A SIMPLE PROOF FOR CALLEBAUT'S INEQUALITY

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Abstract. In this paper is presented a simple proof for Callebaut's inequality and a few applications.

CALLEBAUT'S INEQUALITY ($n = 2$)

If $a_1, a_2, b_1, b_2 > 0; 0 \le x \le y \le 1$ then:

$$(a_1a_2 + b_1b_2)^2 \le (a_1^{1+x}b_1^{1-x} + a_2^{1+x}b_2^{1-x})(a_1^{1-x}b_1^{1+x} + a_2^{1-x}b_2^{1+x}) \le$$
$$\le (a_1^{1+y}b_1^{1-y} + a_2^{1+y}b_2^{1-y})(a_1^{1-y}b_1^{1+y} + a_2^{1-y}b_2^{1+y}) \le$$
$$\le (a_1^2 + a_2^2)(b_1^2 + b_2^2) \tag{1}$$

Proof.

The inequality:

$$(a_1^{1+x}b_1^{1-x} + a_2^{1+x}b_2^{1-x})(a_1^{1-x}b_1^{1+x} + a_2^{1-x}b_2^{1+x}) \le$$
$$\le (a_1^{1+y}b_1^{1-y} + a_2^{1+y}b_2^{1-y})(a_1^{1-y}b_1^{1+y} + a_2^{1-y}b_2^{1+y}) \tag{2}$$

can be written:

$$a_1^2b_1^2 + a_2^2b_2^2 + a_1^{1+x}b_1^{1-x}a_2^{1-x}b_2^{1+x} + a_2^{1+x}b_2^{1-x}a_1^{1-x}b_1^{1+x} \le$$
$$\le a_1^2b_1^2 + a_2^2b_2^2 + a_1^{1+y}b_1^{1-y}a_2^{1-y}b_2^{1+y} + a_2^{1+y}b_2^{1-y}a_1^{1-y}b_1^{1+y}$$
$$a_1^{1+x}b_1^{1-x}a_2^{1-x}b_2^{1+x} + a_2^{1+x}b_2^{1-x}a_1^{1-x}b_1^{1+x} \le$$
$$\le a_1^{1+y}b_1^{1-y}a_2^{1-y}b_2^{1+y} + a_2^{1+y}b_2^{1-y}a_1^{1-y}b_1^{1+y}$$

By dividing with $a_1a_2b_1b_2$ we obtain:

$$a_1^xb_2^xa_2^{-x}b_1^{-x} + a_2^xb_2^{-x}a_1^{-x}b_1^x \le$$
$$\le a_1^yb_1^{-y}a_2^{-y}b_2^y + a_2^yb_2^{-y}a_1^{-y}b_1^y$$
$$\left(\frac{a_1b_2}{a_2b_1}\right)^x + \left(\frac{a_2b_1}{a_1b_2}\right)^x \le \left(\frac{a_1b_2}{a_2b_1}\right)^y + \left(\frac{a_2b_1}{a_1b_2}\right)^y$$

Denote:

$$p = \frac{a_1b_2}{a_2b_1}$$

Remains to prove:

$$p^x + p^{-x} \le p^y + p^{-y}$$

Let be $f : [0, \infty) \to \mathbb{R}; f(x) = p^x + p^{-x}$

$$f'(x) = \log p \cdot (p^x - p^{-x})$$

If $p > 1; \log p > 0$ and by $x \ge 0; x \ge -x$

$$p^x \ge p^{-x} \Rightarrow p^x - p^{-x} \ge 0 \Rightarrow f'(x) \ge 0$$

If $p \in (0,1); \log p < 0$ and by $x \geq 0; x \geq -x$

$$p^x \leq p^{-x} \Rightarrow p^x - p^{-x} \leq 0 \Rightarrow f'(x) \geq 0$$

f increasing on $[0,\infty) \Rightarrow f(x) \leq f(y)$ which is equivalent with (2).
We take in (2): $x = 0$

$$(a_1a_2 + b_1b_2)(a_1a_2 + b_1b_2) \leq$$

$$\leq (a_1^{1+y}b_1^{1-y} + a_2^{1+y}b_2^{1-y})(a_1^{1-y}b_1^{1+y} + a_2^{1-y}b_2^{1+y}) \tag{3}$$

We take in (3): $y = x$

$$(a_1a_2 + b_1b_2)^2 \leq (a_1^{1+x}b_1^{1-x} + a_2^{1+x}b_2^{1-x})(a_1^{1-x}b_1^{1+x} + a_2^{1-x}b_2^{1+x})$$

We take in (2): $y = 1$

$$(a_1^{1+x}b_1^{1-x} + a_2^{1+x}b_2^{1-x})(a_1^{1-x}b_1^{1+x} + a_2^{1-x}b_2^{1+x}) \leq (a_1^2 + a_2^2)(b_1^2 + b_2^2) \tag{4}$$

