoulos-Messolonghi-Greece

JP.379. If $ABCD$ tetrahedron $AB=a, AD=b, AC=c, BD=d, DC=e, CB=f$

F –total area, then

$$a^4+b^4+c^4+d^4+e^4+f^4 \geq 2F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

JP.380. If $a, b, c, d > 0, \sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}=4$ then:

$$\sum_{cyc} \frac{1}{a^4+b^4+c^4+5} \leq \frac{1}{2\sqrt{abcd}}$$

Proposed by Daniel Sitaru-Romania

JP.381. If ABC and UVW are two triangles then :

$$\sum_{cyc}\frac{\cos\frac{A}{2}}{1+\sin\frac{A}{2}}\left(1+\sin\frac{U}{2}\right)\geq\sum_{cyc}\cos\frac{U}{2}$$

Proposed by Cristian Miu-Romania

JP.382 In acute $\Delta ABC, D\in(BC), E\in(AC), F\in(AB)$. Prove that:

$$\sqrt{\frac{AD^3+BE^3}{AD^5+BE^5}}+\sqrt{\frac{BE^3+CF^3}{BE^5+CF^5}}+\sqrt{\frac{CF^3+AD^3}{CF^5+AF^5}}\leq\frac{1}{r}$$

Proposed by Marian Ursărescu-Romania

JP.383 In ΔABC the following relationship holds:

$$\sum_{cyc}\sqrt{\frac{1}{\sin^2 A}+\frac{1}{\sin^2 B}+\frac{1}{\sin^2 C}}\geq 6$$

Proposed by Marian Ursărescu-Romania

JP.384. Solve for real numbers:

$$2^x+9^{\frac{1}{x}}+2^x\cdot 9^{\frac{1}{x}}=19$$

Proposed by Marian Ursărescu-Romania

JP.385 If $a,b,c>0$ are such that $\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}=\frac{3}{4}$ then:

$$\sum_{cyc}\frac{a+2b}{a^2+2b^2}+\sum_{cyc}\frac{b+2a}{b^2+2a^2}\leq 3$$

Proposed by Daniel Sitaru-Romania

JP.386 Let ΔABC be any triangle. Prove that for any $n\in\mathbb{N}\setminus\{0,1\}$ the following relationship holds:

$$\frac{|a-b||a-c|^n a^{3n}}{b^{3n-3}|b-c|^{n-1}}+\frac{|b-c||b-a|^n b^{3n}}{c^{3n-3}|c-a|^{n-1}}+\frac{|a-c||c-b|^n c^{3n}}{a^{3n-3}|a-b|^{n-1}}\geq$$
$$\geq 16sr^2(4R^2+5Rr+r^2-s^2)$$

Proposed by George Florin Şerban-Romania

JP.387 Let ΔABC be any triangle. Prove that for any $n\in\mathbb{N}\setminus\{0,1\}$ the following relationship holds:

$$\frac{|a-b||a-c|^n a^{2n}}{b^{2n-2}|b-c|^{n-1}}+\frac{|b-c||b-a|^n b^{2n}}{c^{2n-2}|c-a|^{n-1}}+\frac{|a-c||c-b|^n c^{2n}}{a^{2n-2}|a-b|^{n-1}}\geq$$
$$\geq 4r^2[(4R+r)^2-3s^2]$$

Proposed by George Florin Şerban-Romania

JP.388 Solve for complex numbers:

$$\begin{cases}|x-y|\geq\sqrt{3}|z|\\|y-z|\geq\sqrt{3}|x|\\|z-x|\geq\sqrt{3}|y|\end{cases}$$

Proposed by Ionuț Florin Voinea-Romania

JP.389 A right parallelipiped $ABCDA'B'C'D'$ has the basis $ABCD$ rhombus, and areas of the two diagonals sections of the parallelipiped are F_1 and F_2 respectively. Let R be the circumradius of ΔABC, R_2circumradius of ΔABD and V volume of the right parallelipiped.

Prove that: $R_1R_2F_1F_2\geq V^2$.

