Transactions of the Nigerian Association of Mathematical Physics
*Volume 16, (July–Sept. 2021 Issue), pp*73 – 98
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New Integral Representations for the Bessel Functions

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Abstract

In this paper, new integral representations for the Bessel J and I functions were presented and their results were used to derive an expression for the Modified Bessel K function.

Keywords: Bessel functions, Modified Bessel functions, Hypergeometric function, Partial differential equation, Gamma function, Legendre's duplication formula, beta functions.

Mathematics Subject Classification (2020): 33C10, 33C05, 33B15.

1 Introduction

The theory of Bessel functions is intimately connected with the theory of a certain type of differential equation of the first order, known as Riccati's equation. In fact, a Bessel function is usually defined as a particular solution of a linear differential equation of the second order (known as Bessel's equation) which is derived from Riccati's equation by elementary transformation. [1]

Bessel functions, first defined by the mathematician Daniel Bernoulli and then generalized by Friedrich Bessel, are canonical solutions $y(x)$ of Bessel's differential equation

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+x\frac{\mathrm{d}y}{\mathrm{d}x}+(x^2-\alpha^2)y=0$$

Bessel functions of the first kind, denoted as $J_\alpha(x)$, are solutions of Bessel's differential equation. For integer or positive α, Bessel functions of the first kind are finite at the origin; while for negative non-integer α, Bessel functions of the first kind diverge as x approaches zero. It is possible to define the function by its series expansion around $x=0$, which can be found by applying the Frobenius method to Bessel's equation

$$J_\alpha(x)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+\alpha+1)}\left(\frac{x}{2}\right)^{2m+\alpha} \qquad \forall x\in\mathbb{C}$$

where $\Gamma(z)$ is the gamma function, a shifted generalization of the factorial function to non-integer values. [5]

2 Integral representations for $J_\alpha(x)$ and $J_0(x)$

$$J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha}$$

$$\begin{aligned}
J_\alpha(x) &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha} \\
&= \frac{1}{\Gamma(\alpha)} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(m+1)\Gamma(\alpha)}{m!\Gamma(m+1)\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha} \\
&= \frac{1}{\Gamma(\alpha)} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(m+1)\Gamma(\alpha)}{(m!)^2\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha} \\
&= \frac{1}{\Gamma(\alpha)} \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \beta(m+1,\alpha) \left(\frac{x}{2}\right)^{2m+\alpha} \\
&= \frac{1}{\Gamma(\alpha)} \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \int_0^1 t^m (1-t)^{\alpha-1}\, \mathrm{d}t \left(\frac{x}{2}\right)^{2m+\alpha} \\
&= \frac{1}{\Gamma(\alpha)} \left(\frac{x}{2}\right)^{\alpha} \int_0^1 \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} t^m \left(\frac{x}{2}\right)^{2m} (1-t)^{\alpha-1}\, \mathrm{d}t \\
&= \frac{1}{\Gamma(\alpha)} \left(\frac{x}{2}\right)^{\alpha} \int_0^1 \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{x\sqrt{t}}{2}\right)^{2m} (1-t)^{\alpha-1}\, \mathrm{d}t
\end{aligned}$$

We notice that

$$\begin{aligned}
J_0(x) &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+0+1)} \left(\frac{x}{2}\right)^{2m+0} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+1)} \left(\frac{x}{2}\right)^{2m} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{x}{2}\right)^{2m}
\end{aligned}$$

$$\begin{aligned}
\implies J_\alpha(x) &= \frac{1}{\Gamma(\alpha)} \left(\frac{x}{2}\right)^{\alpha} \int_0^1 J_0(x\sqrt{t})\, (1-t)^{\alpha-1}\, \mathrm{d}t \\
&= \frac{1}{\Gamma(\alpha)} \left(\frac{x}{2}\right)^{\alpha} \int_0^1 J_0\left(x\sqrt{1-t}\right) t^{\alpha-1} \mathrm{d}t \qquad (1)
\end{aligned}$$

By Legendre's duplication formula,

$$\Gamma\left(m+\frac{1}{2}\right) = \frac{\Gamma(2m)\sqrt{\pi}}{\Gamma(m)2^{2m-1}}$$

$$\implies \frac{\Gamma\left(m+\frac{1}{2}\right)2^{2m-1}}{\Gamma(2m)\sqrt{\pi}} = \frac{1}{\Gamma(m)}$$

$$\implies J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+1)} \left(\frac{x}{2}\right)^{2m}$$

$$
\begin{aligned}
&= \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1)^2} \left(\frac{x}{2}\right)^{2m} \\
&= \sum_{m=1}^{\infty} \frac{(-1)^m}{\Gamma(m)\Gamma(m)} \left(\frac{x}{2}\right)^{2m-2} \\
&= \sum_{m=1}^{\infty} \frac{(-1)^{m-1}\Gamma\left(m+\frac{1}{2}\right)2^{2m-1}}{\Gamma(2m)\Gamma(m)\sqrt{\pi}} \left(\frac{x}{2}\right)^{2m-2} \\
&= \sum_{m=1}^{\infty} \frac{2m(-1)^{m-1}\Gamma\left(m+\frac{1}{2}\right)2^{2m-1}}{2m\Gamma(2m)\Gamma(m)\sqrt{\pi}} \left(\frac{x}{2}\right)^{2m-2} = \sum_{m=1}^{\infty} \frac{2m(-1)^{m-1}\Gamma\left(m+\frac{1}{2}\right)2^{2m-1}}{\Gamma(2m+1)\Gamma(m)\sqrt{\pi}} \left(\frac{x}{2}\right)^{2m-2} \\
&= \sum_{m=1}^{\infty} \frac{2m(-1)^{m-1}\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(2m+1)} \cdot \frac{2^{2m-1}}{\Gamma\left(m+\frac{1}{2}\right)\Gamma(m)\sqrt{\pi}} \left(\frac{x}{2}\right)^{2m-2} \\
&= \sum_{m=1}^{\infty} 2m(-1)^{m-1}\beta\left(m+\frac{1}{2}, m+\frac{1}{2}\right) \cdot \frac{2^{2m-1}}{\Gamma\left(m+\frac{1}{2}\right)\Gamma(m)\sqrt{\pi}} \left(\frac{x}{2}\right)^{2m-2} \\
&= \sum_{m=1}^{\infty} 2m(-1)^{m-1}\beta\left(m+\frac{1}{2}, m+\frac{1}{2}\right) \cdot \frac{2^{2m-1}}{\frac{\Gamma(2m)\sqrt{\pi}}{\Gamma(m)2^{2m-1}} \cdot \Gamma(m)\sqrt{\pi}} \left(\frac{x}{2}\right)^{2m-2} \\
&= \sum_{m=1}^{\infty} 2m(-1)^{m-1}\beta\left(m+\frac{1}{2}, m+\frac{1}{2}\right) \cdot \frac{2^{4m-2}}{\pi\Gamma(2m)} \left(\frac{x}{2}\right)^{2m-2} \\
&= \frac{1}{2\pi} \sum_{m=1}^{\infty} m(-1)^{m-1}\beta\left(m+\frac{1}{2}, m+\frac{1}{2}\right) \cdot \frac{1}{\Gamma(2m)} (2x)^{2m} \left(\frac{x}{2}\right)^{-2} \\
&= \frac{1}{2\pi} \sum_{m=1}^{\infty} m(-1)^{m-1} \int_0^1 \omega^{m-\frac{1}{2}} (1-\omega)^{m-\frac{1}{2}}\, d\omega \cdot \frac{1}{\Gamma(2m)} (2x)^{2m} \left(\frac{x}{2}\right)^{-2} \\
&= \frac{2}{\pi x^2} \int_0^1 \left(\omega^{-\frac{1}{2}} (1-\omega)^{-\frac{1}{2}} \sum_{m=1}^{\infty} m(-1)^{m-1} \omega^m (1-\omega)^m \cdot \frac{1}{\Gamma(2m)} (2x)^{2m} \right) d\omega \\
&= \frac{2}{\pi x^2} \int_0^1 \left(\omega^{-\frac{1}{2}} (1-\omega)^{-\frac{1}{2}} \sum_{m=1}^{\infty} \frac{(-1)^{m-1} m \left(4x^2\left(\omega-\omega^2\right)\right)^m}{\Gamma(2m)} \right) d\omega
\end{aligned}
$$

Consider

$$
\sum_{m=1}^{\infty} \frac{(-1)^{m-1} m z^m}{\Gamma(2m)}
$$

$$
\begin{aligned}
\sum_{m=1}^{\infty} \frac{(-1)^{m-1} m z^m}{\Gamma(2m)} &= z \sum_{m=1}^{\infty} \frac{(-1)^{m-1} \frac{\partial}{\partial z}(z^m)}{\Gamma(2m)} = z\frac{\partial}{\partial z}\left(\sum_{m=1}^{\infty} \frac{(-1)^{m-1} z^m}{\Gamma(2m)} \right) = z\frac{\partial}{\partial z}\left(\sum_{m=0}^{\infty} \frac{(-1)^m z^{m+1}}{\Gamma(2m+2)} \right) \\
&= z\frac{\partial}{\partial z}\left(\sum_{m=0}^{\infty} \frac{(-1)^m z^{m+1}}{(2m+1)!} \right) = z\frac{\partial}{\partial z}\left(\sqrt{z} \sum_{m=0}^{\infty} \frac{(-1)^m (\sqrt{z})^{2m+1}}{(2m+1)!} \right) \\
&= z\frac{\partial}{\partial z}\left(\sqrt{z}\sin\sqrt{z}\right) = z\left(\frac{1}{2\sqrt{z}}\sin\sqrt{z} + \frac{1}{2}\cos\sqrt{z} \right) = \frac{1}{2}\left(\sqrt{z}\sin\sqrt{z} + z\cos\sqrt{z}\right)
\end{aligned}
$$

$$
\begin{aligned}
\implies J_0(x) &= \frac{2}{\pi x^2} \int_0^1 \left(\omega^{-\frac{1}{2}} (1-\omega)^{-\frac{1}{2}} \sum_{m=1}^{\infty} \frac{(-1)^m m \left(4x^2\left(\omega-\omega^2\right)\right)^m}{\Gamma(2m)} \right) d\omega \\
&= \frac{2}{\pi x^2} \cdot \frac{1}{2} \int_0^1 \left(\omega^{-\frac{1}{2}} (1-\omega)^{-\frac{1}{2}} \left(\sqrt{4x^2\omega(1-\omega)} \sin\left(\sqrt{4x^2\omega(1-\omega)}\right) \right.\right. \\
&\qquad \left.\left. + 4x^2\omega(1-\omega)\cos\left(\sqrt{4x^2\omega(1-\omega)}\right)\right)\right) d\omega \\
&= \frac{1}{\pi x^2} \int_0^1 2x\sin\left(\sqrt{4x^2\omega(1-\omega)}\right) + 4x^2\sqrt{\omega(1-\omega)}\cos\left(\sqrt{4x^2\omega(1-\omega)}\right) d\omega
\end{aligned}
$$

On the transformation $\omega \to \sin^2\theta$,

$$J_0(x) = \frac{1}{\pi x^2}\int_0^{\frac{\pi}{2}} \Big(2x\sin(2x\sin\theta\cos\theta) + 4x^2\sin\theta\cos\theta\cos(2x\sin\theta\cos\theta)\Big) 2\sin\theta\cos\theta \mathrm{d}\theta$$

$$\begin{aligned}
\implies J_0(x) &= \frac{1}{\pi x^2}\int_0^{\frac{\pi}{2}} \Big(2x\sin(2x\sin\theta\cos\theta) + 4x^2\sin\theta\cos\theta\cos(2x\sin\theta\cos\theta)\Big) 2\sin\theta\cos\theta\, \mathrm{d}\theta \\
&= \frac{1}{\pi x^2}\int_0^{\frac{\pi}{2}} \Big(2x\sin(x\sin 2\theta) + 2x^2\sin 2\theta\cos(x\sin 2\theta)\Big)\sin 2\theta\, \mathrm{d}\theta \\
&= \frac{1}{\pi x^2}\int_0^{\pi} \Big(x\sin(x\sin\phi) + x^2\sin\phi\cos(x\sin\phi)\Big)\sin\phi\, \mathrm{d}\phi \\
&= \frac{1}{\pi x}\int_0^{\pi} (\sin(x\sin\phi) + x\sin\phi\cos(x\sin\phi))\sin\phi\, \mathrm{d}\phi
\end{aligned}$$

