Harmonic Number Approximation

Introduction

In this article, we will find a simple expression for the n^{th} harmonic number (H_n) using a special integral, the applications of which can be seen in [1], [2] and [3].

Relation between $G(x)$ and H_n

We know from [3] the definition of $G(x)$, that is,

$$G(x) = \int_0^1 \ln(t)\ln(1-t^x)dt$$

where, x is any non-zero positive real number.
From [2], it is known that,

$$G(x) = \sum_{k=1}^{\infty} \frac{1}{k(kx+1)^2} = \gamma + \psi(1+1/x) - \frac{\psi'(1+1/x)}{x}$$

and

$$\int_0^x G(1/t)dt = \gamma x + 2\ln(x\Gamma(x)) - x\psi(1+x)$$

where, x is any non-zero positive real number, γ is the Euler-Mascheroni constant, $\Gamma(x)$ is the gamma function and $\psi(x)$ is the digamma function.
Substituting $x = 1/2$ in the above integral, we obtain,

$$\int_0^{1/2} G(1/t)dt = \frac{\gamma}{2} + 2\ln\left(\frac{\sqrt{\pi}}{2}\right) - \frac{\psi(3/2)}{2} \implies \int_0^{1/2} \sum_{k=1}^{\infty} \frac{t^2}{k(k+t)^2}dt = \gamma + \ln\left(\frac{\pi}{2}\right) - 1$$

simplifying the above expression further, we have,

$$\sum_{k=1}^{\infty} \left(\frac{1}{2k} + 2\ln\left(\frac{k}{k+1/2}\right) + \frac{1}{2k+1} \right) = \gamma + \ln\left(\frac{\pi}{2}\right) - 1$$

writing the above sum as the limit of a sum, we obtain,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{2k} + 2\ln\left(\frac{k}{k+1/2}\right) + \frac{1}{2k+1} \right) = \gamma + \ln\left(\frac{\pi}{2}\right) - 1$$

using the definition of harmonic numbers, we have,

$$\lim_{n\to\infty}\left(\frac{H_n}{2}+2\ln(n!)-2\ln\left(\frac{(2n+2)!}{2^{2n+1}(n+1)!}\right)+H_{2n+2}-\frac{H_{n+1}}{2}-1\right)=\gamma+\ln\left(\frac{\pi}{2}\right)-1$$

observe that,

$$\frac{H_n}{2}+H_{2n+2}-\frac{H_{n+1}}{2}=H_{2n+1}$$

thus for large values of n, we obtain the following approximation,

$$H_{2n+1}\approx\gamma+\ln\left(\frac{\pi}{2}\right)+2\ln\left(\frac{(2n+2)!}{2^{2n+1}n!(n+1)!}\right)$$

Corollary

Substituting $x=1$ in the integral of $G(1/t)$, we obtain a relatively simpler approximation,

$$H_n\approx\gamma+\ln(n+1)-\frac{1}{2n+2}$$

References

[1] Angad Singh *Infinite Series* (Romanian Mathematical Magazine, 2020).
[2] Angad Singh *An Outstanding Integral* (Romanian Mathematical Magazine, 2020).
[3] Angad Singh *A note on Euler-Mascheroni constant* (Octogon Mathematical Magazine, 2020).

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ANGAD SINGH
email-id: angadsingh1729@gmail.com