A SIMLE PROOF FOR HEINZ'S INEQUALITY

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ABSTRACT. In this paper is presented a simple proof for Heinz's inequality and a few applications.

HEINZ'S INEQUALITY ($n = 2$)

If $x, y > 0; \alpha \in [0, 1]$ then:

$$\sqrt{xy} \leq \frac{x^{1-\alpha}y^{\alpha} + x^{\alpha}y^{1-\alpha}}{2} \leq \frac{x+y}{2} \tag{1}$$

Proof.

$$\text{Let be } f : [0, 1] \to \mathbb{R}; f(\alpha) = x^{1-\alpha}y^{\alpha} + x^{\alpha}y^{1-\alpha}$$

$$f'(\alpha) = -x^{1-\alpha}y^{\alpha}\log x + x^{1-\alpha}y^{\alpha}\log y + x^{\alpha}y^{1-\alpha}\log x - x^{\alpha}y^{1-\alpha}\log y$$

$$f'(\alpha) = x^{1-\alpha}y^{\alpha}(-\log x + \log y) + x^{\alpha}y^{1-\alpha}(\log x - \log y)$$

$$f'(\alpha) = \left(\log\frac{x}{y}\right)\cdot(x^{\alpha}y^{1-\alpha} - x^{1-\alpha}y^{\alpha})$$

$$f'(\alpha) = 0 \Rightarrow y\left(\frac{x}{y}\right)^{\alpha} = x\left(\frac{y}{x}\right)^{\alpha} \Rightarrow \left(\frac{x}{y}\right)^{\alpha}\cdot\left(\frac{y}{x}\right)^{-\alpha} = \frac{x}{y}$$

$$\left(\frac{x}{y}\right)^{2\alpha} = \frac{x}{y} \Rightarrow 2\alpha = 1, \alpha = \frac{1}{2}$$

$$f(0) = \frac{x+y}{2}; f(1) = \frac{x+y}{2}; f\left(\frac{1}{2}\right) = \sqrt{xy}$$

α	0		$\frac{1}{2}$		1
$f'(\alpha)$	$-$	$- - - - - - - -$	0	$+ + + + + + + +$	$+$
$f(\alpha)$	$\frac{x+y}{2}$	$\searrow$	$\sqrt{xy}$	$\nearrow$	$\frac{x+y}{2}$

$$\sqrt{xy} \leq f(\alpha) \leq \frac{x+y}{2}$$

□

HEINZ'S INEQUALITY ($n = 3$)

If $x, y, z > 0; \alpha \in [0, 1]$ then:

$$\sqrt[3]{xyz} \leq \frac{x^{1-\alpha}y^{\alpha} + y^{1-\alpha}z^{\alpha} + z^{1-\alpha}x^{\alpha}}{3} \leq \frac{x+y+z}{3} \tag{2}$$

GENERAL HEINZ'S INEQUALITY

If $x_1, x_2, \ldots, x_n > 0; \alpha \in [0, 1]; n \in \mathbb{N}; n \geq 2$ then:

$$\sqrt[3]{x_1 x_2 \cdot \ldots \cdot x_n} \leq \frac{x_1^{1-\alpha}x_2^{\alpha} + x_2^{1-\alpha}x_3^{\alpha} + \ldots + x_n^{1-\alpha}x_1^{\alpha}}{n} \leq \frac{x_1 + x_2 + \ldots + x_n}{n} \tag{3}$$

Corollary 1:
If $x, y > 0$ then:

$$\sqrt{xy} \leq \frac{\sqrt[3]{x^2y} + \sqrt[3]{xy^2}}{2} \leq \frac{x+y}{2} \tag{4}$$

Proof.
In (1) we take $\alpha = \frac{1}{3}$. □

Corollary 2:
If $x, y, z > 0$ then:

$$\sqrt[3]{xyz} \leq \frac{\sqrt[3]{x^2y} + \sqrt[3]{y^2z} + \sqrt[3]{z^2x}}{3} \leq \frac{x+y+z}{3} \tag{5}$$

Proof.
In (2) we take $\alpha = \frac{1}{3}$. □

Corollary 3:
If $x, y, z > 0$ then:

$$\sqrt[3]{xyz} \leq \frac{\sqrt[4]{x^3y} + \sqrt[4]{y^3z} + \sqrt[4]{z^3x}}{3} \leq \frac{x+y+z}{3} \tag{6}$$

Proof.
In (2) we take $\alpha = \frac{1}{4}$ □

Application 1.
If $a \in (0, \frac{\pi}{2})$ then:

$$\sqrt[3]{\sin^4 a \cos a} + \sqrt[3]{\sin a \cos^4 a} \leq 1$$

Proof.
In (4) we take $x = \sin^2 a; y = \cos^2 a$ □

Application 2.
If $a, b \in [0, \frac{\pi}{2}]$ then:

$$\sqrt[3]{\sin^4 a \cos^2 a \sin^2 b} + \sqrt[3]{\sin^4 b \cos^2 b \cos^6 a} + \sqrt[3]{\sin^2 a \cos^4 a \cos^4 b} \leq 1$$

Proof.
In (5) we take $x = \sin^2 a; y = \cos^2 a \sin^2 b; z = \cos^2 a \cos^2 b$. □

Application 3.
If $x, y, z > 0; x + y + z = 3$ then:

$$3\sqrt[3]{xyz} \leq \sqrt[3]{x^2y} + \sqrt[3]{y^2z} + \sqrt[3]{z^2x} \leq 3$$

Proof.
In (5) replace $x + y = z = 3$. □

Application 4.
If $x, y, z > 0; x + y = z = 3$ then:

$$3\sqrt[3]{xyz} \leq \sqrt[4]{x^3y} + \sqrt[4]{y^3z} + \sqrt[4]{z^3x} \leq 3$$

Proof.
In (6) replace $x + y + z = 3$ □

References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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