Therefore,

$$J_0(x) = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+1)}\left(\frac{x}{2}\right)^{2m} \tag{2}$$

$$= \frac{1}{\pi x}\int_0^{\pi} (\sin(x\sin\phi) + x\sin\phi\cos(x\sin\phi))\sin\phi\, \mathrm{d}\phi \tag{3}$$

$$\begin{aligned}
J_\alpha(x) &= \frac{1}{\Gamma(\alpha)}\left(\frac{x}{2}\right)^{\alpha}\int_0^1 J_0\left(x\sqrt{1-t}\right)t^{\alpha-1}\mathrm{d}t \\
&= \frac{1}{\Gamma(\alpha)}\left(\frac{x}{2}\right)^{\alpha}\int_0^{\frac{\pi}{2}} 2\sin^{2\alpha-1}\vartheta\cos\vartheta J_0(x\cos\vartheta)\,\mathrm{d}\vartheta \\
&= \frac{1}{\pi x\Gamma(\alpha)}\left(\frac{x}{2}\right)^{\alpha}\int_0^{\frac{\pi}{2}}\int_0^{\pi}\frac{1}{\cos\vartheta}\Big[2\sin^{2\alpha-1}\vartheta\sin\phi\cos\vartheta\,[\sin(x\cos\vartheta\sin\phi) \\
&\qquad + x\cos\vartheta\sin\phi\cos(x\cos\vartheta\sin\phi)]\Big]\,\mathrm{d}\phi\mathrm{d}\vartheta \\
&= \frac{1}{\pi x\Gamma(\alpha)}\left(\frac{x}{2}\right)^{\alpha}\int_0^{\frac{\pi}{2}}\int_0^{\pi}\Big(2\sin^{2\alpha-1}\vartheta\sin\phi\,[\sin(x\cos\vartheta\sin\phi) + x\cos\vartheta\sin\phi\cos(x\cos\vartheta\sin\phi)]\Big)\,\mathrm{d}\phi\mathrm{d}\vartheta
\end{aligned}$$

3 Using the new integral representation to establish Lipschitz's result

It was shown by Lipschitz that [2]

$$\int_0^{\infty} e^{-at}J_0(bt)\,\mathrm{d}t = \frac{1}{\sqrt{a^2+b^2}}$$

The simplest method of establishing this result is to replace the Bessel coefficient by Parseval's integral representation and then change the order of integration by Fubini's theorem.

However, implementing the integral representation I discovered, Lipschitz's result can also be established.

$$\begin{aligned}
J_0(x) &= \frac{1}{\pi x}\int_0^{\pi} (\sin(x\sin\phi) + x\sin\phi\cos(x\sin\phi))\sin\phi\,\mathrm{d}\phi \\
\implies J_0(bt) &= \frac{1}{\pi bt}\int_0^{\pi} (\sin(bt\sin\phi) + bt\sin\phi\cos(bt\sin\phi))\sin\phi\,\mathrm{d}\phi
\end{aligned}$$

$$\begin{aligned}\int_0^\infty e^{-at}J_0(bt)\,\mathrm{d}t &= \int_0^\infty \frac{e^{-at}}{\pi bt}\int_0^\pi \left(\sin(bt\sin\phi)+bt\sin\phi\cos(bt\sin\phi)\right)\sin\phi\,\mathrm{d}\phi\,\mathrm{d}t\\ &= \int_0^\pi\int_0^\infty e^{-at}\left(\frac{\sin(bt\sin\phi)}{\pi bt}+\frac{1}{\pi}\sin\phi\cos(bt\sin\phi)\right)\sin\phi\,\mathrm{d}t\,\mathrm{d}\phi\\ &= \int_0^\pi\left(\int_0^\infty e^{-at}\frac{\sin(bt\sin\phi)\sin\phi}{\pi bt}\mathrm{d}t+\int_0^\infty\frac{e^{-at}}{\pi}\sin^2\phi\cos(bt\sin\phi)\,\mathrm{d}t\right)\mathrm{d}\phi\end{aligned}$$

Let

$$\Phi(b,\phi)=\int_0^\infty e^{-at}\frac{\sin(bt\sin\phi)}{\pi bt}\mathrm{d}t$$

$$\begin{aligned}\Phi(b,\phi) &= \frac{1}{\pi b}\int_0^\infty e^{-at}\frac{\sin(bt\sin\phi)}{t}\mathrm{d}t\\ &= \frac{1}{\pi b}\int_0^\infty e^{-at}\int_0^{b\sin\phi}\cos(\tau t)\,\mathrm{d}\tau\,\mathrm{d}t\\ &= \frac{1}{\pi b}\int_0^{b\sin\phi}\int_0^\infty e^{-at}\cos(\tau t)\,\mathrm{d}t\,\mathrm{d}\tau\\ &= \frac{1}{\pi b}\int_0^{b\sin\phi}\mathcal{L}\{\cos(\tau t)\}(a)\,\mathrm{d}\tau\\ &= \frac{1}{\pi b}\int_0^{b\sin\phi}\frac{a}{\tau^2+a^2}\mathrm{d}\tau\\ &= \frac{1}{\pi b}\arctan\left(\frac{\tau}{a}\right)\Big|_0^{b\sin\phi}\\ &= \frac{1}{\pi b}\arctan\left(\frac{b\sin\phi}{a}\right)\end{aligned}$$

Let

$$\eta(b,\phi)=\int_0^\infty\frac{e^{-at}}{\pi}\cos(bt\sin\phi)\,\mathrm{d}t$$

$$\begin{aligned}\eta(b,\phi) &= \int_0^\infty\frac{e^{-at}}{\pi}\cos(bt\sin\phi)\,\mathrm{d}t\\ &= \frac{1}{\pi}\int_0^\infty e^{-at}\cos(bt\sin\phi)\,\mathrm{d}t\\ &= \frac{1}{\pi}\mathcal{L}\{\cos(bt\sin\phi)\}(a)\\ &= \frac{1}{\pi}\frac{a}{b^2\sin^2\phi+a^2}\end{aligned}$$

$$\begin{aligned}\int_0^\infty e^{-at}J_0(bt)\,\mathrm{d}t &= \int_0^\pi\left(\Phi(b,\phi)\sin\phi+\eta(b,\phi)\sin^2\phi\right)\mathrm{d}\phi\\ &= \frac{1}{\pi b}\int_0^\pi\arctan\left(\frac{b\sin\phi}{a}\right)\sin\phi\,\mathrm{d}\phi+\frac{1}{\pi}\int_0^\pi\frac{a\sin^2\phi}{b^2\sin^2\phi+a^2}\mathrm{d}\phi\end{aligned}$$

$$\begin{aligned}\frac{1}{\pi}\int_0^\pi\frac{a\sin^2\phi}{b^2\sin^2\phi+a^2}\mathrm{d}\phi &= \frac{a}{\pi b^2}\int_0^\pi\frac{b^2\sin^2\phi}{b^2\sin^2\phi+a^2}\mathrm{d}\phi\\ &= \frac{a}{\pi b^2}\int_0^\pi\frac{b^2\sin^2\phi+a^2-a^2}{b^2\sin^2\phi+a^2}\mathrm{d}\phi\end{aligned}$$

$$= \frac{a}{\pi b^2}\int_0^{\pi} \mathrm{d}\phi - \frac{a}{\pi b^2}\int_0^{\pi} \frac{a^2}{b^2\sin^2\phi + a^2}\mathrm{d}\phi$$
$$= \frac{\pi a}{\pi b^2} - \frac{a^3}{\pi b^2}\int_0^{\pi} \frac{1}{b^2\sin^2\phi + a^2}\mathrm{d}\phi$$

$$\int_0^{\pi} \frac{1}{b^2\sin^2\phi + a^2}\mathrm{d}\phi \overset{v\to\tan\left(\frac{\phi}{2}\right)}{=} \int_0^{\infty} \frac{1}{b^2\left(\frac{2v}{1+v^2}\right)^2 + a^2}\frac{2\mathrm{d}v}{1+v^2}$$
$$= 2\int_0^{\infty} \frac{1+v^2}{4b^2v^2 + a^2\left(1+v^2\right)^2}\mathrm{d}v$$
$$= 2\int_0^{\infty}\left(\frac{1}{2a(av^2+a+2ibv)} + \frac{1}{2a(av^2+a-2ibv)}\right)\mathrm{d}v$$
$$= \frac{1}{a^2}\int_0^{\infty}\left(\frac{1}{v^2+1+\frac{2ibv}{a}} + \frac{1}{v^2+1-\frac{2ibv}{a}}\right)\mathrm{d}v$$
$$= \frac{1}{a^2}\int_0^{\infty}\left(\frac{1}{\left(v+\frac{ib}{a}\right)^2+1-\left(\frac{ib}{a}\right)^2} + \frac{1}{\left(v-\frac{ib}{a}\right)^2+1-\left(\frac{ib}{a}\right)^2}\right)\mathrm{d}v$$
$$= \frac{1}{a^2}\int_0^{\infty}\left(\frac{1}{\left(v+\frac{ib}{a}\right)^2+1+\left(\frac{b}{a}\right)^2} + \frac{1}{\left(v-\frac{ib}{a}\right)^2+1+\left(\frac{b}{a}\right)^2}\right)\mathrm{d}v$$
$$= \frac{1}{a^2}\cdot\frac{1}{\sqrt{1+\frac{b^2}{a^2}}}\left(\arctan\left(\frac{v+\frac{ib}{a}}{\sqrt{1+\frac{b^2}{a^2}}}\right) + \arctan\left(\frac{v-\frac{ib}{a}}{\sqrt{1+\frac{b^2}{a^2}}}\right)\right)\Bigg|_0^{\infty}$$
$$= \frac{1}{a^2}\cdot\frac{1}{\sqrt{1+\frac{b^2}{a^2}}}\left(\frac{\pi}{2}+\frac{\pi}{2}\right)$$
$$= \frac{\pi}{a^2}\cdot\frac{1}{\sqrt{1+\frac{b^2}{a^2}}} = \frac{\pi}{a\sqrt{a^2+b^2}}$$

$$\therefore \frac{1}{\pi}\int_0^{\pi} \frac{a\sin^2\phi}{b^2\sin^2\phi + a^2}\mathrm{d}\phi = \frac{\pi a}{\pi b^2} - \frac{a^3\pi}{a\pi b^2\sqrt{a^2+b^2}}$$
$$= \frac{a}{b^2} - \frac{a^2}{b^2\sqrt{a^2+b^2}}$$

$$\frac{1}{\pi b}\int_0^{\pi} \arctan\left(\frac{b\sin\phi}{a}\right)\sin\phi\,\mathrm{d}\phi = \frac{1}{\pi b}\int_0^{\pi}\arctan\left(\frac{b\sin\phi}{a}\right)\mathrm{d}\left(-\cos\phi\right)$$
$$= -\frac{\cos\phi}{\pi b}\arctan\left(\frac{b\sin\phi}{a}\right)\Bigg|_0^{\pi} + \frac{1}{\pi b}\int_0^{\pi}\frac{\frac{b\cos\phi}{a}}{1+\left(\frac{b\sin\phi}{a}\right)^2}\cos\phi\mathrm{d}\phi$$
$$= \frac{1}{\pi a}\int_0^{\pi}\frac{\cos^2\phi}{1+\left(\frac{b\sin\phi}{a}\right)^2}\mathrm{d}\phi$$
$$= \frac{a}{\pi}\int_0^{\pi}\frac{\cos^2\phi}{a^2+b^2\sin^2\phi}\mathrm{d}\phi$$