By (2);(3);(4) it is obtained (1).
CALLEBAUT'S INEQUALITY ($n = 3$)
If $a_1, a_2, a_3, b_1, b_2, b_3 > 0; 0 \leq x \leq y \leq 1$ then:

$$(a_1b_1 + a_2b_2 + a_3b_3)^2 \leq$$

$$\leq (a_1^{1+x}b_1^{1-x} + a_2^{1+x}b_2^{1-x} + a_3^{1+x}b_3^{1-x})(a_1^{1-x}b_1^{1+x} + a_2^{1-x}b_2^{1+x} + a_3^{1-x}b_3^{1+x}) \leq$$

$$\leq (a_1^{1+y}b_1^{1-y} + a_2^{1+y}b_2^{1-y} + a_3^{1+y}b_3^{1-y})(a_1^{1-y}b_1^{1+y} + a_2^{1-y}b_2^{1+y} + a_3^{1-y}b_3^{1+y}) \leq$$

$$\leq (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) \tag{5}$$

□

GENERAL CALLEBAUT'S INEQUALITY
If $a_i > 0; b_i > 0; i \in \overline{1,n}; 0 \leq x \leq y \leq 1$ then:

$$\left(\sum_{i=1}^n a_ib_i\right)^2 \leq \left(\sum_{i=1}^n a_i^{1+x}b_i^{1-x}\right)\left(\sum_{i=1}^n a_i^{1-x}b_i^{1+x}\right) \leq$$

$$\leq \left(\sum_{i=1}^n a_i^{1+y}b_i^{1-y}\right)\left(\sum_{i=1}^n a_i^{1-y}b_i^{1+y}\right) \leq \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right)$$

Corollary 1
If $a_1, a_2, b_1, b_2 > 0$ then:

$$(a_1a_2 + b_1b_2)^2 \leq (a_1\sqrt{a_1b_1} + a_2\sqrt{a_2b_2})(b_1\sqrt{a_1b_1} + b_2\sqrt{a_2b_2}) \leq$$

$$\leq (a_1^2 + a_2^2)(b_1^2 + b_2^2)$$

Proof.
We take $x = y = \frac{1}{2}$ in (1) □

Corollary 2
If $a_1, a_2, b_1, b_2 > 0$ then

$$(a_1a_2 + b_1b_2)^2 \leq (\sqrt[3]{a_1^4b_1^2} + \sqrt[3]{a_2^4b_1^2})(\sqrt[3]{a_1^2b_1^4} + \sqrt[3]{a_2^2b_2^4}) \leq (a_1^2 + a_2^2)(b_1^2 + b_2^2)$$

Proof.
We take $x = y = \frac{1}{3}$ in (1)

□

Corollary 3

If $a_1, a_2, b_1, b_2 > 0$ then:

$$(a_1a_2 + b_1b_2)^2 \le (\sqrt[3]{a_1^4b_1^2} + \sqrt[3]{a_2^4b_1^2})(\sqrt[3]{a_1^2b_1^4} + \sqrt[3]{a_2^2b_2^4}) \le$$
$$\le (a_1\sqrt{a_1b_1} + a_2\sqrt{a_2b_2})(b_1\sqrt{a_1b_1} + b_2\sqrt{a_2b_2}) \le$$
$$\le (a_1^2 + a_2^2)(b_1^2 + b_2^2)$$

Proof.

We take $x = \frac{1}{3}; y = \frac{1}{2}$ in (1).

□

Corollary 4

If $a_1, a_2, a_3, b_1, b_2, b_3 > 0$ then:

$$(a_1b_1 + a_2b_2 + a_3b_3)^2 \le$$
$$\le (a_1\sqrt{a_1b_1} + a_2\sqrt{a_2b_2} + a_3\sqrt{a_3b_3})(b_1\sqrt{a_1b_1} + b_2\sqrt{a_2b_2} + b_3\sqrt{a_3b_3}) \le$$
$$\le (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$$

Proof.

We take $x = \frac{1}{2}$ in (5).

□

Corollary 5

If $a_1, a_2, a_3, b_1, b_2, b_3 > 0$ then:

$$(a_1b_1 + a_2b_2 + a_3b_3)^2 \le$$
$$\le (\sqrt[3]{a_1^4b_1^2} + \sqrt[3]{a_2^4b_2^2} + \sqrt[3]{a_3^4b_3^2})(\sqrt[3]{a_1^2b_1^4} + \sqrt[3]{a_2^2b_2^4} + \sqrt[3]{a_3^2b_3^4}) \le$$
$$\le (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$$

Proof.

We take $x = \frac{1}{3}$ in (5).

□

Corollary 6

If $a_1, a_2, a_3, b_1, b_2, b_3 > 0$ then:

$$(a_1b_1 + a_2b_2 + a_3b_3)^2 \le$$
$$\le (\sqrt[3]{a_1^4b_1^2} + \sqrt[3]{a_2^4b_2^2} + \sqrt[3]{a_3^4b_3^2})(\sqrt[3]{a_1^2b_1^4} + \sqrt[3]{a_2^2b_2^4} + \sqrt[3]{a_3^2b_3^4}) \le$$
$$\le (a_1\sqrt{a_1b_1} + a_2\sqrt{a_2b_2} + a_3\sqrt{a_3b_3})(b_1\sqrt{a_1b_1} + b_2\sqrt{a_2b_2} + b_3\sqrt{a_3b_3}) \le$$
$$\le (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$$

Proof.

We take $x = \frac{1}{3}; y = \frac{1}{2}$ in (5).

□

References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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