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NEW INEQUALITIES WITH TWO TRIANGLES

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1. Introduction

The purpose of this work is to offer applications which can use all of the teaching methods of mathematics: i.e., inductive, deductive, heuristic or discovery, analytic, synthetic, problem solving and other methods.

In this paper we generalize formulas based on various patterns of geometric inequalities and we give new elementary proofs for some old triangle inequalities, and also present additional proofs for new inequalities in triangles.

2. New result

We consider two triangles ABC and UVW with the length of sides a, b, c and u, v, w respectively, and areas S and F, respectively.

Proposition 1. (New inequality)

If σ is a permutation of the set $\{u, v, w\}$ and $x, y, z > 0$, then:

$$\frac{y+z}{x}a\cdot\sigma(u)+\frac{z+x}{y}b\cdot\sigma(v)+\frac{x+y}{z}c\cdot\sigma(w)\geq 8\sqrt{3SF}\quad (1)$$

Proof. From AM-GM inequality, we have $d+e\geq 2\sqrt{de}$; $(*)$ $\ d+e+f\geq 3\sqrt[4]{def}$; $(**)$ and from Carlitz inequality in triangles, i.e. $\sqrt[3]{(abc)^2}\geq\frac{4S\sqrt{3}}{3}$; $(***)$ and $\sqrt[3]{(uvw)^2}\geq\frac{4F\sqrt{3}}{3}$, respectively we obtain

$$\frac{y+z}{x}a\cdot\sigma(u)+\frac{z+x}{y}b\cdot\sigma(v)+\frac{x+y}{z}c\cdot\sigma(w)\overset{(*)}{\geq}$$

$$\geq\frac{2\sqrt{yz}}{x}a\cdot\sigma(u)+\frac{2\sqrt{zx}}{y}b\cdot\sigma(v)+\frac{2\sqrt{xy}}{z}c\cdot\sigma(w)\overset{(**)}{\geq}$$

$$\geq 2\cdot 3\cdot\sqrt[3]{\frac{\sqrt{yz}}{x}\cdot\frac{\sqrt{zx}}{y}\cdot\frac{\sqrt{xy}}{z}\cdot abc\cdot\sigma(u)\sigma(v)\sigma(w)}=6\sqrt[3]{abc}\cdot\sqrt[3]{\sigma(u)\sigma(v)\sigma(w)}=$$

$$= 6 \cdot \sqrt{\sqrt[3]{(abc)^2}} \cdot \sqrt{\sqrt[3]{(uvw)^2}} \overset{(***)}{\geq} 6\sqrt{\frac{4S\sqrt{3}}{3}} \cdot \sqrt{\frac{4F\sqrt{3}}{3}} = 8\sqrt{3SF}$$

Remarks:

If $x = y = z$, then the inequality (1) becomes:

$$a \cdot \sigma(u) + b \cdot \sigma(v) + c \cdot \sigma(w) \geq 4\sqrt{3SF}; \quad (2)$$

If σ is the identity permutation, then inequality (2) becomes:

$au + bv + cw \geq 4\sqrt{3SF}$; (3), i.e. one of the Tsintsifas' inequality.

If $\Delta ABC \equiv \Delta UVW$, then inequality (3) becomes:

$a^2 + b^2 + c^2 \geq 4S\sqrt{3}$; (4),i.e. Ionescu-Weitzenbock's inequality.

If $\sigma(u) = v, \sigma(v) = w, \sigma(w) = u$, then inequality (2) becomes:

$$av + bw + cu \geq 4\sqrt{3SF}; \quad (5)$$

If $\Delta ABC \equiv \Delta VWU$, then inequality (5) becomes:

$ab + bc + ca \geq 4S\sqrt{3}$; (6), i.e. Gordon's inequality.

If σ is the identity permutation, then (1) becomes:

$$\frac{y+z}{x}au + \frac{z+x}{y}bv + \frac{x+y}{z}cw \geq 8\sqrt{3SF}; \quad (7)$$

If $\Delta ABC \equiv \Delta VWU$, then (7) becomes:

$\frac{y+z}{x}a^2 + \frac{z+x}{y}b^2 + \frac{x+y}{z}c^2 \geq 8S\sqrt{3}$; (8) , i.e. Bătinețu-Giurgiu's inequality.

In the next, we consider the triangles $A_1B_1C_1$ and $A_2B_2C_2$ with the lengths of sides a_1, b_1, c_1 and a_2, b_2, c_2 respectively, the semiperimeters s_1, s_2 respectively and the areas F_1 and F_2, respectively.

Proposition 2. (New inequality)

$$\sqrt{a_1a_2} + \sqrt{b_1b_2} + \sqrt{c_1c_2} \geq 2\sqrt[4]{27} \cdot \sqrt[4]{F_1F_2}$$

Proof. As in Proposition 1. We use AM-GM inequality and Carlitz's inequality in triangles and we obtain

$$\sum_{cyc}\sqrt{a_1a_2} \overset{(**)}{\geq} 3\sqrt[3]{\prod_{cyc}\sqrt{a_1a_2}} = 3\sqrt[3]{\sqrt{a_1b_1c_1} \cdot \sqrt{a_2b_2c_2}} = 3\sqrt{\sqrt[3]{a_1b_1c_1} \cdot \sqrt[3]{a_2b_2c_2}} \overset{(***)}{\geq}$$

$$\geq 3\sqrt{\sqrt{\frac{4F_1}{\sqrt{3}}}\cdot\sqrt{\frac{4F_2}{\sqrt{3}}}}=2\cdot 3\cdot\frac{\sqrt[4]{F_1F_2}}{\sqrt[4]{3}}=2\cdot\sqrt[4]{27}\cdot\sqrt[4]{F_1F_2}$$