But
$$\int_0^\pi \frac{1}{b^2\sin^2\phi + a^2}\,\mathrm{d}\phi = \int_0^\infty \frac{1}{b^2\left(\frac{2v}{1+v^2}\right)^2 + a^2}\frac{2\mathrm{d}v}{1+v^2} = \frac{\pi}{a\sqrt{a^2+b^2}}$$

$$\begin{aligned}
\Longrightarrow \frac{a}{\pi}\int_0^\pi \frac{\cos^2\phi}{a^2+b^2\sin^2\phi}\,\mathrm{d}\phi &= \frac{a}{\pi}\int_0^\pi \frac{1-\sin^2\phi}{a^2+b^2\sin^2\phi}\,\mathrm{d}\phi \\
&= \frac{a}{\pi}\int_0^\pi \frac{1}{b^2\sin^2\phi+a^2}\,\mathrm{d}\phi - \frac{1}{\pi}\int_0^\pi \frac{a\sin^2\phi}{a^2+b^2\sin^2\phi}\,\mathrm{d}\phi \\
&= \frac{a}{\pi}\cdot\frac{\pi}{a\sqrt{a^2+b^2}} - \left(\frac{a}{b^2} - \frac{a^2}{b^2\sqrt{a^2+b^2}}\right) \\
&= \frac{1}{\sqrt{a^2+b^2}} - \frac{a}{b^2} + \frac{a^2}{b^2\sqrt{a^2+b^2}}
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow \int_0^\infty e^{-at}J_0(bt)\,\mathrm{d}t &= \int_0^\pi \Big(\Phi(b,\phi)\sin\phi + \eta(b,\phi)\sin^2\phi\Big)\,\mathrm{d}\phi \\
&= \frac{1}{\pi b}\int_0^\pi \arctan\left(\frac{b\sin\phi}{a}\right)\sin\phi\,\mathrm{d}\phi + \frac{1}{\pi}\int_0^\pi \frac{a\sin^2\phi}{b^2\sin^2\phi+a^2}\,\mathrm{d}\phi \\
&= \frac{1}{\sqrt{a^2+b^2}} - \frac{a}{b^2} + \frac{a^2}{b^2\sqrt{a^2+b^2}} + \frac{a}{b^2} - \frac{a^2}{b^2\sqrt{a^2+b^2}} \\
&= \frac{1}{\sqrt{a^2+b^2}}
\end{aligned}$$

Hence,
$$\int_0^\infty e^{-at}J_0(bt)\,\mathrm{d}t = \frac{1}{\sqrt{a^2+b^2}}$$

4 Evaluating $\int_0^\infty e^{-at}J_\alpha(bt)t^{-\alpha}\,\mathrm{d}t$, $\int_0^\infty J_{\frac{1}{2}}(bt)t^{-\frac{1}{2}}\,\mathrm{d}t$, $\int_0^\infty e^{-at}J_{\frac{1}{2}}(bt)t^{-\frac{1}{2}}\,\mathrm{d}t$, $\int_0^\infty e^{-at}J_1(bt)t^{-1}\,\mathrm{d}t$, $\int_0^\infty J_1(bt)t^{-1}\,\mathrm{d}t$ and $\int_0^\infty J_\alpha(bt)t^{-\alpha}\,\mathrm{d}t$ with the new integral representation

4.1 $\int_0^\infty e^{-at}J_\alpha(bt)t^{-\alpha}\,\mathrm{d}t$

It has been established previously that

$$J_\alpha(x) = \frac{1}{\pi x\Gamma(\alpha)}\left(\frac{x}{2}\right)^\alpha \int_0^{\frac{\pi}{2}}\int_0^\pi \Big(2\sin^{2\alpha-1}\vartheta\sin\phi\,[\sin(x\cos\vartheta\sin\phi) + x\cos\vartheta\sin\phi\cos(x\cos\vartheta\sin\phi)]\Big)\,\mathrm{d}\phi\,\mathrm{d}\vartheta$$

$$\Longrightarrow J_\alpha(bt) = \frac{1}{\pi bt\Gamma(\alpha)}\left(\frac{bt}{2}\right)^\alpha \int_0^{\frac{\pi}{2}}\int_0^\pi \Big[2\sin^{2\alpha-1}\vartheta\sin\phi\Big(\sin(bt\cos\vartheta\sin\phi)$$

$$+ \ bt\cos\vartheta\sin\phi\cos\left(bt\cos\vartheta\sin\phi\right)\Big]\Big)\,\mathrm{d}\phi\mathrm{d}\vartheta$$

$$\Longrightarrow \int_0^\infty e^{-at}J_\alpha(bt)t^{-\alpha}\,\mathrm{d}t = \int_0^\infty \frac{1}{\pi b\Gamma(\alpha)}\left(\frac{b}{2}\right)^\alpha t^{-1}e^{-at}\left(\int_0^{\frac{\pi}{2}}\int_0^\pi \Big[2\sin^{2\alpha-1}\vartheta\sin\phi\left[\sin\left(bt\cos\vartheta\sin\phi\right)\right.\right.$$
$$\left.\left.+\ bt\cos\vartheta\sin\phi\cos\left(bt\cos\vartheta\sin\phi\right)\right]\Big]\,\mathrm{d}\phi\mathrm{d}\vartheta\,\mathrm{d}t\right)$$
$$= \frac{1}{\pi b\Gamma(\alpha)}\left(\frac{b}{2}\right)^\alpha \int_0^{\frac{\pi}{2}}\int_0^\pi\int_0^\infty t^{-1}e^{-at}\left(\Big[2\sin^{2\alpha-1}\vartheta\sin\phi\left[\sin\left(bt\cos\vartheta\sin\phi\right)\right.\right.$$
$$\left.\left.+\ bt\cos\vartheta\sin\phi\cos\left(bt\cos\vartheta\sin\phi\right)\right]\Big]\,\mathrm{d}t\mathrm{d}\phi\,\mathrm{d}\vartheta\right)$$

From our previous calculation on Lipschitz's result,

$$\int_0^\pi\left(\int_0^\infty e^{-at}\frac{\sin\left(bt\sin\phi\right)\sin\phi}{\pi bt}\mathrm{d}t + \int_0^\infty \frac{e^{-at}}{\pi}\sin^2\phi\cos\left(bt\sin\phi\right)\,\mathrm{d}t\right)\mathrm{d}\phi = \frac{1}{\sqrt{a^2+b^2}} \tag{4}$$

Replacing b with $b\cos\vartheta$ in (4) above,

$$\Longrightarrow \int_0^\pi\int_0^\infty \frac{1}{b\pi\cos\vartheta}t^{-1}e^{-at}\left(\sin\phi\left[\sin\left(bt\cos\vartheta\sin\phi\right)+bt\cos\vartheta\sin\phi\cos\left(bt\cos\vartheta\sin\phi\right)\right]\right)\mathrm{d}t\mathrm{d}\phi = \frac{1}{\sqrt{b^2\cos^2\vartheta+a^2}}$$
$$\Longrightarrow \int_0^\pi\int_0^\infty \frac{1}{b\pi}t^{-1}e^{-at}\left(\sin\phi\left[\sin\left(bt\cos\vartheta\sin\phi\right)+bt\cos\vartheta\sin\phi\cos\left(bt\cos\vartheta\sin\phi\right)\right]\right)\mathrm{d}t\mathrm{d}\phi = \frac{\cos\vartheta}{\sqrt{b^2\cos^2\vartheta+a^2}}$$

$$\begin{aligned}
\int_0^\infty e^{-at}J_\alpha(bt)t^{-\alpha}\,\mathrm{d}t &= \frac{1}{\Gamma(\alpha)}\left(\frac{b}{2}\right)^\alpha\int_0^{\frac{\pi}{2}}\frac{2\sin^{2\alpha-1}\vartheta\cos\vartheta}{\sqrt{b^2\cos^2\vartheta+a^2}}\mathrm{d}\vartheta\\
&= \frac{1}{\Gamma(\alpha)}\left(\frac{b}{2}\right)^\alpha\int_0^{\frac{\pi}{2}}\frac{2\sin^{2\alpha-1}\vartheta}{\sqrt{b^2+a^2-b^2\sin^2\vartheta}}\mathrm{d}\left(\sin\vartheta\right)\\
&= \frac{1}{\Gamma(\alpha)}\left(\frac{b}{2}\right)^\alpha\int_0^1\frac{2u^{2\alpha-1}}{\sqrt{b^2+a^2-b^2u^2}}\mathrm{d}u\\
&= \frac{1}{\Gamma(\alpha)\sqrt{b^2+a^2}}\left(\frac{b}{2}\right)^\alpha\int_0^1\frac{2u^{2\alpha-1}}{\sqrt{1-\frac{b^2u^2}{b^2+a^2}}}\mathrm{d}u \qquad (5)\\
&= \frac{1}{b\Gamma(\alpha)}\left(\frac{b}{2}\right)^\alpha\left(\frac{\sqrt{b^2+a^2}}{b}\right)^{2\alpha-1}\int_0^{\frac{b}{\sqrt{b^2+a^2}}}\frac{2u^{2\alpha-1}}{\sqrt{1-u^2}}\mathrm{d}u\\
&= \frac{1}{2b\Gamma(\alpha)}\left(\frac{b}{2}\right)^\alpha\left(\frac{\sqrt{b^2+a^2}}{b}\right)^{2\alpha-1}\int_0^{\frac{b}{\sqrt{b^2+a^2}}}\frac{2u^{\alpha-1}}{\sqrt{1-u}}\mathrm{d}u\\
&= \frac{1}{b\Gamma(\alpha)}\left(\frac{b}{2}\right)^\alpha\left(\frac{\sqrt{b^2+a^2}}{b}\right)^{2\alpha-1}\int_0^{\frac{b}{\sqrt{b^2+a^2}}}u^{\alpha-1}(1-u)^{-\frac{1}{2}}\mathrm{d}u\\
&= \frac{1}{b\Gamma(\alpha)}\left(\frac{b}{2}\right)^\alpha\left(\frac{\sqrt{b^2+a^2}}{b}\right)^{2\alpha-1}B\left(\frac{b}{\sqrt{b^2+a^2}};\alpha,\frac{1}{2}\right)\\
&= \frac{\left(b^2+a^2\right)^\alpha}{2^\alpha b^{\alpha+1}\Gamma(\alpha)}\cdot\frac{b}{\sqrt{b^2+a^2}}B\left(\frac{b}{\sqrt{b^2+a^2}};\alpha,\frac{1}{2}\right) \quad \text{valid for all } a\geq 0,\ \alpha>0, \qquad (6)
\end{aligned}$$

where $B(x;a,b)$ is the incomplete beta function.