Proposed by Radu Diaconu-Romania

JP.390 Let $x\in\mathbb{R}$ and ABC a triangle with F area.Prove that:

$$\frac{a^3}{\sqrt{b^2\sin^2x+c^2\cos^2x}}+\frac{b^3}{\sqrt{c^2\sin^2x+a^2\cos^2x}}+\frac{c^3}{\sqrt{a^2\sin^2x+b^2\cos^2x}}\geq 4\sqrt{3}F$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

PROBLEMS FOR SENIORS

SP.376 Let r, r_a, r_b, r_c and R be, respectively, the inradius, the exradii, and the circumradius of triangle ABC with side lengths a, b, c. Prove that:

$$36\sqrt{3}\frac{r^3}{R^2}\leq\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a}+4r\left(\frac{r_a}{b+c}+\frac{r_b}{c+a}+\frac{r_c}{a+b}\right)\leq\frac{9\sqrt{3}R^2}{4r}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

SP.377 Let w_a, w_b, w_c be the internal bisectors, r_a, r_b, r_c the exradii, r the inradius and R the circumradius of a triangle ABC. Prove that:

$$\left(\frac{w_a}{r_a}\right)^2+\left(\frac{w_b}{r_b}\right)^2+\left(\frac{w_c}{r_c}\right)^2\leq 27\left(\frac{R}{2r}\right)^4-24$$

Proposed by George Apostolopoulos-Messolonghi-Greece

SP.378. Let m_a, m_b, m_c be the lengths of the medians of a triangle ABC with area F.

Prove that:

$m_a^n+m_b^n+m_c^n\geq 3^{\frac{n}{4}+1}\cdot F^{\frac{n}{2}}$ for each integer $n\geq 1$.

Proposed by George Apostolopoulos-Messolonghi-Greece

SP.379 If $x, y, z\in(0,1)$ then:

$$\frac{x}{(y+z)^2(1-x^2)}+\frac{y}{(z+x)^2(1-y^2)}+\frac{z}{(x+y)^2(1-z^2)}\geq\frac{9\sqrt{3}}{8}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

SP.380 Let a, b, c be the sides of an arbitrary triangle . Denote by m_a, w_a, h_a the lengths of the median , the internal bisector and the altitude corresponding to the side a and ω its Brocard angle . Prove that:

$$\frac{1}{\sin\omega} \ge 2\sqrt[4]{\frac{m_a^2 m_b^2 m_c^2}{w_a w_b w_c h_a h_b h_c}}$$

Proposed by Vasile Jiglău-Romania

SP.381 Find all continuous functions $f:(0,\infty) \to \mathbb{R}$ such that

$$f(a^x) + f(a^{2x}) + f(a^{4x}) = x, \forall x \in \mathbb{R}, a > 0, a \neq 1 - \text{fixed.}$$

Proposed by Marian Ursărescu-Romania

SP.382. $z_1, z_2, z_3 \in \mathbb{C}^*$ –different in pairs such that

$|z_1| = |z_2| = |z_3| = 1, A(z_1), B(z_2), C(z_3)$. Prove that:

$$\sum_{cyc} \frac{z_2 z_3}{(z_2 - z_3)^2[z_2(z_1 - z_3)^2 - z_3(z_1 + z_2)^2]} = \frac{1}{4z_1 z_2 z_3} \Rightarrow AB = BC = CA$$

Proposed by Marian Ursărescu-Romania

SP.383 If $a, b, c > 0$ then:

$$\frac{a^{10}c^5 + b^{10}a^5 + c^{10}b^5}{a^2b + b^2c + c^2a} \ge a^4b^4c^4$$

Proposed by Daniel Sitaru-Romania

SP.384 Let $(a_n)_{n\ge1}$ be sequence of real numbers with $a_n > 0, \forall n \in \mathbb{N}$ and

$a_0 = 1,\ a_n^2 + a_n e^{a_n} = (n+1)(n+1+e^{a_n})$.Find:

$$\Omega = \lim_{n\to\infty} a_n \cdot \sqrt[n]{a_n \cdot \sin 1 \cdot \sin\frac{1}{2} \cdot \ldots \cdot \sin\frac{1}{n}}$$

