FAMOUS INEQUALITIES REDESIGNED IN THE TRIANGLE WITH SIDES SUM BY SIDES

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ABSTRACT. In this paper it is considered an initial triangle with sides a, b, c. We take the triangle with side $a' = b + c; b' = c + a; c' = a + b$ and we find the form of a few famous inequalities in triangle obtained in this new triangle.

Notations:
a, b, c, s, R, r, F - sides, semiperimeter, circumradii, inradii and area of the initial triangle.
$a', b', c', s', R', r', F'$ - sides, semiperimeter, circumradii, inradii and area of the new triangle.
r_a, r_b, r_c - exradii in the initial triangle
r'_a, r'_b, r'_c - exradii in the new triangle
h_a, h_b, h_c - altitudes in the new triangle
h'_a, h'_b, h'_c - altitudes in the new triangle
A, B, C - angles in the initial triangle
A', B', C' - angles in the new triangle
Preliminaries:

$$s' = \frac{1}{2}(a' + b' + c') = \frac{1}{2}(2a + 2b + 2c) = a + b + c = 2s$$

$$F' = \sqrt{s'(s' - a')(s' - b')(s' - c')} = \sqrt{(a + b + c)abc} =$$

$$= \sqrt{2s \cdot 4RF} = \sqrt{2s \cdot 4Rrs} = 2s\sqrt{2Rr}$$

$$R' = \frac{a'b'c'}{4F'} = \frac{(a + b)(b + c)(c + a)}{4 \cdot 2s\sqrt{2Rr}} = \frac{(a + b)(b + c)(c + a)}{8s\sqrt{2Rr}} =$$

$$= \frac{2s(s^2 + r^2 + 2Rr)}{4 \cdot 2s\sqrt{2Rr}} = \frac{s^2 + r^2 + 2Rr}{4\sqrt{2Rr}}$$

$$r' = \frac{F'}{s'} = \frac{2s\sqrt{2Rr}}{2s} = \sqrt{2Rr}$$

$$r'_a = \frac{F'}{s' - a'} = \frac{\sqrt{(a + b + c)abc}}{a + b + c - b - c} = \sqrt{\frac{(a + b + c)abc}{a^2}} = \sqrt{\frac{(a + b + c)bc}{a}}$$

$$r'_b = \frac{F'}{s' - b'} = \sqrt{\frac{(a + b + c)ca}{b}}$$

$$r'_c = \frac{F'}{s' - c'} = \sqrt{\frac{(a + b + c)ab}{c}}$$

$$r'_a + r'_b + r'_c = \frac{F'}{s' - a'} + \frac{F'}{s' - b'} + \frac{F'}{s' - c'} = F'\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) =$$

$$= F' \cdot \frac{ab + bc + ca}{abc} = F' \cdot \frac{ab + bc + ca}{4R'F'} = \frac{ab + bc + ca}{4R'} =$$

$$=\frac{ab+bc+ca}{4\cdot\frac{s^2+r^2+2Rr}{4\sqrt{2Rr}}}=\frac{(ab+bc+ca)\sqrt{2Rr}}{s^2+r^2+2Rr}$$

$$\frac{1}{r'_a}+\frac{1}{r'_b}+\frac{1}{r'_c}=\frac{s'-a'}{F'}+\frac{s'-b'}{F'}+\frac{s'-c'}{F'}=$$

$$=\frac{3s'-2s'}{F'}=\frac{s'}{F'}=\frac{1}{r'}=\frac{1}{\sqrt{2Rr}}$$

$$\sum_{cyc}(m'_a)^2=\frac{3}{4}\sum_{cyc}(a')^2=\frac{3}{4}\sum_{cyc}(b+c)^2=$$

$$=\frac{3}{4}\sum_{cyc}(b^2+c^2+2bc)=$$

$$=\frac{3}{4}\cdot 2\sum_{cyc}a^2+\frac{3}{2}\sum_{cyc}bc=$$

$$=\frac{3}{2}\cdot\frac{4}{3}\sum_{cyc}m_a^2+\frac{3}{2}\sum_{cyc}bc=$$

$$=\sum_{cyc}m_a^2+\sum_{cyc}m_a^2+\frac{3}{2}\sum_{cyc}bc=$$

$$=\sum_{cyc}m_a^2+\frac{3}{4}\sum_{cyc}a^2+\frac{3}{2}\sum_{cyc}bc=$$

$$=\sum_{cyc}m_a^2+\frac{3}{4}\cdot 2(s^2-r^2-4Rr)+\frac{3}{2}(s^2+r^2+4Rr)=$$

$$=\sum_{cyc}m_a^2+\frac{3}{2}(s^2-r^2-4Rr+s^2+r^2+4Rr)=$$

$$=3s^2+\sum_{cyc}m_a^2$$

$$\sum_{cyc}(m'_a)^2=3s^2+\sum_{cyc}m_a^2=\frac{3}{4}(a+b+c)^2+\frac{3}{4}(a^2+b^2+c^2)$$