4.2 $\int_0^{\infty} J_{\frac{1}{2}}(bt)t^{-\frac{1}{2}}\,\mathrm{d}t$

From equation (5),

$$\int_0^{\infty} e^{-at}J_{\alpha}(bt)t^{-\alpha}\,\mathrm{d}t = \frac{1}{\Gamma(\alpha)\sqrt{b^2+a^2}}\left(\frac{b}{2}\right)^{\alpha}\int_0^1 \frac{2u^{2\alpha-1}}{\sqrt{1-\frac{b^2u^2}{b^2+a^2}}}\mathrm{d}u$$

Let $\alpha = \frac{1}{2}$ and $a = 0$

$$\begin{aligned}
\implies \int_0^{\infty} J_{\frac{1}{2}}(bt)t^{-\frac{1}{2}}\,\mathrm{d}t &= \frac{1}{b\Gamma\left(\frac{1}{2}\right)}\left(\frac{b}{2}\right)^{\frac{1}{2}}\int_0^1 \frac{2}{\sqrt{1-u^2}}\mathrm{d}u \\
&= \frac{2}{b\sqrt{\pi}}\sqrt{\frac{b}{2}}\arcsin u\Big|_0^1 \\
&= \frac{2}{b\sqrt{\pi}}\sqrt{\frac{b}{2}}\cdot\frac{\pi}{2} \\
&= \sqrt{\frac{\pi}{2b}}
\end{aligned}$$

$$\therefore \int_0^{\infty} J_{\frac{1}{2}}(bt)t^{-\frac{1}{2}}\,\mathrm{d}t = \sqrt{\frac{\pi}{2b}} \quad \forall\, \Re(b) > 0$$

4.3 $\int_0^{\infty} e^{-at}J_{\frac{1}{2}}(bt)t^{-\frac{1}{2}}\,\mathrm{d}t$

$$\begin{aligned}
\int_0^{\infty} e^{-at}J_{\frac{1}{2}}(bt)t^{-\frac{1}{2}}\,\mathrm{d}t &= \frac{1}{\Gamma\left(\frac{1}{2}\right)\sqrt{b^2+a^2}}\left(\frac{b}{2}\right)^{\frac{1}{2}}\int_0^1 \frac{2}{\sqrt{1-\frac{b^2u^2}{b^2+a^2}}}\mathrm{d}u \\
&= \frac{2}{\sqrt{\pi}\sqrt{b^2+a^2}}\left(\frac{b}{2}\right)^{\frac{1}{2}}\frac{\sqrt{b^2+a^2}}{b}\arcsin\left(\frac{b}{\sqrt{b^2+a^2}}u\right)\Bigg|_0^1 \\
&= \sqrt{\frac{2}{\pi b}}\arcsin\left(\frac{b}{\sqrt{b^2+a^2}}\right) \quad \forall\, \Re(b) > 0,\ \Re(a) \geq 0
\end{aligned}$$

4.4 $\int_0^{\infty} e^{-at}J_1(bt)t^{-1}\,\mathrm{d}t$

$$\begin{aligned}
\int_0^{\infty} e^{-at}J_1(bt)t^{-1}\,\mathrm{d}t &= \frac{1}{\Gamma(1)\sqrt{b^2+a^2}}\left(\frac{b}{2}\right)\int_0^1 \frac{2u}{\sqrt{1-\frac{b^2u^2}{b^2+a^2}}}\mathrm{d}u \\
&= \frac{b}{2}\int_0^1 \frac{2u}{\sqrt{b^2+a^2-b^2u^2}}\mathrm{d}u \\
&= \frac{b}{2}\int_0^1 \frac{1}{\sqrt{b^2+a^2-b^2u}}\mathrm{d}u \\
&= \frac{b}{2}\cdot\frac{2}{-b^2}\sqrt{b^2+a^2-b^2u}\Big|_0^1 \\
&= -\frac{1}{b}\left(a-\sqrt{b^2+a^2}\right) = \frac{1}{b}\left(\sqrt{b^2+a^2}-a\right) \quad \forall\, \Re(b) > 0,\ \Re(a) \geq 0
\end{aligned}$$

4.4.1 Special result

$$\int_0^\infty e^{-3t} J_1(4t)t^{-1}\,\mathrm{d}t = \frac{1}{4}\left(\sqrt{4^2+3^2}-3\right) = \frac{1}{2}$$
$$\implies 2\int_0^\infty e^{-3t} J_1(4t)t^{-1}\,\mathrm{d}t = 1$$

4.5 $\int_0^\infty J_1(bt)t^{-1}\,\mathrm{d}t$

It has been established previously that

$$\int_0^\infty e^{-at} J_1(bt)t^{-1}\,\mathrm{d}t = \frac{1}{b}\left(\sqrt{b^2+a^2}-a\right)$$

At $a=0$,

$$\int_0^\infty J_1(bt)t^{-1}\,\mathrm{d}t = \frac{1}{b}\left(\sqrt{b^2+0^2}-0\right) = 1$$
$$\therefore \int_0^\infty J_1(bt)t^{-1}\,\mathrm{d}t = 1 \;\; \forall\, \Re(b) > 0$$

4.6 $\int_0^\infty e^{-at} J_\alpha(bt)t^{-\alpha}\,\mathrm{d}t$

From equation (6)

$$\int_0^\infty e^{-at} J_\alpha(bt)t^{-\alpha}\,\mathrm{d}t = \frac{\left(b^2+a^2\right)^\alpha}{2^\alpha b^{\alpha+1}\Gamma(\alpha)} \cdot \frac{b}{\sqrt{b^2+a^2}} B\left(\frac{b}{\sqrt{b^2+a^2}};\alpha,\frac{1}{2}\right) \quad \forall\, \alpha > 0,\; a \geq 0$$

At $a=0$,

$$\begin{aligned}
\int_0^\infty J_\alpha(bt)t^{-\alpha}\,\mathrm{d}t &= \frac{b^{\alpha-1}}{2^\alpha\Gamma(\alpha)} B\left(1;\alpha,\frac{1}{2}\right) \\
&= \frac{b^{\alpha-1}}{2^\alpha\Gamma(\alpha)} \beta\left(\alpha,\frac{1}{2}\right) \\
&= \frac{b^{\alpha-1}}{2^\alpha\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\alpha+\frac{1}{2}\right)} \\
&= \frac{b^{\alpha-1}\sqrt{\pi}}{2^\alpha\Gamma\left(\alpha+\frac{1}{2}\right)} \\
&= \frac{b^{\alpha-1}\sqrt{\pi}}{2^\alpha\left(\frac{\Gamma(2\alpha)\sqrt{\pi}}{\Gamma(\alpha)2^{2\alpha-1}}\right)} = \frac{b^{\alpha-1}}{\frac{\Gamma(2\alpha)}{\Gamma(\alpha)2^{\alpha-1}}} \\
&= \frac{(2b)^{\alpha-1}\Gamma(\alpha)}{\Gamma(2\alpha)}
\end{aligned}$$

$$\therefore \int_0^\infty J_\alpha(bt)t^{-\alpha}\,\mathrm{d}t = \begin{cases} \frac{(2b)^{\alpha-1}\Gamma(\alpha)}{\Gamma(2\alpha)} & \text{for } \alpha > 0 \\ \frac{b^{0-1}\sqrt{\pi}}{2^0\Gamma\left(0+\frac{1}{2}\right)} = \frac{1}{b} & \text{for } \alpha = 0 \end{cases}$$

4.6.1 Special results

$$\int_0^\infty J_2(3t)t^{-2}\,dt = \frac{(2\times 3)^1\Gamma(2)}{\Gamma(4)} = 1$$

$$\int_0^\infty J_0(bt)\,dt = \frac{1}{b} \quad \forall\,\Re(b) > 0$$

$$\int_0^\infty J_0(t)\,dt = 1 \;\text{ for } b = 1.$$

5 Integral representations for $I_\alpha(x)$ and $I_0(x)$

The Bessel functions are valid even for complex arguments x, and an important special case is that of a purely imaginary argument. In this case, the solutions to the Bessel equation are called the modified Bessel functions (or occasionally the hyperbolic Bessel functions) of the first and second kind and are defined as [5]

$$I_\alpha(x) = i^{-\alpha}J_\alpha(ix) = \sum_{m=0}^{\infty}\frac{1}{m!\Gamma(m+\alpha+1)}\left(\frac{x}{2}\right)^{2m+\alpha} \qquad \forall\, x\in\mathbb{C}$$

$$K_\alpha(x) = \frac{\pi}{2}\frac{I_{-\alpha}(x)-I_\alpha(x)}{\sin(\alpha\pi)} \qquad \text{when } \alpha \text{ is not an integer.} \tag{7}$$

Now, at $\alpha = 0$,

$$I_0(x) = J_0(ix) \tag{8}$$

By equation (3),

$$J_0(x) = \frac{1}{\pi x}\int_0^\pi (\sin(x\sin\phi) + x\sin\phi\cos(x\sin\phi))\sin\phi\,d\phi$$

$$\begin{aligned} I_0(x) = J_0(ix) &= \frac{1}{\pi ix}\int_0^\pi (\sin(ix\sin\phi) + ix\sin\phi\cos(ix\sin\phi))\sin\phi\,d\phi \\ &= \frac{1}{\pi x}\int_0^\pi (-i\sin(ix\sin\phi) + x\sin\phi\cosh(x\sin\phi))\sin\phi\,d\phi \\ &= \frac{1}{\pi x}\int_0^\pi (\sinh(x\sin\phi) + x\sin\phi\cosh(x\sin\phi))\sin\phi\,d\phi \end{aligned}$$

$$\implies I_0(x) = \frac{1}{\pi x}\int_0^\pi (\sinh(x\sin\phi) + x\sin\phi\cosh(x\sin\phi))\sin\phi\,d\phi \tag{9}$$

By equation (1),

$$J_\alpha(ix) = \frac{1}{\Gamma(\alpha)}\left(\frac{ix}{2}\right)^\alpha \int_0^1 J_0\left(ix\sqrt{1-t}\right)t^{\alpha-1}dt$$

$$\implies i^{-\alpha}J_\alpha(ix) = \frac{1}{\Gamma(\alpha)}\left(\frac{x}{2}\right)^\alpha \int_0^1 J_0\left(ix\sqrt{1-t}\right)t^{\alpha-1}dt$$

By equation (8), $I_0(x) = J_0(ix)$.

$$\implies i^{-\alpha}J_\alpha(ix) = I_\alpha(x) = \frac{1}{\Gamma(\alpha)}\left(\frac{x}{2}\right)^\alpha \int_0^1 I_0\left(x\sqrt{1-t}\right)t^{\alpha-1}dt$$

$$\implies I_\alpha(x) = \frac{1}{\Gamma(\alpha)}\left(\frac{x}{2}\right)^\alpha \int_0^1 I_0\left(x\sqrt{1-t}\right)t^{\alpha-1}dt \tag{10}$$

Inserting (9) in (10),

$$I_\alpha(x) = \frac{1}{\pi x\Gamma(\alpha)}\left(\frac{x}{2}\right)^\alpha \int_0^{\frac{\pi}{2}}\int_0^\pi \left[2\sin^{2\alpha-1}\vartheta\sin\phi\left[\sinh(x\cos\vartheta\sin\phi)+x\cos\vartheta\sin\phi\cosh(x\cos\vartheta\sin\phi)\right]\right]\mathrm{d}\phi\mathrm{d}\vartheta$$

6 Simplifications of the new integral representations

$$\begin{aligned}
J_0(x) &= \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+1)}\left(\frac{x}{2}\right)^{2m}\\
&= \frac{1}{\pi x}\int_0^\pi (\sin(x\sin\phi)+x\sin\phi\cos(x\sin\phi))\sin\phi\,\mathrm{d}\phi\\
&= \frac{1}{\pi x}\int_0^\pi \left(\sin(x\sin\phi)+x\frac{\partial}{\partial x}(\sin(x\sin\phi))\right)\sin\phi\,\mathrm{d}\phi\\
&= \frac{1}{\pi x}\int_0^\pi \sin\phi\frac{\partial}{\partial x}(x\sin(x\sin\phi))\,\mathrm{d}\phi
\end{aligned}$$

$$\implies J_0(x) = \frac{1}{\pi x}\int_0^\pi \sin\phi\frac{\partial}{\partial x}(x\sin(x\sin\phi))\,\mathrm{d}\phi$$

Since

$$I_0(x) = \frac{1}{\pi x}\int_0^\pi (\sinh(x\sin\phi)+x\sin\phi\cosh(x\sin\phi))\sin\phi\,\mathrm{d}\phi$$

$$\implies I_0(x) = \frac{1}{\pi x}\int_0^\pi \sin\phi\frac{\partial}{\partial x}(x\sinh(x\sin\phi))\,\mathrm{d}\phi$$

$$I_\alpha(x) = \frac{1}{\Gamma(\alpha)}\left(\frac{x}{2}\right)^\alpha\int_0^{\frac{\pi}{2}} 2\sin^{2\alpha-1}\vartheta\cos\vartheta I_0(x\cos\vartheta)\,\mathrm{d}\vartheta$$

$$\begin{aligned}
\implies I_\alpha(x) &= \frac{1}{\pi x\Gamma(\alpha)}\left(\frac{x}{2}\right)^\alpha\int_0^{\frac{\pi}{2}}\int_0^\pi 2\sin^{2\alpha-1}\vartheta\cos\vartheta\frac{\sin\phi}{\cos\vartheta}\frac{\partial}{\partial(x\cos\vartheta)}(x\cos\vartheta\sinh(x\cos\vartheta\sin\phi))\,\mathrm{d}\phi\mathrm{d}\vartheta\\
&= \frac{1}{\pi x\Gamma(\alpha)}\left(\frac{x}{2}\right)^\alpha\int_0^{\frac{\pi}{2}}\int_0^\pi 2\sin^{2\alpha-1}\vartheta\sin\phi\frac{\partial}{\partial(x\cos\vartheta)}(x\cos\vartheta\sinh(x\cos\vartheta\sin\phi))\,\mathrm{d}\phi\mathrm{d}\vartheta
\end{aligned}$$

