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ROMANIAN MATHEMATICAL MAGAZINE

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Find:

$$\Omega = \int_0^\infty \frac{\log(1+x)}{x(x^2+x+1)}\,dx$$

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Solution by Togrul Ehmedov-Azerbaijan

$$\Omega = \int_0^\infty \frac{\log(1+x)}{x(x^2+x+1)}\,dx = \int_0^1\int_0^\infty \frac{dxdy}{(1+xy)(x^2+x+1)} =$$

$$= \int_0^1 \frac{1}{1-y+y^2}\int_0^\infty \left[\frac{-yx+1-y}{x^2+x+1} + \frac{y^2}{1+xy}\right] dx\,dy =$$

$$= \int_0^1 \frac{1}{1-y+y^2}\left[y\log(y) - \frac{\sqrt{3}}{9}\pi y + \frac{2\sqrt{3}}{9}\pi\right] dy =$$

$$= \int_0^1 \frac{y\log(y)}{1-y+y^2}\,dy - \frac{\sqrt{3}}{9}\pi\int_0^1 \frac{y}{1-y+y^2}\,dy + \frac{2\sqrt{3}}{9}\pi\int_0^1 \frac{1}{1-y+y^2}\,dy =$$

$$=\frac{1}{6}\left(\frac{5\pi^2}{6}-\varphi'\left(\frac{1}{3}\right)\right)-\frac{\sqrt{3}}{9}\pi\left(\frac{\pi}{3\sqrt{3}}\right)+\frac{2\sqrt{3}}{9}\pi\left(\frac{2\pi}{3\sqrt{3}}\right)=\frac{\pi^2}{4}-\frac{1}{6}\varphi'\left(\frac{1}{3}\right)$$

Note: $\int_0^1 \frac{y\log(y)}{1-y+y^2}dy=\frac{1}{6}\left(\frac{5\pi^2}{6}-\varphi'\left(\frac{1}{3}\right)\right)$