INEQUALITIES IN TRIANGLES

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Abstract. In this paper we will demonstrate some general inequalities. By particularization, we obtain new but also known inequalities.

1. Introduction

In this section, we recall some notations and results we will use in this paper.
For a given triangle ABC, we note by a, b, c the side lengths, s the semiperimeter, F the area, R the radius of the circumscribed circle, r the radius of the inscribed circle and r_a, r_b, r_c the radii of the exinscribed circles of triangle.

The following inequalities are well known.

Theorem 1.1 (Bătineţu-Giurgiu's Inequality). *If $x, y, z > 0$, then in the triangle ABC the inequality*

$$\frac{x+y}{z}a^2 + \frac{y+z}{x}b^2 + \frac{z+x}{y}c^2 \geq 8\sqrt{3}\,F \tag{1.1}$$

holds.

Theorem 1.2 (Bergström's Inequality). *If $x_n \in \mathbb{R}$ and $a_k > 0$, $k \in \{1, 2, ..., n\}$, then*

$$\frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} + \cdots + \frac{x_n^2}{a_n} \geq \frac{(x_1 + x_2 + \cdots + x_n)^2}{a_1 + a_2 + \cdots + a_n}. \tag{1.2}$$

The equality holds if and only if $\dfrac{x_1}{a_1} = \dfrac{x_2}{a_2} = \cdots = \dfrac{x_n}{a_n}$.

Theorem 1.3 (Doucet's Inequality). *In the triangle ABC, the inequality*

$$s\sqrt{3} \leq 4R + r \tag{1.3}$$

holds.

Theorem 1.4 (Ionescu-Weitzenböck's Inequality). *In the triangle ABC, the inequality*

$$a^2 + b^2 + c^2 \geq 4F\sqrt{3} \tag{1.4}$$

holds.

Theorem 1.5 (Klamkin's Inequality). *If $x, y, z > 0$ and a, b, c are the sides lengths of the triangle ABC, then*

$$(xa^2 + yb^2 + zc^2)^2 \geq 16(xy + yz + zx)F^2 \tag{1.5}$$

2010 *Mathematics Subject Classification.* 51M04, 51M15.
Key words and phrases. Inequalities in triangles.

Theorem 1.6 (Tsintsifas's Inequality)**.** *If $x, y, z > 0$, in the triangle ABC, the inequality*

$$\frac{x}{y+z}\,a^2 + \frac{y}{z+x}\,b^2 + \frac{z}{x+y}\,c^2 \geq 2\sqrt{3}\,F \tag{1.6}$$

holds.

2. Main results and applications

In the problem 11857 from the American Mathematical Monthly 2015, page 700, problem proposed by Mehmet Sahin, the triangle ABC it is associated with a triangle $A_1B_1C_1$ with the side lengths $a_1 = \sqrt{a}$, $b_1 = \sqrt{b}$, $c_1 = \sqrt{c}$ and F_1 his area. In this problem it proves that

$$F_1 = \frac{1}{2}\sqrt{r(r_a + r_b + r_c)}\,. \tag{2.1}$$

Theorem 2.1. *The following inequality*

$$F_1 \geq \frac{\sqrt[4]{3}}{2}\sqrt{F} \tag{2.2}$$

holds. The inequality takes place if and only if the triangle ABC is equilateral.

The first proof. Taking that $r_a + r_b + r_c = 4R + r$ and Doucet's Inequality into account, we have

$$F_1 = \frac{1}{2}\sqrt{r(r_a + r_b + r_c)} = \frac{1}{2}\sqrt{r(4R + r)} \geq \frac{1}{2}\sqrt{r\,s\sqrt{3}} = \frac{\sqrt[4]{3}}{2}\sqrt{rs}\,,$$

from where (2.2) is obtained.

The second proof. In this solution, we will use that

$$(r_a + r_b + r_c)^2 \geq 3(r_ar_b + r_br_c + r_cr_a) \quad\text{and}\quad r_ar_b + r_br_c + r_cr_a = s^2.$$

We have

$$F_1 = \frac{1}{2}\sqrt{r(r_a + r_b + r_c)} = \frac{1}{2}\sqrt[4]{r^2(r_a + r_b + r_c)^2} \geq$$

$$\geq \frac{1}{2}\sqrt[4]{r^2\cdot 3(r_ar_b + r_br_c + r_cr_a)} = \frac{1}{2}\sqrt[4]{3}\,\sqrt[4]{r^2s^2} = \frac{\sqrt[4]{3}}{2}\sqrt{rs}\,,$$

from where (2.2) follows.

Corollary 2.1. *If $x, y, z > 0$, then the inequality*

$$\frac{x}{y+z}\,a + \frac{y}{z+x}\,b + \frac{z}{x+y}\,c \geq \sqrt[4]{27}\,\sqrt{F} \tag{2.3}$$

occurs in the triangle ABC.

Proof. In the triangle $A_1B_1C_1$ associated with the triangle ABC, where $a_1 = \sqrt{a}$, $a_2 = \sqrt{b}$, $a_3 = \sqrt{c}$, using Tsintsifas's Inequality and inequality (2.2), we have $\sum\limits_{cyc}\frac{x}{y+z}a = \sum\limits_{cyc}\frac{x}{y+z}a_1^2 \geq 2\sqrt{3}\,F_1 \geq 2\sqrt{3}\frac{\sqrt[4]{3}}{2}\sqrt{F}$, from where (2.3) follows. □

Corollary 2.2. *For any $x, y, z > 0$ and any triangle ABC, the inequality*

$$\frac{x}{y+z}\,a^2 + \frac{y}{z+x}\,b^2 + \frac{z}{x+y}\,c^2 \geq 2\sqrt{3}\,F \tag{2.4}$$

holds.