Similarly,

$$J_\alpha(x) = \frac{1}{\pi x\Gamma(\alpha)}\left(\frac{x}{2}\right)^\alpha\int_0^{\frac{\pi}{2}}\int_0^\pi 2\sin^{2\alpha-1}\vartheta\sin\phi\frac{\partial}{\partial(x\cos\vartheta)}(x\cos\vartheta\sin(x\cos\vartheta\sin\phi))\,\mathrm{d}\phi\mathrm{d}\vartheta$$

7 An expression for $K_{\frac{1}{2}}(x)\ \forall\,\Re(x)>0$ with the new integral representations

$$I_\alpha(x) = \frac{1}{\Gamma(\alpha)}\left(\frac{x}{2}\right)^\alpha\int_0^{\frac{\pi}{2}} 2\sin^{2\alpha-1}\vartheta\cos\vartheta I_0(x\cos\vartheta)\,\mathrm{d}\vartheta$$

$$\begin{aligned}
I_{\frac{1}{2}}(x) &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^{\frac{1}{2}}\int_0^{\frac{\pi}{2}} 2\sin^{2\left(\frac{1}{2}\right)-1}\vartheta\cos\vartheta I_0(x\cos\vartheta)\,\mathrm{d}\vartheta\\
&= \frac{2}{\sqrt{\pi}}\left(\frac{x}{2}\right)^{\frac{1}{2}}\int_0^{\frac{\pi}{2}}\cos\vartheta I_0(x\cos\vartheta)\,\mathrm{d}\vartheta
\end{aligned}$$

$$J_{-\frac{1}{2}}(x) = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m-\frac{1}{2}+1)}\left(\frac{x}{2}\right)^{2m-\frac{1}{2}}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\sqrt{\frac{2}{x}} + \sum_{m=1}^{\infty} \frac{(-1)^m}{(m+1)!\Gamma(m-\frac{1}{2}+1)}\left(\frac{x}{2}\right)^{2m-\frac{1}{2}} \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\sqrt{\frac{2}{x}} + \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{(m+1)!\Gamma(m+\frac{1}{2}+1)}\left(\frac{x}{2}\right)^{2m+\frac{3}{2}} \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\sqrt{\frac{2}{x}} - \sum_{m=0}^{\infty} \frac{(-1)^m\Gamma(m+1)\Gamma\left(\frac{1}{2}\right)}{\Gamma(m+2)\Gamma(m+1)\Gamma(m+\frac{1}{2}+1)}\left(\frac{x}{2}\right)^{2m+\frac{3}{2}} \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\sqrt{\frac{2}{x}} - \frac{1}{\Gamma\left(\frac{1}{2}\right)}\sum_{m=0}^{\infty} \frac{(-1)^m\Gamma(m+1)\Gamma\left(\frac{1}{2}\right)}{m!\Gamma(m+2)\Gamma(m+\frac{1}{2}+1)}\left(\frac{x}{2}\right)^{2m+\frac{3}{2}} \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\sqrt{\frac{2}{x}} - \frac{1}{\Gamma\left(\frac{1}{2}\right)}\sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+2)}\beta\left(m+1,\frac{1}{2}\right)\left(\frac{x}{2}\right)^{2m+\frac{3}{2}} \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\sqrt{\frac{2}{x}} - \frac{1}{\Gamma\left(\frac{1}{2}\right)}\sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+2)}\int_0^1 t^m(1-t)^{\frac{1}{2}-1}\,\mathrm{d}t\left(\frac{x}{2}\right)^{2m+\frac{3}{2}} \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\sqrt{\frac{2}{x}} - \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^{\frac{1}{2}}\int_0^1 \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+2)}t^m\left(\frac{x}{2}\right)^{2m+1}(1-t)^{-\frac{1}{2}}\,\mathrm{d}t \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\sqrt{\frac{2}{x}} - \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^{\frac{1}{2}}\int_0^1 \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+2)}\left(\frac{x\sqrt{t}}{2}\right)^{2m+1} t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}\,\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
\Longrightarrow J_{-\frac{1}{2}}(x) &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\sqrt{\frac{2}{x}} - \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^{\frac{1}{2}}\int_0^1 J_1(x\sqrt{t})t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}\,\mathrm{d}t \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\sqrt{\frac{2}{x}} - \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^{\frac{1}{2}}\int_0^1 J_1\left(x\sqrt{1-t}\right)t^{\frac{1}{2}-1}(1-t)^{-\frac{1}{2}}\,\mathrm{d}t
\end{aligned}
$$

Since, $I_\alpha(x) = i^{-\alpha}J_\alpha(ix)$,

$$
\Longrightarrow I_{-\frac{1}{2}}(x) = i^{\frac{1}{2}}J_{\frac{1}{2}}(ix)
$$

$$
I_1(x) = i^{-1}J_1(ix) \Longrightarrow J_1(ix) = iI_1(x) \Longrightarrow iJ_1(ix) = -I_1(x)
$$

$$
\begin{aligned}
\Longrightarrow i^{\frac{1}{2}}J_{\frac{1}{2}}(ix) &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\sqrt{\frac{2}{x}} - \frac{i}{\Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^{\frac{1}{2}}\int_0^1 J_1\left(ix\sqrt{1-t}\right)t^{\frac{1}{2}-1}(1-t)^{-\frac{1}{2}}\,\mathrm{d}t \\
\Longrightarrow I_{-\frac{1}{2}}(x) &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\sqrt{\frac{2}{x}} + \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^{\frac{1}{2}}\int_0^1 I_1\left(x\sqrt{1-t}\right)t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}\,\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
I_{-\frac{1}{2}}(x) &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\sqrt{\frac{2}{x}} + \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^{\frac{1}{2}}\int_0^{\frac{\pi}{2}} 2\sin^{2\left(\frac{1}{2}\right)-1}\vartheta\cos\vartheta\,\frac{1}{\cos\vartheta}I_1(x\cos\vartheta)\,\mathrm{d}\vartheta \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\sqrt{\frac{2}{x}} + \frac{2}{\Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^{\frac{1}{2}}\int_0^{\frac{\pi}{2}} I_1(x\cos\vartheta)\,\mathrm{d}\vartheta
\end{aligned}
$$

$$
\begin{aligned}
\int_0^{\frac{\pi}{2}} \cos\vartheta I_0(x\cos\vartheta)\,\mathrm{d}\vartheta &= \int_0^1 I_0\left(x\sqrt{1-u^2}\right)\mathrm{d}u \\
&= \frac{1}{2}\int_0^1 u^{-\frac{1}{2}}I_0\left(x\sqrt{1-u}\right)\mathrm{d}u
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\int_0^1 (1-u)^{-\frac{1}{2}} I_0\left(x\sqrt{u}\right)\mathrm{d}u \\
&= \frac{1}{2}\sum_{m=0}^{\infty}\frac{1}{m!\Gamma(m+1)}\int_0^1 (1-u)^{-\frac{1}{2}}\left(\frac{x\sqrt{u}}{2}\right)^{2m}\mathrm{d}u \\
&= \frac{1}{2}\sum_{m=0}^{\infty}\frac{1}{m!\Gamma(m+1)}\left(\frac{x}{2}\right)^{2m}\int_0^1 u^m (1-u)^{-\frac{1}{2}}\mathrm{d}u \\
&= \frac{1}{2}\sum_{m=0}^{\infty}\frac{1}{m!\Gamma(m+1)}\left(\frac{x}{2}\right)^{2m}\beta\left(m+1,\frac{1}{2}\right) \\
&= \frac{1}{2}\sum_{m=0}^{\infty}\frac{1}{m!\Gamma(m+1)}\left(\frac{x}{2}\right)^{2m}\frac{\Gamma(m+1)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{3}{2}\right)} \\
&= \frac{1}{2}\sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{x}{2}\right)^{2m}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{3}{2}\right)} \\
&= \frac{1}{2}\sum_{m=0}^{\infty}\frac{1}{\Gamma(m+1)}\left(\frac{x}{2}\right)^{2m}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{3}{2}\right)}
\end{aligned}
$$

By Legendre's duplication formula,

$$
\frac{2^{2m+1}}{\Gamma(2m+2)} = \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{3}{2}\right)\Gamma(m+1)}
$$

$$
\begin{aligned}
\Longrightarrow \int_0^{\frac{\pi}{2}} \cos\vartheta I_0(x\cos\vartheta)\,\mathrm{d}\vartheta &= \frac{1}{2}\sum_{m=0}^{\infty}\frac{2^{2m+1}}{\Gamma(2m+2)}\left(\frac{x}{2}\right)^{2m} \\
&= \sum_{m=0}^{\infty}\frac{x^{2m}}{\Gamma(2m+2)} = \sum_{m=0}^{\infty}\frac{x^{2m}}{(2m+1)!} \\
&= \frac{1}{x}\sum_{m=0}^{\infty}\frac{x^{2m+1}}{(2m+1)!} \\
&= \frac{\sinh(x)}{x}
\end{aligned}
$$

$$
\begin{aligned}
\Longrightarrow I_{\frac{1}{2}}(x) &= \frac{2}{\sqrt{\pi}}\left(\frac{x}{2}\right)^{\frac{1}{2}}\int_0^{\frac{\pi}{2}}\cos\vartheta I_0(x\cos\vartheta)\,\mathrm{d}\vartheta \\
&= \sqrt{\frac{2x}{\pi}}\frac{\sinh(x)}{x} \\
&= \sqrt{\frac{2}{\pi x}}\sinh(x)
\end{aligned}
$$

$$
\begin{aligned}
\int_0^{\frac{\pi}{2}} I_1(x\cos\vartheta)\,\mathrm{d}\vartheta &= \int_0^1 \frac{I_1\left(x\sqrt{1-u^2}\right)}{\sqrt{1-u^2}}\mathrm{d}u \\
&= \frac{1}{2}\int_0^1 u^{-\frac{1}{2}}(1-u)^{-\frac{1}{2}} I_1\left(x\sqrt{1-u}\right)\mathrm{d}u \\
&= \frac{1}{2}\int_0^1 u^{-\frac{1}{2}}(1-u)^{-\frac{1}{2}} I_1\left(x\sqrt{u}\right)\mathrm{d}u \\
&= \frac{1}{2}\sum_{m=0}^{\infty}\frac{1}{m!\Gamma(m+2)}\int_0^1 u^{-\frac{1}{2}}(1-u)^{-\frac{1}{2}}\left(\frac{x\sqrt{u}}{2}\right)^{2m+1}\mathrm{d}u
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{m=0}^{\infty}\frac{1}{m!\Gamma(m+2)}\left(\frac{x}{2}\right)^{2m+1}\int_0^1 u^m(1-u)^{-\frac{1}{2}}\,\mathrm{d}u \\
&= \frac{1}{2}\sum_{m=0}^{\infty}\frac{1}{m!\Gamma(m+2)}\left(\frac{x}{2}\right)^{2m+1}\beta\left(m+1,\frac{1}{2}\right) \\
&= \frac{1}{2}\sum_{m=0}^{\infty}\frac{1}{m!(m+1)\Gamma(m+1)}\left(\frac{x}{2}\right)^{2m+1}\frac{\Gamma(m+1)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{3}{2}\right)} \\
&= \frac{1}{2}\sum_{m=0}^{\infty}\frac{1}{m!(m+1)}\left(\frac{x}{2}\right)^{2m+1}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{3}{2}\right)} \\
&= \frac{1}{2}\sum_{m=0}^{\infty}\frac{1}{\Gamma(m+2)}\left(\frac{x}{2}\right)^{2m+1}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{3}{2}\right)} \\
&= \frac{1}{2}\sum_{m=0}^{\infty}\frac{m+\frac{3}{2}}{\Gamma(m+2)}\left(\frac{x}{2}\right)^{2m+1}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{5}{2}\right)} \\
&= \frac{1}{4}\sum_{m=0}^{\infty}\frac{2m+3}{\Gamma(m+2)}\left(\frac{x}{2}\right)^{2m+1}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{5}{2}\right)}
\end{aligned}
$$