Proposed by Florică Anastase-Romania

SP.385 Let $(a_n)_{n\ge1}$ be sequence of positive real numbers such that:

$$a_{n+1}^3 - (a_n + a_1)a_{n+1}^2 + (a_{n+1} - a_1)a_n^2 + a_1 a_n a_{n+1} = 0, \forall n \in \mathbb{N}^*, n > 1$$

Prove that:

$$\sum_{k=1}^{n} \log_3\left(\left(\frac{a_k}{a_{k+1}}\right)^2 + \left(\frac{a_k}{a_{k+1}}\right) + 1\right) \ge n$$

Proposed by Florică Anastase-Romania

SP.386 Solve for real numbers: $\log_2(5^x - 3) = \log_7(3^x + 4)$

Proposed by Ionuț Florin Voinea-Romania

SP387. Given $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=-2x^2+6x-3$, then find

$$\Omega=\lim_{n\to\infty}\int_1^2 f^n(x)\,dx$$

(where $f^n(x)=\underbrace{f\left(f\left(f\left(\dots f(x)\right)\right)\right)}_{n-times}$

Proposed by Rajeev Rastogi-India

SP.388 Given $\{a_n\}$ is a sequence of real numbers satisfying $a_1=0$ and

$$\frac{4}{4-a_{n+1}}-\frac{4}{4-a_n}=2n+1;\forall n\geq 1$$

Define $b_n=\frac{4-a_n}{4}$ for $n\geq 1$, then find

$$\Omega=\lim_{n\to\infty}\left[4\cdot\sin^{-1}\frac{b_{n+1}}{b_n}-\left(\pi-\frac{1}{2}+\tan h^{-1}\sqrt{b_n}\right)\left(\prod_{k=2}^{n}a_k\right)\right]$$

Proposed by Rajeev Rastogi-India

SP.389 Given $f:\mathbb{R}\to\mathbb{R}$ is a continuous function satisfying the functional equation

$f(x+y)-3^y f(x)=3^x f(y);\forall x,y\in\mathbb{R}$ then find

$$\Omega=\lim_{x\to\infty}\left[\frac{f(x)}{f'(x)}+\frac{f'(x)}{f''(x)}+\dots+\frac{f^{n-1}(x)}{f^n(x)}\right]$$

(where $f^n(x)$ denotes the n^{th} derivative of $f(x)$ with respect to x)

Proposed by Rajeev Rastogi-India

SP.390 Given $f(x)$ be a non-constant function satisfying the integral equation

$$f(x)=2x^2-\int_0^2(f(t)-x)^2\,dt$$

then find:

$$\Omega=\lim_{x\to\infty}\left[\lim_{n\to\infty}\left(\frac{\sum_{r=0}^{n}f\left(\frac{x}{2^r}\right)}{x}\right)\right]$$

Proposed by Rajeev Rastogi-India

UNDERGRADUATE PROBLEMS

UP.376 If $a,b>0$ then find:

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{\sqrt[n]{(2n-1)!!}}\cdot\sum_{k=1}^{n}\sqrt{\frac{1}{b^2}+\frac{1}{(a+bn)^2}+\frac{1}{\left(a+b(n+1)\right)^2}}\right)$$

Proposed by D.M.Bătinețu-Giurgiu, Daniel Sitaru-Romania

UP.377 Let $(x_n)_{n\geq 0}$ sequence of positive real numbers such that

$$nx_n^2 = ax_{n+1}^2 + (an-1)x_{n+1}x_n; a > 0, x_0 > 0 \text{ –fixed.Find:}$$

$$\Omega = \lim_{n\to\infty}\left(\lim_{m\to\infty}\left(\frac{\sum_{k=1}^{n} x_k^{\frac{1}{\sqrt{m}}}}{n}\right)^{\frac{1}{\tan\left(\frac{1}{\sqrt{m}}\right)}}\right)$$

