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Find:

$$\Omega = \int_0^{\infty} \frac{\sqrt{x}\log(x)}{x^2+1}\,dx$$

Proposed by Vasile Mircea Popa-Romania

Solution 1 by Togrul Ehmedov-Azerbaijan, Solution 2 by Hikmat Mammadov-Azerbaijan, Solution 3 by Adrian Popa-Romania, Solution 4 by Fao Ler-Iraq

Solution 1 by Togrul Ehmedov-Azerbaijan

$$\Omega = \int_0^{\infty} \frac{\sqrt{x}\log(x)}{x^2+1}\,dx = \int_0^{1} \frac{\sqrt{x}\log(x)}{x^2+1}\,dx + \int_1^{\infty} \frac{\sqrt{x}\log(x)}{x^2+1}\,dx\Bigg]_{x=\frac{1}{y}} =$$

$$= \int_0^{1} \frac{\sqrt{x}\log(x)}{x^2+1}\,dx - \int_0^{1} \frac{\log(y)}{\sqrt{y}(y^2+1)}\,dy =$$

$$= \sum_{k=0}^{\infty}(-1)^k \int_0^1 x^{2k+\frac{1}{2}}\log(x)\,dx - \sum_{k=0}^{\infty}(-1)^k \int_0^1 y^{2k-\frac{1}{2}}\log(y)\,dy =$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{\left(2k+\frac{3}{2}\right)^2}-\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{\left(2k+\frac{1}{2}\right)^2}=$$

$$=\sum_{k=0}^{\infty}\left[\frac{1}{\left(4k+\frac{7}{2}\right)^2}-\frac{1}{\left(4k+\frac{3}{2}\right)^2}\right]-\sum_{k=0}^{\infty}\left[\frac{1}{\left(4k+\frac{5}{2}\right)^2}-\frac{1}{\left(4k+\frac{1}{2}\right)^2}\right]=$$

$$=\frac{1}{16}\left[\sum_{k=0}^{\infty}\left[\frac{1}{\left(k+\frac{7}{8}\right)^2}-\frac{1}{\left(k+\frac{3}{8}\right)^2}\right]-\sum_{k=0}^{\infty}\left[\frac{1}{\left(k+\frac{5}{8}\right)^2}-\frac{1}{\left(k+\frac{1}{8}\right)^2}\right]\right]=$$

$$=\frac{1}{16}\left[\varphi^{(1)}\left(\frac{7}{8}\right)-\varphi^{(1)}\left(\frac{3}{8}\right)-\varphi^{(1)}\left(\frac{5}{8}\right)+\varphi^{(1)}\left(\frac{1}{8}\right)\right]$$

We know that

$$\varphi^{(1)}(1-z)+\varphi^{(1)}(z)=\frac{\pi^2}{\sin^2(\pi z)}\Rightarrow\begin{cases}\varphi^{(1)}\left(\frac{7}{8}\right)+\varphi^{(1)}\left(\frac{1}{8}\right)=\frac{\pi^2}{\sin^2\left(\frac{\pi}{8}\right)}\\ \varphi^{(1)}\left(\frac{5}{8}\right)+\varphi^{(1)}\left(\frac{3}{8}\right)=\frac{\pi^2}{\sin^2\left(\frac{3\pi}{8}\right)}\end{cases}$$

$$\Omega=\frac{1}{16}\left[\frac{\pi^2}{\sin^2\left(\frac{\pi}{8}\right)}-\frac{\pi^2}{\sin^2\left(\frac{3\pi}{8}\right)}\right]=\frac{1}{16}\left[\frac{2\pi^2}{1-\cos\left(\frac{\pi}{4}\right)}-\frac{2\pi^2}{1-\cos\left(\frac{3\pi}{4}\right)}\right]=$$

$$=\frac{\pi^2}{8}\left[\frac{1}{1-\cos\left(\frac{\pi}{4}\right)}-\frac{1}{1+\cos\left(\frac{\pi}{4}\right)}\right]=\frac{\sqrt{2}\pi^2}{4}$$

Solution 2 by Hikmat Mammadov-Azerbaijan

$$\Omega=\int_0^{\infty}\frac{\sqrt{x}\log x}{x^2+1}dx=\frac{\partial}{\partial a}\int_0^{\infty}\frac{x^a}{1+x^2}dx\Big|_{a=\frac{1}{2}}$$

$$\int_0^{\infty}\frac{x^a}{1+x^2}dx=\int_0^{\infty}\frac{v^{\frac{a}{2}}}{1+v}\cdot\frac{dv}{2\sqrt{v}}=\frac{1}{2}\int_0^{\infty}\frac{v^{\frac{a+1}{2}-1}}{1+v}dv=f(a)$$

$$\beta(x,y)=\int_0^{\infty}\frac{u^{x-1}}{(1+v)^{x+y}}dx$$

$$f(a)=\frac{\beta\left(\frac{a+1}{2};\frac{1-a}{2}\right)}{2}=\frac{1}{2}\left[\beta\left(\frac{1+a}{2};1-\frac{1+a}{2}\right)\right]=$$

$$=\frac{1}{2}\cdot\frac{\pi}{\sin\left(\frac{1+a}{2}\pi\right)}=\frac{\pi}{2\cos\left(\frac{\pi a}{2}\right)}$$

$$\Omega=\frac{\pi^2}{4}\cdot\frac{\sin\left(\frac{\pi a}{2}\right)}{\cos^2\left(\frac{\pi a}{2}\right)}\Big|_{a=\frac{1}{2}}=\frac{\pi^2}{4}\cdot\sqrt{2}=\frac{\pi^2}{2\sqrt{2}}$$