$$
\begin{aligned}
\Longrightarrow \int_0^{\frac{\pi}{2}}\cos\vartheta I_0(x\cos\vartheta)\,\mathrm{d}\vartheta &= \frac{1}{4}\sum_{m=0}^{\infty}\frac{2^{2m+3}}{\Gamma(2m+4)}(2m+3)\left(\frac{x}{2}\right)^{2m+1} \\
&= 2\cdot\frac{x}{2}\sum_{m=0}^{\infty}\frac{x^{2m}(2m+3)}{(2m+3)\Gamma(2m+3)} = x\sum_{m=0}^{\infty}\frac{x^{2m}}{(2m+2)!} \\
&= \frac{1}{x}\sum_{m=0}^{\infty}\frac{x^{2m+2}}{(2m+2)!} \\
&= \frac{\cosh(x)-1}{x}
\end{aligned}
$$

$$
\begin{aligned}
I_{-\frac{1}{2}}(x) &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\sqrt{\frac{2}{x}}+\sqrt{\frac{2x}{\pi}}\left(\frac{\cosh(x)-1}{x}\right) \\
&= \sqrt{\frac{2}{\pi x}}+\sqrt{\frac{2x}{\pi}}\left(\frac{\cosh(x)-1}{x}\right) \\
&= \sqrt{\frac{2}{\pi x}}\cosh(x)
\end{aligned}
$$

$$
\begin{aligned}
I_{-\frac{1}{2}}(x)-I_{\frac{1}{2}}(x) &= \sqrt{\frac{2}{\pi x}}\cosh(x)-\sqrt{\frac{2}{\pi x}}\sinh(x) \\
&= \sqrt{\frac{2}{\pi x}}e^{-x}
\end{aligned}
$$

At $\alpha = \frac{1}{2}$ in equation (7),

$$
\begin{aligned}
K_{\frac{1}{2}}(x) &= \frac{\pi}{2}\frac{I_{-\frac{1}{2}}(x)-I_{\frac{1}{2}}(x)}{\sin\left(\frac{\pi}{2}\right)} \\
&= \frac{\pi}{2}\left(I_{-\frac{1}{2}}(x)-I_{\frac{1}{2}}\right)
\end{aligned}
$$

$$= \frac{\pi}{2} \cdot \sqrt{\frac{2}{\pi x}} e^{-x} = \sqrt{\frac{\pi}{2x}} e^{-x}$$

$$\therefore K_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}} e^{-x} \tag{11}$$

8 Deriving another integral representation for $K_\alpha(xz)$ directly from Basset's formula

By Basset's formula [3],

$$K_\alpha(xz) = \frac{\Gamma\left(\alpha+\frac{1}{2}\right)\cdot(2z)^\alpha}{x^\alpha \Gamma\left(\frac{1}{2}\right)} \int_0^\infty \frac{\cos(xu)}{\left(u^2+z^2\right)^{\alpha+\frac{1}{2}}} \mathrm{d}u$$

By elmentary substitution, Basset's formula can be rewritten to $K_\alpha(z)$.

$$\begin{aligned}
K_\alpha(xz) &= \frac{\Gamma\left(\alpha+\frac{1}{2}\right)\cdot(2z)^\alpha}{x^\alpha \Gamma\left(\frac{1}{2}\right)} \int_0^\infty \frac{\cos(xu)}{\left(u^2+z^2\right)^{\alpha+\frac{1}{2}}} \mathrm{d}u \\
&= \frac{\Gamma\left(\alpha+\frac{1}{2}\right)\cdot(2z)^\alpha}{x^\alpha \Gamma\left(\frac{1}{2}\right)} z \int_0^\infty \frac{\cos(xzu)}{\left(z^2u^2+z^2\right)^{\alpha+\frac{1}{2}}} \mathrm{d}u \\
&= \frac{\Gamma\left(\alpha+\frac{1}{2}\right)\cdot(2z)^\alpha}{(z^{2\alpha}\cdot z) x^\alpha \Gamma\left(\frac{1}{2}\right)} z \int_0^\infty \frac{\cos(xzu)}{\left(u^2+1\right)^{\alpha+\frac{1}{2}}} \mathrm{d}u \\
&= \frac{\Gamma\left(\alpha+\frac{1}{2}\right)\cdot(2)^\alpha}{z^\alpha x^\alpha \Gamma\left(\frac{1}{2}\right)} \int_0^\infty \frac{\cos(xzu)}{\left(u^2+1\right)^{\alpha+\frac{1}{2}}} \mathrm{d}u \\
\implies K_\alpha(z) &= \frac{\Gamma\left(\alpha+\frac{1}{2}\right)\cdot(2)^\alpha}{z^\alpha \Gamma\left(\frac{1}{2}\right)} \int_0^\infty \frac{\cos(zu)}{\left(u^2+1\right)^{\alpha+\frac{1}{2}}} \mathrm{d}u
\end{aligned}$$

Consider the following integral

$$H(\beta,p,R) = \frac{2\Gamma(p)}{\sqrt{\pi}} \int_0^\infty \frac{\cos(2Rx)}{\left(\beta^2+x^2\right)^p} \mathrm{d}x$$

$$\begin{aligned}
H(\beta,p,R) &= \frac{2\Gamma(p)}{\sqrt{\pi}} \int_0^\infty \frac{\cos(2Rx)}{\left(\beta^2+x^2\right)^p} \mathrm{d}x \\
&= \frac{2}{\sqrt{\pi}} \int_0^\infty \int_0^\infty t^{p-1} e^{-\left(\beta^2+x^2\right)t} \cos(2Rx)\, \mathrm{d}t\, \mathrm{d}x \\
&= \frac{2}{\sqrt{\pi}} \int_0^\infty \int_0^\infty t^{p-1} e^{-\left(\beta^2+x^2\right)t} \cos(2Rx)\, \mathrm{d}x\, \mathrm{d}t \\
&= \frac{2}{\sqrt{\pi}} \int_0^\infty t^{p-1} e^{-\beta^2 t} \int_0^\infty e^{-x^2 t} \cos(2Rx)\, \mathrm{d}x\, \mathrm{d}t
\end{aligned}$$

Let

$$f(R,t) = \int_0^\infty e^{-x^2 t} \cos(2Rx)\, \mathrm{d}x$$

$$\begin{aligned}
\frac{\partial f(R,t)}{\partial R} &= -2 \int_0^\infty x e^{-x^2 t} \sin(2Rx)\, \mathrm{d}x \\
&= \frac{1}{t} \int_0^\infty \sin(2Rx)\, \mathrm{d}\left(e^{-x^2 t}\right)
\end{aligned}$$

$$= -\frac{2R}{t}\int_0^\infty e^{-x^2t}\cos(2Rx)\,\mathrm{d}x = -\frac{2R}{t}f(R,t)$$

$$\begin{aligned}
\frac{\partial f(R,t)}{\partial R} &= -\frac{2R}{t}f(R,t)\\
\implies \frac{1}{f(R,t)}\frac{\partial f(R,t)}{\partial R} &= -\frac{2R}{t}\\
\implies \frac{\partial(\ln(f(R,t)))}{\partial R} &= -\frac{2R}{t}\\
\implies \partial(\ln(f(R,t))) &= -\frac{2R}{t}\partial R\\
\implies \int \mathrm{d}(\ln(f(R,t))) &= -\int\frac{2R}{t}\mathrm{d}R\\
\implies \ln(f(R,t)) &= -\frac{R^2}{t}+C(t)\\
\implies f(R,t) &= A(t)e^{-\frac{R^2}{t}}
\end{aligned}$$

$$\implies f(R,t) = \int_0^\infty e^{-x^2t}\cos(2Rx)\,\mathrm{d}x = A(t)e^{-\frac{R^2}{t}}$$

$$\therefore f(0,t) = \int_0^\infty e^{-x^2t}\mathrm{d}x = A(t).$$

$$\begin{aligned}
\int_0^\infty e^{-x^2t}\mathrm{d}x &\overset{x\to\sqrt{u}}{=} \frac{1}{2}\int_0^\infty u^{-\frac{1}{2}}e^{-ut}\,\mathrm{d}u\\
&= \frac{1}{2}\mathcal{L}\left(u^{-\frac{1}{2}}\right)(t) = \frac{\Gamma\left(\frac{1}{2}\right)}{2t^{\frac{1}{2}}} = \frac{1}{2}\sqrt{\frac{\pi}{t}}
\end{aligned}$$

$$\implies f(R,t) = \int_0^\infty e^{-x^2t}\cos(2Rx)\,\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{t}}e^{-\frac{R^2}{t}}$$

$$\begin{aligned}
\implies H(\beta,p,R) &= \frac{2}{\sqrt{\pi}}\int_0^\infty t^{p-1}e^{-\beta^2t}\int_0^\infty e^{-x^2t}\cos(2Rx)\,\mathrm{d}x\,\mathrm{d}t\\
&= \frac{2}{\sqrt{\pi}}\int_0^\infty t^{p-1}e^{-\beta^2t}\left(\frac{1}{2}\sqrt{\frac{\pi}{t}}e^{-\frac{R^2}{t}}\right)\mathrm{d}t\\
&= \int_0^\infty t^{p-\frac{3}{2}}e^{-\left(\beta^2t+\frac{R^2}{t}\right)}\mathrm{d}t
\end{aligned}$$

$$\implies H(\beta,p,R) = \int_0^\infty t^{p-\frac{3}{2}}e^{-\left(\beta^2t+\frac{R^2}{t}\right)}\mathrm{d}t = \frac{2\Gamma(p)}{\sqrt{\pi}}\int_0^\infty \frac{\cos(2Rt)}{\left(\beta^2+t^2\right)^p}\mathrm{d}t$$

Replacing β^2 with a, R^2 with z,

$$\int_0^\infty t^{p-\frac{3}{2}}e^{-\left(at+\frac{z}{t}\right)}\mathrm{d}t = \frac{2\Gamma(p)}{\sqrt{\pi}}\int_0^\infty \frac{\cos(2\sqrt{z}t)}{\left(a^2+t^2\right)^p}\mathrm{d}t$$

Replacing p with $\frac{1}{2}+\alpha$, z with $\frac{z^2}{4}$, a with 1,

$$\int_0^\infty t^{\alpha-1}e^{-\left(t+\frac{z^2}{4t}\right)}\mathrm{d}t = \frac{2\Gamma\left(\frac{1}{2}+\alpha\right)}{\sqrt{\pi}}\int_0^\infty \frac{\cos(zt)}{\left(1+t^2\right)^{\frac{1}{2}+\alpha}}\mathrm{d}t$$

Multiplying both sides by $\frac{2^{\alpha-1}}{z^{\alpha}}$,

$$\frac{\Gamma\left(\alpha+\frac{1}{2}\right)\cdot(2)^{\alpha}}{z^{\alpha}\Gamma\left(\frac{1}{2}\right)}\int_0^{\infty}\frac{\cos(zt)}{\left(t^2+1\right)^{\alpha+\frac{1}{2}}}\mathrm{d}t=\frac{1}{z^{\alpha}}\int_0^{\infty}(2t)^{\alpha-1}e^{-\left(t+\frac{z^2}{4t}\right)}\mathrm{d}t$$