If $\Delta A_1B_1C_1\equiv\Delta A_2B_2C_2$, then from Proposition 2, we have $a_1+b_1+c_1\geq 2\sqrt[4]{27}\cdot\sqrt{F_1}$, so in either triangle we have the following inequality:

$$a+b+c\geq 2\sqrt[4]{27}\cdot\sqrt{F};\ \ (9)\Leftrightarrow 2s\geq 2\sqrt[4]{27}\cdot\sqrt{F}\Leftrightarrow s\geq\sqrt[4]{27}\cdot\sqrt{sr}\Leftrightarrow$$
$$\sqrt{s}\geq\sqrt[4]{27}\cdot\sqrt{r}\Leftrightarrow s\geq r\sqrt{27}\Leftrightarrow p\geq 3r\sqrt{3};\ (Mitrinovic)$$

Proposition 3. (Old inequality).

$$a_1a_2+b_1b_2+c_1c_2\geq 4\sqrt{3}\cdot\sqrt{F_1F_2}\ (Tsintsifas')$$

New proof. From Bergstrom's inequality and Proposition 2. We deduce that

$$\sum_{cyc}a_1a_2=\sum_{cyc}\left(\sqrt{a_1a_2}\right)^2\overset{Bergstrom}{\geq}\frac{\left(\sum\sqrt{a_1a_2}\right)^2}{3}\geq\frac{\left(2\sqrt[4]{27}\cdot\sqrt[4]{F_1F_2}\right)^2}{3}=4\sqrt{3}\cdot\sqrt{F_1F_2}$$

Remarks:

If $\Delta A_1B_1C_1\equiv\Delta A_2B_2C_2\equiv\Delta ABC$, then in Proposition 3, we obtain the inequality

$$a^2+b^2+c^2\geq 4F\sqrt{3};\ \ (10)(Ionescu-Weitzenbock)$$

If $\Delta A_1B_1C_1\equiv\Delta ABC$ and $\Delta A_2B_2C_2\equiv\Delta BCA$, then in Proposition 3, we deduce that

$$ab+bc+ca\geq 4F\sqrt{3};(11)\ \ (Gordon's)$$

Proposition 4. (New inequality).

$$a^{m+1}+b^{m+1}+c^{m+1}\geq 2^{m+1}\cdot 3^{\frac{3-m}{4}}\cdot F^{\frac{m+1}{2}},\forall m\geq 0$$

Proof. By Radon's inequality we obtain that:

$$\sum_{cyc}a^{m+1}\overset{Radon}{\geq}\frac{\left(\sum a\right)^{m+1}}{3^m}\geq\frac{\left(2\sqrt[4]{27}\cdot\sqrt{F}\right)^{m+1}}{3^m}=2^{m+1}\cdot 3^{\frac{3-m}{4}}\cdot F^{\frac{m+1}{2}}$$

Remarks:

If $m=1$, then we obtain again the Ionescu-Weitzenbock's inequality.

If $m=3$, then we obtain $a^4+b^4+c^4\geq 16F^2;(12)\ (Goldner's)$

Proposition 5. (New inequality).

$$\left(\sqrt{ab}\right)^{m+1}+\left(\sqrt{bc}\right)^{m+1}+\left(\sqrt{ca}\right)^{m+1}\geq 2^{m+1}\left(\sqrt[4]{3}\right)^{3-m}\cdot F^{\frac{m+1}{2}},\forall m\geq 0$$

Proof. If $\Delta A_1B_1C_1 \equiv \Delta ABC$ and $\Delta A_2B_2C_2 \equiv \Delta BCA$, then from Proposition 2, we obtain $\sqrt{ab}+\sqrt{bc}+\sqrt{ca} \geq 2\sqrt[4]{27}\cdot\sqrt{F}$, so from Radon's inequality, we deduce that

$$\left(\sqrt{ab}\right)^{m+1}+\left(\sqrt{bc}\right)^{m+1}+\left(\sqrt{ca}\right)^{m+1} \overset{Radon}{\geq} \frac{\left(\sum\sqrt{ab}\right)^{m+1}}{3^m} \geq$$

$$\geq \frac{\left(2\cdot\sqrt[4]{27}\cdot\sqrt{F}\right)^{m+1}}{3^m} = 2^{m+1}\left(\sqrt[4]{3}\right)^{3-m}\cdot F^{\frac{m+1}{2}}$$

Remarks:

If $m=1$, then we obtain again the inequality of Gordon.

If $m=3$, then we obtain an other inequality of Gordon, i.e.

$$a^2b^2+b^2c^2+c^2a^2 \geq 16F^2 \text{ (4.12 from [1])}$$

REFERENCES

[1] O. Bottema, R. Ž. Djordjević, R. R. Janić, D. S. Mitrinović, P. M. Vasić, *Geometric inequalities*, Wolters-Noordhoff Publishing, Groningen, 1969.

[2] D.M. Bătineţu-Giurgiu, N. Stanciu, *Asupra unor inegalităţi într-un triunghi*, Recreaţii Matematice, Nr. 1, Ianuarie - Iunie, 2021, pp. 31-32.

[3] Romanian Mathematical Magazine-www.ssmrmh.ro