Therefore,

$$\Omega=\int_0^\infty\frac{\sqrt{x}\log x}{x^2+1}dx=\frac{\pi^2}{2\sqrt{2}}$$

Solution 3 by Adrian Popa-Romania

$$\Omega=\int_0^\infty\frac{\sqrt{x}\log x}{x^2+1}dx=\frac{\partial}{\partial a}\int_0^\infty\frac{x^a}{1+x^2}dx|_{a=\frac{1}{2}}$$

$$=\frac{\partial}{\partial a}\left(\frac{1}{2}\int_0^\infty\frac{\left(t^{\frac{1}{2}}\right)^{a+\frac{1}{2}}}{t+1}\cdot t^{-\frac{1}{2}}dt\right)\Bigg|_{a=0}=\frac{\partial}{\partial a}\cdot\frac{1}{2}\int_0^\infty\frac{t^{\frac{a}{2}-\frac{1}{4}}}{t+1}\Big|_{a=0}dt$$

$$\beta(m,n)=\int_0^\infty\frac{x^{m-1}}{(1+x)^{m+n}}dx$$

$$m-1=\frac{a}{2}-\frac{1}{4}\Rightarrow m=\frac{a}{2}+\frac{3}{4}$$

$$m+n=1\Rightarrow\frac{a}{2}+\frac{3}{4}+n=1\Rightarrow n=-\frac{a}{2}+\frac{1}{4}$$

$$\Omega=\frac{\partial}{\partial a}\cdot\frac{1}{2}\beta\left(\frac{a}{2}+\frac{3}{4};-\frac{a}{2}+\frac{1}{4}\right)_{|a=0}$$

$$\beta\left(\frac{a}{2}+\frac{3}{4};-\frac{a}{2}+\frac{1}{4}\right)=\frac{\Gamma\left(\frac{2a+3}{4}\right)\Gamma\left(\frac{-2a+1}{4}\right)}{\Gamma(1)}=\Gamma\left(\frac{2a+3}{4}\right)\Gamma\left(1-\frac{2a+3}{4}\right)$$

$$\Gamma(x)\Gamma(1-x)=\frac{\pi}{\sin\pi x}\Rightarrow\beta\left(\frac{2a+3}{4};\frac{-2a+1}{4}\right)=\frac{\pi}{\sin\frac{\pi(2a+3)}{4}}$$

$$\frac{\partial}{\partial a}\beta\left(\frac{2a+3}{4};\frac{-2a+1}{4}\right)\Big|_{a=0}=\frac{\partial}{\partial a}\frac{\pi}{\sin\left(\frac{\pi a}{2}+\frac{3\pi}{4}\right)}=\frac{-\pi\cos\left(\frac{\pi a}{2}+\frac{3\pi}{4}\right)\cdot\frac{\pi}{2}}{\sin^2\left(\frac{\pi a}{2}+\frac{3\pi}{4}\right)}=\frac{\pi^2\sqrt{2}}{2}$$

Therefore,

$$\Omega = \frac{1}{2} \cdot \frac{\pi^2\sqrt{2}}{2} = \frac{\pi^2}{2\sqrt{2}}$$

Solution 4 by Fao Ler-Iraq

$$\Omega = \int_0^\infty \frac{\sqrt{x}\log x}{x^2+1}dx = \int_0^\infty \frac{\sqrt{\sqrt{x}}\log\sqrt{x}}{x+1}d(\sqrt{x}) =$$

$$= \frac{1}{4}\int_0^\infty \frac{x^{\frac{1}{4}-\frac{1}{2}}\log x}{x+1}dx = \frac{1}{4}\frac{d}{dx}\int_0^\infty \frac{x^y}{x+1}dx\,;\left(y = -\frac{1}{4}\right)$$

$$= \frac{1}{4}\frac{d}{dy}\pi\csc\left(\pi(y+1)\right) = \frac{\pi}{4}\left(-\pi\cot\left(\pi(y+1)\right)\csc\left(\pi(y+1)\right)\right) =$$

$$= -\frac{\pi^2}{4}\cot\left(\frac{3\pi}{4}\right)\csc\left(\frac{3\pi}{4}\right) = \frac{\pi^2}{4}\sqrt{2}$$