Thus if

$$K_{\alpha}(z)=\frac{\Gamma\left(\alpha+\frac{1}{2}\right)\cdot(2)^{\alpha}}{z^{\alpha}\Gamma\left(\frac{1}{2}\right)}\int_0^{\infty}\frac{\cos(zt)}{\left(t^2+1\right)^{\alpha+\frac{1}{2}}}\mathrm{d}t$$

Then

$$K_{\alpha}(z)=\frac{1}{z^{\alpha}}\int_0^{\infty}(2t)^{\alpha-1}e^{-\left(t+\frac{z^2}{4t}\right)}\mathrm{d}t,$$

$$\Longrightarrow K_{\alpha}(z)=\frac{\Gamma\left(\alpha+\frac{1}{2}\right)\cdot(2)^{\alpha}}{z^{\alpha}\Gamma\left(\frac{1}{2}\right)}\int_0^{\infty}\frac{\cos(zt)}{\left(t^2+1\right)^{\alpha+\frac{1}{2}}}\mathrm{d}t=\frac{1}{z^{\alpha}}\int_0^{\infty}(2t)^{\alpha-1}e^{-\left(t+\frac{z^2}{4t}\right)}\mathrm{d}t$$

9 Another variant of Hardy's integral in relation to partial differential equations

Prof. Godfrey Harold Hardy, FRS gave the following integral reprsentation [4]

$$\int_0^{\infty}\sin\left(au+\frac{b}{u}\right)\frac{\mathrm{d}u}{u}=\pi J_0\left\{2\sqrt{ab}\right\}$$

On the transformation $u\longrightarrow u^2$, this is equivalent to

$$\int_0^{\infty}\sin\left(au^2+\frac{b}{u^2}\right)\frac{\mathrm{d}u}{u}=\frac{\pi}{2}J_0\left\{2\sqrt{ab}\right\}$$

Sequel to the above, a similar integral representation for $K_{\frac{1}{2}}(z)$ can be proposed in what follows

Proposition.

$$\int_0^{\infty}\sin\left(au^2-\frac{b}{u^2}\right)\mathrm{d}u=\left(\frac{b}{2a}\right)^{\frac{1}{4}}K_{\frac{1}{2}}\left\{2\sqrt{ab}\right\}$$

Proof. We can take the partial derivatives of the integrand wrt. a or b by differentiating underthe integral sign since the partial derivatives are continuous in the limits of the integral.

$$\frac{\partial}{\partial a}\left(\int_0^{\infty}\sin\left(au^2-\frac{b}{u^2}\right)\mathrm{d}u\right)=\int_0^{\infty}u^2\cos\left(au^2-\frac{b}{u^2}\right)\mathrm{d}u$$

$$\begin{aligned}\frac{\partial^2}{\partial a\partial b}\left(\int_0^{\infty}\sin\left(au^2-\frac{b}{u^2}\right)\mathrm{d}u\right)&=\int_0^{\infty}-\frac{1}{u^2}\cdot u^2-\sin\left(au^2-\frac{b}{u^2}\right)\mathrm{d}u\\&=\int_0^{\infty}\sin\left(au^2-\frac{b}{u^2}\right)\mathrm{d}u\end{aligned}$$

$$\frac{\partial^2}{\partial a\partial b}\left(\int_0^{\infty}\sin\left(au^2-\frac{b}{u^2}\right)\mathrm{d}u\right)=\int_0^{\infty}\sin\left(au^2-\frac{b}{u^2}\right)\mathrm{d}u$$

Let

$$\int_0^{\infty}\sin\left(au^2-\frac{b}{u^2}\right)\mathrm{d}u=F(a,b)$$

$$\implies \frac{\partial^2}{\partial a \partial b} F(a,b) = F(a,b)$$

Let $F(a,b)$ be the product of two functions $A(a)$, $B(b)$,

$$\implies F(a,b) = A(a)B(b)$$

$$\implies A'(a)B'(b) = A(a)B(b)$$

$$\implies \frac{A'(a)}{A(a)} = \frac{B(b)}{B'(b)}$$

Let

$$\frac{A'(a)}{A(a)} = \frac{B(b)}{B'(b)} = \lambda(a,b)$$

$$\implies \frac{A'(a)}{A(a)} = \frac{B(b)}{B'(b)} = \lambda(a,b)$$

$$\implies \frac{A'(a)}{A(a)} = \lambda(a,b), \quad \frac{B(b)}{B'(b)} = \lambda(a,b)$$

$$\implies \frac{A'(a)}{A(a)} = \lambda(a,b), \quad \frac{B'(b)}{B(b)} = \frac{1}{\lambda(a,b)}$$

$$\implies \frac{\mathrm{d}(A(a))}{A(a)} = \lambda(a,b)\,\mathrm{d}a, \quad \frac{\mathrm{d}(B(b))}{B(b)} = \frac{1}{\lambda(a,b)}\mathrm{d}b$$

$$\implies \int \frac{\mathrm{d}(A(a))}{A(a)} = \int \lambda(a,b)\,\mathrm{d}a, \quad \int \frac{\mathrm{d}(B(b))}{B(b)} = \int \frac{1}{\lambda(a,b)}\mathrm{d}b$$

$$\implies \ln(A(a)) = \int \lambda(a,b)\,\mathrm{d}a + c_1, \quad \ln(B(b)) = \int \frac{\mathrm{d}b}{\lambda(a,b)} + c_2$$

$$\implies A(a) = C_1 e^{\int \lambda(a,b)\,\mathrm{d}a}, \quad B(b) = C_2 e^{\int \frac{\mathrm{d}b}{\lambda(a,b)}}$$

Since $F(a,b) = A(a)B(b)$

$$\implies F(a,b) = C_1 C_2 e^{\int \lambda(a,b)\,\mathrm{d}a + \int \frac{\mathrm{d}b}{\lambda(a,b)}} = C_3 \exp\left(\int \lambda(a,b)\,\mathrm{d}a + \int \frac{\mathrm{d}b}{\lambda(a,b)}\right)$$

$$\begin{aligned} F(a,0) &= C_3 \exp\left(\int \lambda(a,0)\,\mathrm{d}a + \int \frac{\mathrm{d}b}{\lambda(a,0)}\right) \\ &= \int_0^\infty \sin\left(au^2 - \frac{0}{u^2}\right)\mathrm{d}u = \int_0^\infty \sin\left(au^2\right)\mathrm{d}u \\ &= \frac{1}{2\sqrt{a}}\sqrt{\frac{\pi}{2}}\exp(0) \end{aligned}$$

$$\implies C_3 = \frac{1}{2\sqrt{a}}\sqrt{\frac{\pi}{2}}, \qquad \int \lambda(a,0)\,\mathrm{d}a + \int \frac{\mathrm{d}b}{\lambda(a,0)} = 0$$

Let $\lambda(a,0) = f(a)$

$$\int f(a)\,\mathrm{d}a + \int \frac{\mathrm{d}b}{f(a)} = 0$$

$$\int f(a)\,\mathrm{d}a = -\int \frac{\mathrm{d}b}{f(a)} = -\frac{b}{f(a)} + \text{constant}$$

Taking the derivative at a,

$$\frac{\mathrm{d}}{\mathrm{d}a}\left(\int f(a)\,\mathrm{d}a\right) = -\frac{\mathrm{d}}{\mathrm{d}a}\left(\frac{b}{f(a)}\right)$$

$$\implies f(a) = -b\cdot\frac{-f'(a)}{f(a)^2} = \frac{bf'(a)}{f(a)^2}$$

$$\implies f'(a) = \frac{f(a)^3}{b}$$

$$\implies \frac{\mathrm{d}(f(a))}{f(a)^3} = \frac{\mathrm{d}a}{b}$$

$$\implies \int\frac{\mathrm{d}(f(a))}{f(a)^3} = \int\frac{\mathrm{d}a}{b}$$

For λ_1, γ_1 constant,

$$\implies -\frac{1}{2f(a)^2} = \frac{a}{b} + \lambda_1$$

$$\implies \frac{1}{f(a)^2} = -\frac{2a}{b} + \lambda_1 = \frac{b\gamma_1 - 2a}{b}$$

$$\implies f(a) = \sqrt{\frac{b}{b\gamma_1 - 2a}}$$

$$\begin{aligned} F(0,b) &= C_3\exp\left(\int\lambda(0,b)\,\mathrm{d}a + \int\frac{\mathrm{d}b}{\lambda(0,b)}\right) \\ &= \int_0^\infty \sin\left((0)u^2 - \frac{b}{u^2}\right)\mathrm{d}u = \int_0^\infty \sin\left(-\frac{b}{u^2}\right)\mathrm{d}u \\ &= -\sqrt{b}\sqrt{\frac{\pi}{2}}\exp(0) \end{aligned}$$

$$\implies C_3 = -\sqrt{b}\sqrt{\frac{\pi}{2}}, \qquad \int\lambda(0,b)\,\mathrm{d}a + \int\frac{\mathrm{d}b}{\lambda(0,b)} = 0$$

Let $\lambda(0,b) = f(b)$

$$\int f(b)\,\mathrm{d}a + \int\frac{\mathrm{d}b}{f(b)} = 0$$

$$af(b) + \int\frac{\mathrm{d}b}{f(b)} = \text{constant}$$

$$af(b) = -\int\frac{\mathrm{d}b}{f(b)} + \text{constant}$$

Taking the derivative at b,

$$\frac{\mathrm{d}}{\mathrm{d}b}(af(b)) = \frac{\mathrm{d}}{\mathrm{d}b}\left(-\int\frac{\mathrm{d}b}{f(b)}\right)$$

$$\implies af'(b) = -\frac{1}{f(b)}$$

$$\implies f'(b)f(b) = \frac{-1}{a}$$

$$\implies f(b)\frac{\mathrm{d}(f(b))}{\mathrm{d}b} = \frac{-1}{a}$$

$$\implies \frac{1}{2}\mathrm{d}\left(f(b)^2\right) = \frac{-\mathrm{d}b}{a}$$
$$\implies \frac{1}{2}\int \mathrm{d}\left(f(b)^2\right) = \int \frac{-\mathrm{d}b}{a}$$

For λ_2, γ_2 constant,

$$\implies \frac{f(b)^2}{2} = -\frac{b}{a} + \lambda_2$$
$$\implies f(b) = \sqrt{\gamma_2 - \frac{2b}{a}} = \sqrt{\frac{a\gamma_2 - 2b}{a}}$$

Since $f(b) = \lambda(0,b)$ and $f(a) = \lambda(a,0)$,

$$\implies \lambda(0,b) = \sqrt{\frac{a\gamma_2 - 2b}{a}} = \phi_1\sqrt{\phi_2 - 2b} \text{ and } \lambda(a,0) = \sqrt{\frac{b}{b\gamma_1 - 2a}} = \frac{\phi_3}{\sqrt{\phi_4 - 2a}}$$

$\lambda(a,b)$ can be written from $\lambda(0,b)$ and $\lambda(a,0)$

$$\implies \lambda(a,b) = \phi\sqrt{\frac{\phi_2 - 2b}{\phi_4 - 2a}} \ \ni \phi = \frac{\phi_3}{\sqrt{\phi_2}} \text{ or } \phi = \phi_1\sqrt{\phi_4}.$$

$\phi \neq 0$ since we are looking for non-trivial solutions of the PDE.

$$\implies \lambda(a,b) = \phi\sqrt{\frac{\phi_2 - 2b}{\phi_4 - 2a}}, \qquad \phi \neq 0$$

$$\begin{aligned} F(a,b) &= C_3 \exp\left(\phi\int\sqrt{\frac{\phi_2 - 2b}{\phi_4 - 2a}}\mathrm{d}a + \int\frac{1}{\phi}\sqrt{\frac{\phi_4 - 2b}{\phi_2 - 2a}}\mathrm{d}b\right) \\ &= C_3 \exp\left(-\phi\sqrt{(\phi_2 - 2b)(\phi_4 - 2a)} - \frac{1}{\phi}\sqrt{(\phi_2 - 2b)(\phi_4 - 2a)}\right) \\ &= C_3 \exp\left(-\left(\phi + \frac{1}{\phi}\right)\sqrt{(\phi_2 - 2b)(\phi_4 - 2a)}\right) \end{aligned}$$