Proposed by Florică Anastase-Romania

UP.378 If $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ are positive real sequences such that

$$b_n = a_1 \cdot \sqrt{a_2!} \cdot \sqrt[3]{a_3!} \cdot \ldots \cdot \sqrt[n]{a_n!} \text{ and } \lim_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n} = \pi. \text{ Find:}$$

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{b_{n+1}}} - \frac{n^2}{\sqrt[n]{b_n}}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

UP.379 If $S_n = -2\sqrt{n} + \sum_{k=1}^{n}\frac{1}{\sqrt{k}}$ is the Ioachimescu's sequence with $\lim_{n\to\infty} s_n = s$, then

compute $\lim_{n\to\infty}(s_n - s)\sqrt[2n]{(2n-1)!!}$.

Proposed by D.M. Bătinețu-Giurgu, Neculai Stanciu-Romania

UP.380 Let be $E(n) = \Gamma\left(\frac{1}{n}\right)\cdot\Gamma\left(\frac{2}{n}\right)\cdot\ldots\cdot\Gamma\left(\frac{n-1}{n}\right), n \geq 2, n \in \mathbb{N}^*$.Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{E(n)}{\sin\frac{1}{\sqrt{n}}}\cdot\sin\frac{1}{\sqrt{(2\pi)^{n-1}}}\right)$$

Proposed by Florică Anastase-Romania

UP.381 If $a, b \in \mathbb{R}_+, \gamma_n(a,b) = -\log(n+a) + \sum_{k=1}^{n}\frac{1}{k+b}, \lim_{n\to\infty}\gamma_n(a,b) = \gamma(a,) \in \mathbb{R}$,

then find:

$$\Omega = \lim_{n\to\infty}\left(\log\left(\frac{e}{n+a}\right) + \sum_{k=1}^{n}\frac{1}{k+b} - \gamma(a,b)\right)^n$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

UP.382 For $t > 0$ find:

$$\Omega = \lim_{n\to\infty} n^{1-t}\left(\frac{\left(\sqrt[n+1]{(n+1)!}\right)^{2t}}{(n+1)^t} - \frac{\left(\sqrt[n]{n!}\right)^{2t}}{n^t}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

UP. 383 Let R be the circumradius of ΔABC having the length of the sides a, b, c.
Prove that:

$$\Delta = \begin{vmatrix} 3\sqrt{3}R & a & b & c \\ a & 3\sqrt{3}R & c & b \\ b & c & 3\sqrt{3}R & a \\ c & b & a & 3\sqrt{3}R \end{vmatrix} > 0$$

Proposed by Daniel Sitaru-Romania

UP. 384 Find:

$$\Omega = \lim_{n\to\infty}\left(1+\left(1+\frac{1}{n}\right)^{n+1}-e\right)^{n}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

UP.385 In any ΔABC let x, y, z be the distances from the incentre to the sides of triangle and $u \geq 1$ −fixed. Prove that:

$$u^x + u^y + u^z \leq u^{\sqrt{\frac{bc(s-a)}{s}}} + u^{\sqrt{\frac{ca(s-b)}{s}}} u^{\sqrt{\frac{ab(s-c)}{s}}}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

UP.386 If $0 < a \leq b$ then:

$$\int_a^b \int_a^b \frac{dxdy}{xy+1} \leq \frac{2(b-a)^2}{(a+1)(b+1)}$$

Proposed by Daniel Sitaru-Romania

UP.387 If in $\Delta ABC, 2s = 3$ then:

$$\frac{m_a + m_b}{m_c} + \frac{a^2 b(m_b + m_c)}{m_a} + \frac{bc^2(m_c + m_a)}{m_b} \geq 8\sqrt{3}F$$

Proposed by Daniel Sitaru-Romania

UP.388 If $x, y, z, p, q, r > 0; x + y + z = p + q + r = 3$ then:

$$x^p + x^q + x^r + y^p + y^q + y^r + z^p + z^q + z^r \geq 9$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

UP.389 If $0 < a \leq b$ then:

$$\int_a^b \int_a^b \frac{dxdy}{(xy+1)^2} \leq \frac{4(b-a)^2(a^2+b^2+ab+3a+3b+3)}{3(a+1)^3(b+1)^3}$$