Proof. Applying Bergström's Inequality and Klamkin's Inequality, respectively, in triangle $A_1B_1C_1$ we we have

$$\sum_{cyc}\frac{x}{y+z}a^2=\sum_{cyc}\frac{x^2}{xy+xz}a_1^4=\sum_{cyc}\frac{(xa_1^2)^2}{xy+xz}\geq\frac{\left(\sum_{cyc}xa_1^2\right)^2}{\sum_{cyc}(xy+z)}$$

$$=\frac{(xa_1^2+yb_1^2+zc_1^2)^2}{2(xy+yz+zx)^2}\geq\frac{16(xy+yz+zx)F_1^2}{2(xy+yz+zx)}=8F_1^2.$$

Taking inequality from (2.2) into account, we obtain inequality from the hypothesis. □

Remark 2.1. The proof in the Corollary 2.2. is a new proof of Tsintsifas's Inequality.

Corollary 2.3. *For any $x,y,z>0$ and any triangle ABC, the inequality*

$$\frac{x+y}{z}a+\frac{y+z}{x}b+\frac{z+x}{y}c\geq 4\sqrt[4]{27}\cdot\sqrt{F} \tag{2.5}$$

holds.

Proof. In the proof, we will use Bătineţu-Giurgiu's Inequality and inequality from (2.2) in triangle $A_1B_1C_1$. We have

$$\sum_{cyc}\frac{x+y}{z}a=\sum_{cyc}\frac{x+y}{z}a_1^2\geq 8\sqrt{3}\,F_1\geq 8\sqrt{3}\frac{\sqrt[4]{3}}{2}\sqrt{F},$$

from where (2.5) follows. □

Theorem 2.2. *Let $\alpha,\beta,\gamma,\delta$ be a real numbers. Prove that*

a) $\frac{p}{q+r}\alpha+\frac{q}{r+p}\beta+\frac{r}{p+q}\gamma\geq\delta$, *for any $p,q,r>0$ equivalent to*
$\frac{x+y}{z}\alpha+\frac{y+z}{x}\beta+\frac{z+x}{y}\gamma\geq 2\delta+\alpha+\beta+\gamma$, *for any $x,y,z>0$ with $-x+y+z>0$, $x-y+z>0$ and $x+y-z>0$.*

b) $\frac{p}{q+r}\alpha+\frac{q}{r+\beta}\beta+\frac{r}{p+q}\gamma\leq\delta$, *for any $p,q,r>0$ equivalent to*
$\frac{x+y}{z}\alpha+\frac{y+z}{x}\beta+\frac{z+x}{y}\gamma\leq 2\delta+\alpha+\beta+\gamma$, *for any $x,y,z>0$ with $-x+y+z>0$, $x-y+z>0$ and $x+y-z>0$.*

Proof. *a)* We note $q+r=z$, $r+p=x$, $p+q=y$ and then $p=\frac{x+y-z}{2}$, $q=\frac{-x+y+z}{2}$, $r=\frac{x-y+z}{2}$.

Suppose that the inequality $\frac{p}{q+r}\alpha+\frac{q}{r+p}\beta+\frac{r}{p+q}r\geq\delta$, for any $p,q,r>0$ holds. Let be any numbers $x,y,z>0$ with $-x+y+z>0$, $x-y+z>0$, $x+y-z>0$. From the relations above, it results that $p,q,r>0$ and than the inequality $\frac{p}{q+r}x+\frac{q}{r+p}\beta+\frac{r}{p+q}\gamma\geq\delta$ holds.

In this inequality replacing p, q, r with the relations above and after calculations, we obtain the inequality

$$\frac{x+y}{z}\alpha + \frac{y+z}{x}\beta + \frac{z+x}{y}\gamma \ge 2\delta + \alpha + \beta + \gamma.$$

Reciprocally, we suppose that the inequality $\frac{x+y}{z}\alpha + \frac{y+z}{x}\beta + \frac{z+x}{y}\gamma \ge 2\delta + \alpha + \beta + \gamma$, for any $x, y, z > 0$ with $-x+y+z > 0$, $x-y+z > 0$, $x+y-z > 0$ is true.

Let p, q, r arbitrary real numbers strictly positive. Then, taking the relations above into account, we have $x = r+p > 0$, $y = p+q > 0$, $z = q+r$ and $-x+y+z = 2q > 0$, $x-y+z = 2r > 0$ and $x+y-z = 2p > 0$.

Then, the inequality $\frac{x+y}{z}\alpha + \frac{y+z}{x}\beta + \frac{z+x}{y}\gamma \ge 2\delta + \alpha + \beta + \gamma$ holds.

In this true inequality, replacing x, y, z with the relations above and after calculations, the inequality $\frac{p}{q+r}\alpha + \frac{q}{r+p}\beta + \frac{r}{p+q}\gamma \ge \delta$ follows.

Similarly is demonstrated b). □

Theorem 2.3. *In the triangle ABC, the following inequality*

$$\frac{x+y}{z}a^2 + \frac{y+z}{x}b^2 + \frac{z+x}{y}c^2 \ge 4\sqrt{3}\,F + a^2 + b^2 + c^2, \tag{2.6}$$

holds for any $x, y, z > c$, with $-x+y+z > 0$, $x-y+z > 0$, $x+y-z > 0$.

Proof. Taking Tsintsifas's Inequality and Theorem 2.2 into account, for $\alpha = a^2$, $\beta = b^2$, $\gamma = c^2$, the inequality (2.6) follows. □

Remark 2.2. In the (2.6) inequality, taking Ionescu–Weitzenböck's Inequality into account, we obtain Bătineţu-Giurgiu's Inequality.

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