Since $\phi \neq 0$, $\phi + \frac{1}{\phi} \neq 0$. So, let $\delta = \phi + \frac{1}{\phi} \ni \delta \neq 0$.

$$\implies F(a,b) = C_3 \exp\left(-\delta\sqrt{(\phi_2 - 2b)(\phi_4 - 2a)}\right)$$

$$\begin{aligned} F(a,0) &= C_3 \exp\left(-\delta\sqrt{(\phi_2)(\phi_4 - 2a)}\right) \\ &= \frac{1}{2\sqrt{a}}\sqrt{\frac{\pi}{2}}\exp(0) \end{aligned}$$

$$\implies C_3 = \frac{1}{2\sqrt{a}}, \ \phi_2 = 0.$$

$\phi_4 \neq 2a$, since ϕ_4 is an arbitrary constant, it cannot be a function of a.

$$F(0,b) = C_3 \exp\left(-\delta\sqrt{(-2b)(\phi_4 - 2(0))}\right)$$

$$= C_3 \exp\left(-\delta\sqrt{(-2b)(\phi_4)}\right)$$
$$= -\sqrt{b}\sqrt{\frac{\pi}{2}}\exp(0)$$

$\implies C_3 = -\sqrt{b},\ \sqrt{\phi_4(-2b)} = 0 \implies \phi_4 = 0$, since $b \neq 0$. $b \neq 0$ since $F(a,b) = C_3$ does not satisfy the PDE.

$$\implies F(a,b) = \frac{1}{2\sqrt{a}}\sqrt{\frac{\pi}{2}}\exp\left(-\delta\sqrt{-2b(\phi_4 - 2a)}\right)$$

or

$$F(a,b) = -\sqrt{b}\sqrt{\frac{\pi}{2}}\exp\left(-\delta\sqrt{-2a(\phi_2 - 2b)}\right)$$

Determining which of the two satisfies the PDE, let

$$F(a,b) = \frac{1}{2\sqrt{a}}\sqrt{\frac{\pi}{2}}\exp\left(-\delta\sqrt{-2b(\phi_4 - 2a)}\right)$$

$$\implies \frac{\partial^2}{\partial a\partial b}\left(\frac{1}{2\sqrt{a}}\sqrt{\frac{\pi}{2}}\exp\left(-\delta\sqrt{-2b(\phi_4 - 2a)}\right)\right)$$
$$= \frac{1}{2}\sqrt{\frac{\pi}{2}}\left(\frac{\delta\exp\left(\delta(-\sqrt{4ab - 2b\phi_4})\right)\left(4a\delta\sqrt{b(2a-\phi_4)} - \sqrt{2}\phi_4\right)}{4a\sqrt{a}\sqrt{b(2a-\phi_4)}}\right).$$

At $\phi_4 = 0$,

$$\frac{\partial^2}{\partial a\partial b}\left(\frac{1}{2\sqrt{a}}\sqrt{\frac{\pi}{2}}\exp\left(-2\delta\sqrt{ab}\right)\right) = \frac{1}{2}\sqrt{\frac{\pi}{2}}\left(\frac{\delta\exp\left(-\delta\sqrt{4ab}\right)(4a\delta)}{4a\sqrt{a}}\right)$$
$$= \frac{1}{2}\sqrt{\frac{\pi}{2}}\left(\frac{\delta^2\exp\left(-\delta\sqrt{4ab}\right)}{\sqrt{a}}\right) = \frac{\delta^2}{2\sqrt{a}}\sqrt{\frac{\pi}{2}}\exp\left(-2\delta\sqrt{ab}\right)$$

If $\delta = 1$, then $\frac{\partial^2}{\partial a\partial b}\left(\frac{1}{2\sqrt{a}}\sqrt{\frac{\pi}{2}}\exp\left(-2\delta\sqrt{ab}\right)\right) = \frac{1}{2\sqrt{a}}\sqrt{\frac{\pi}{2}}\exp\left(-2\sqrt{ab}\right)$.

Also, $F(a,b) = \frac{1}{2\sqrt{a}}\sqrt{\frac{\pi}{2}}\exp\left(-2\sqrt{ab}\right)$.

It implies that $F(a,b) = \frac{1}{2\sqrt{a}}\sqrt{\frac{\pi}{2}}\exp\left(-2\sqrt{ab}\right)$ satisfies the PDE.

Also let $F(a,b) = -\sqrt{b}\sqrt{\frac{\pi}{2}}\exp\left(-\delta\sqrt{-2a(\phi_2 - 2b)}\right)$

$$\implies \frac{\partial^2}{\partial a\partial b}\left(-\sqrt{b}\sqrt{\frac{\pi}{2}}\exp\left(-\delta\sqrt{-2a(\phi_2 - 2b)}\right)\right)$$
$$= \sqrt{\frac{\pi}{2}}\left(\frac{\delta\exp\left(-\delta\sqrt{-2a(\phi_2 - 2b)}\right)\left(\sqrt{2}(4b-\phi_2) - 4b\delta\sqrt{a(2b-\phi_2)}\right)}{4\sqrt{b}\sqrt{a(2b-\phi_2)}}\right).$$

At $\phi_2 = 0$,

$$\frac{\partial^2}{\partial a\partial b}\left(-\sqrt{b}\sqrt{\frac{\pi}{2}}\exp\left(-2\delta\sqrt{ab}\right)\right) = \left(-\delta^2\sqrt{b}\exp\left(-2\delta\sqrt{ab}\right) + \frac{\delta}{\sqrt{a}}\exp\left(-2\delta\sqrt{ab}\right)\right).$$

But

$$\sqrt{\frac{\pi}{2}}\left(-\delta^2\sqrt{b}\exp\left(-2\delta\sqrt{ab}\right)+\frac{\delta}{\sqrt{a}}\exp\left(-2\delta\sqrt{ab}\right)\right)\neq-\sqrt{b}\sqrt{\frac{\pi}{2}}\exp\left(-2\delta\sqrt{ab}\right)$$

for constant values of δ and this implies that $F(a,b)\neq-\sqrt{b}\sqrt{\frac{\pi}{2}}\exp\left(-2\delta\sqrt{ab}\right)$ does not satisfy the PDE for all constant values of δ.

Hence,

$$F(a,b)=\frac{1}{2\sqrt{a}}\sqrt{\frac{\pi}{2}}\exp\left(-2\sqrt{ab}\right)$$

By the result in (11),

$$K_{\frac{1}{2}}(x)=\sqrt{\frac{\pi}{2x}}e^{-x}$$

$$\Longrightarrow F(a,b)=\frac{1}{2\sqrt{a}}\sqrt{\frac{\pi}{2}}\exp\left(-2\sqrt{ab}\right)=\sqrt{\sqrt{\frac{b}{2a}}}\sqrt{\frac{\pi}{2\cdot2\sqrt{ab}}}e^{-2\sqrt{ab}}$$
$$=\left(\frac{b}{2a}\right)^{\frac{1}{4}}K_{\frac{1}{2}}\left\{2\sqrt{ab}\right\}$$

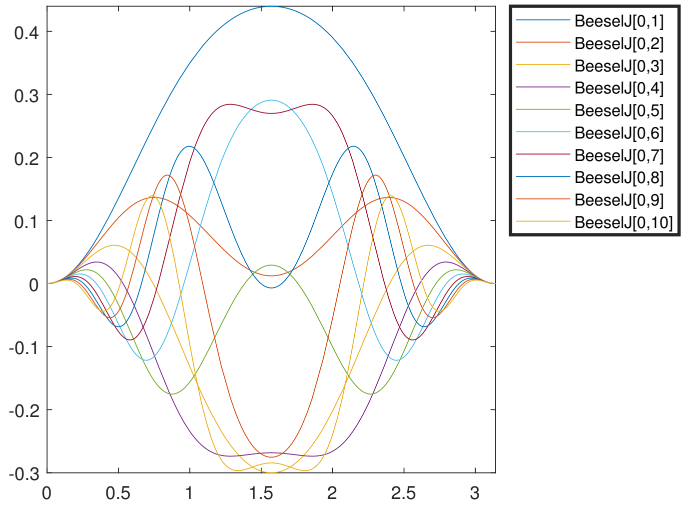
$$\therefore\int_0^\infty\sin\left(au^2-\frac{b}{u^2}\right)\mathrm{d}u=\left(\frac{b}{2a}\right)^{\frac{1}{4}}K_{\frac{1}{2}}\left\{2\sqrt{ab}\right\}$$

■

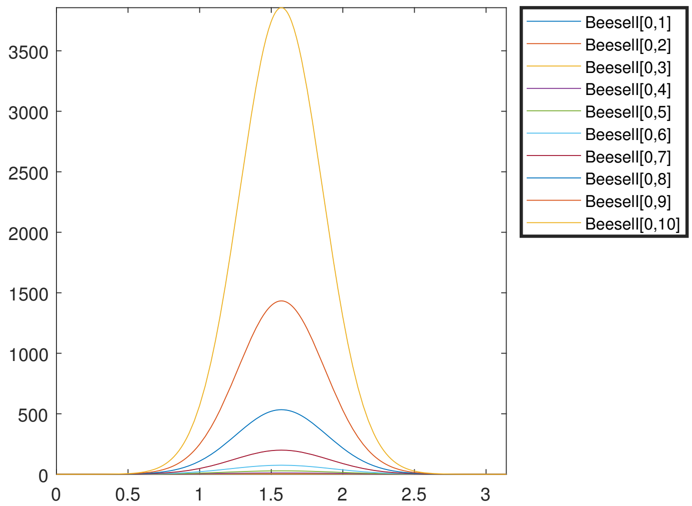
10 Main Results/Propositions

$$
\begin{aligned}
J_\alpha(x) &= \frac{1}{\Gamma(\alpha)}\left(\frac{x}{2}\right)^\alpha \int_0^1 J_0\left(x\sqrt{1-t}\right) t^{\alpha-1}\mathrm{d}t \\
&= \frac{1}{\pi x\Gamma(\alpha)}\left(\frac{x}{2}\right)^\alpha \int_0^{\frac{\pi}{2}}\int_0^\pi 2\sin^{2\alpha-1}\vartheta\sin\phi\frac{\partial}{\partial\left(x\cos\vartheta\right)}\left(x\cos\vartheta\sin\left(x\cos\vartheta\sin\phi\right)\right)\mathrm{d}\phi\mathrm{d}\vartheta \\
I_\alpha(x) &= \frac{1}{\Gamma(\alpha)}\left(\frac{x}{2}\right)^\alpha \int_0^1 I_0\left(x\sqrt{1-t}\right) t^{\alpha-1}\mathrm{d}t \\
&= \frac{1}{\pi x\Gamma(\alpha)}\left(\frac{x}{2}\right)^\alpha \int_0^{\frac{\pi}{2}}\int_0^\pi 2\sin^{2\alpha-1}\vartheta\sin\phi\frac{\partial}{\partial\left(x\cos\vartheta\right)}\left(x\cos\vartheta\sinh\left(x\cos\vartheta\sin\phi\right)\right)\mathrm{d}\phi\mathrm{d}\vartheta \\
J_0(x) &= \frac{1}{\pi x}\int_0^\pi \sin\phi\frac{\partial}{\partial x}\left(x\sin\left(x\sin\phi\right)\right)\mathrm{d}\phi \\
I_0(x) &= \frac{1}{\pi x}\int_0^\pi \sin\phi\frac{\partial}{\partial x}\left(x\sinh\left(x\sin\phi\right)\right)\mathrm{d}\phi \\
&\int_0^\infty \sin\left(au^2-\frac{b}{u^2}\right)\mathrm{d}u = \left(\frac{b}{2a}\right)^{\frac{1}{4}} K_{\frac{1}{2}}\left\{2\sqrt{ab}\right\}
\end{aligned}
$$

10.1 The area between each of the curves and the x-axis gives the values of $J_0(z)$ from $z = 1$ to 10



10.2 The area between each of the curves and the x-axis gives the values of $I_0(z)$ from $z = 1$ to 10



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