Proposed by Daniel Sitaru-Romania

UP.390 If $x, y, z \geq 1; x + y + z = 6$ then in ΔABC the following relationship holds:

$$(x^x + y^x + z^x)a^4 + (x^y + y^y + z^y)b^4 + (x^z + y^z + z^z)c^4 \geq 5184r^4$$

Proposed by D.M. Bătinețu-Gurgiu, Daniel Sitaru-Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

INDEX OF AUTHORS RMM-34

Nr.crt.	Numele și prenumele	Nr.crt.	Numele și prenumele
1	DANIEL SITARU-ROMANIA	30	SRINIVASA RAGHAVA-INDIA
2	D.M.BĂTINEȚU-GIURGIU-ROMANIA	31	ADIL ABDULLAYEV-AZERBAIJAN
3	CLAUDIA NĂNUȚI-ROMANIA	32	ALEX SZOROS-ROMANIA
4	MARIAN URSĂRESCU-ROMANIA	33	MEHMET ȘAHIN-TURKIYE
5	MARIN CHIRCIU-ROMANIA	34	GHAZALY ABIODUN-NIGERIA
6	FLORICĂ ANASTASE-ROMANIA	35	VASILE MIRCEA POPA-ROMANIA
7	NECULAI STANCIU-ROMANIA	36	GEORGE APOSTOLOPOULOS-GREECE
8	BOGDAN FUȘTEI-ROMANIA	37	CRISTIAN MIU-ROMANIA
9	DAN NĂNUȚI-ROMANIA	38	GEORGE FLORIN ȘERBAN-ROMANIA
10	MARIUS DRĂGAN-ROMANIA	39	IONUȚ FLORIN VOINEA-ROMANIA
11	NARENDRA BHANDARI-NEPAL	40	VASILE JIGLĂU-ROMANIA
12	LONG HUYNH HUU-VIETNAM	41	MOHAMMED BOURAS-MOROCCO
13	SEYRAN IBRAHIMOV-AZERBAIJAN	42	ORLANDO IRAHOLA ORTEGA-BOLIVIA
14	RADU DIACONU-ROMANIA	43	JALIL HAJIMIR-CANADA
15	ERTAN YILDIRIM-TURKIYE	44	NGUYEN VAN CANH-VIETNAM
16	ADRIAN POPA-ROMANIA	45	AMERUL HASSAN-MYANMAR
17	MOHAMED NASERY-AFGHANISTAN	46	RUXANDRA TONILĂ-ROMANIA
18	SAFAL DAS BISWAS-INDIA	47	SOUMAVA CHAKRABORTY-INDIA
19	GHULAM NASERI-INDIA	48	TY HALPEN-USA
20	SERGIO ESTEBAN-ARGENTINA	49	RAJEEV RASTOGI-INDIA
21	AKERELE OLOFIN-NIGERIA	50	ASMAT QATEA-AFGHANISTAN
22	AJETUNMOBI ABDULQOYYUM-NIGERIA	51	NIKOS NTORVAS-GREECE
23	ARSLAN AHMED-YEMEN	52	PRECIOUS ITSUOKOR-NIGERIA
24	KUNIHIKO CHIKAYA-JAPAN	53	FARID KHELILI-ALGERIE
25	MADAN MASTERMIND-INDIA	54	MOHAMAD ARAHMAN JAMA-SOMALIA
26	CORNEL IOAN VĂLEAN-ROMANIA	55	MOKHTAR KHASSANI-ALGERIE
27	QUSAY YOUSEF-ALGERIE	56	VICKY CHAUDARY-INDIA
28	DENISA LEPĂDATU-ROMANIA	57	MARIN IONESCU-ROMANIA
29	ILIR DEMIRI-AZERBAIJAN	58	MINH NHAT NGUYEN-VIETNAM

NOTĂ: Pentru a publica probleme propuse, articole și note matematice în RMM puteți trimite materialele pe mailul: dansitaru63@yahoo.com