$$h'_a=\frac{2F'}{b+c}=\frac{2\sqrt{(a+b+c)abc}}{b+c}$$

$$h'_b=\frac{2F'}{c+a}=\frac{2\sqrt{(a+b+c)abc}}{c+a}$$

$$h'_c=\frac{2F'}{a+b}=\frac{2\sqrt{(a+b+c)abc}}{a+b}$$

$$\sin A'=\frac{a'}{2R'}=\frac{b+c}{2\cdot\frac{(a+b)(b+c)(c+a)}{8s\sqrt{2Rr}}}=\frac{4s\sqrt{2Rr}}{(a+b)(a+c)}$$

$$\sin B'=\frac{4s\sqrt{2Rr}}{(b+c)(b+a)};\sin C'=\frac{4s\sqrt{2Rr}}{(c+a)(c+b)}$$

$$\cos A'=\frac{(b')^2+(c')^2-(a')^2}{2b'c'}=\frac{(a+c)^2+(a+b)^2-(b+c)^2}{2(a+b)(a+c)}=$$

$$=\frac{a^2+ac+ab-bc}{(a+b)(a+c)}=\frac{2sa-bc}{(a+b)(a+c)}$$

$$\cos B' = \frac{2sb - ca}{(b+a)(b+c)}; \cos C' = \frac{2sc - ab}{(c+a)(c+b)}$$
$$\sin \frac{A'}{2} = \sqrt{\frac{(s'-b')(s'-c')}{b'c'}} = \sqrt{\frac{bc}{(a+b)(a+c)}}$$
$$\sin \frac{B'}{2} = \sqrt{\frac{ca}{(b+c)(b+a)}}; \sin \frac{C'}{2} = \sqrt{\frac{ab}{(c+a)(c+b)}}$$
$$\cos \frac{A'}{2} = \sqrt{\frac{s'(s'-a')}{b'c'}} = \sqrt{\frac{(a+b+c)a}{(a+b)(a+c)}}$$
$$\cos \frac{B'}{2} = \sqrt{\frac{(a+b+c)b}{(b+a)(b+c)}}; \cos \frac{C'}{2} = \sqrt{\frac{(a+b+c)c}{(c+a)(c+b)}}$$

1. MITRINOVIC'S INEQUALITY
In $\Delta A'B'C'$ the following relationship holds:

$$3\sqrt{3}r' \le s' \le \frac{3\sqrt{3}}{2}R'$$

Redesigned:

$$s' \ge 3\sqrt{3}r' \Leftrightarrow a+b+c \ge 3\sqrt{3} \cdot \sqrt{2Rr}$$
$$a+b+c \ge 3\sqrt{6Rr}$$
$$s' \le \frac{3\sqrt{3}}{2}R' \Leftrightarrow a+b+c \le \frac{3\sqrt{3}}{2} \cdot \frac{(a+b)(b+c)(c+a)}{8s\sqrt{2Rr}} =$$
$$= \frac{3\sqrt{6}(a+b)(b+c)(c+a)}{16s\sqrt{Rr}}$$
$$a+b+c \le \frac{3\sqrt{6}(a+b)(b+c)(c+a)}{16s\sqrt{Rr}}$$

2. IONESCU-WEITZENBOCK'S INEQUALITY
In $\Delta A'B'C'$ the following relationship holds:

$$(a')^2 + (b')^2 + (c')^2 \ge 4\sqrt{3}F'$$

Redesigned: In ΔABC holds:

$$(b+c)^2 + (c+a)^2 + (a+b)^2 \ge 4\sqrt{3} \cdot \sqrt{(a+b+c)abc}$$
$$(b+c)^2 + (c+a)^2 + (a+b)^2 \ge 4\sqrt{3abc(a+b+c)}$$

3. ZETEL'S INEQUALITY
In $\Delta A'B'C'$ the following relationship holds:

$$h'_a \cdot h'_b \cdot h'_c \ge 27(r')^3$$

Redesigned: In ΔABC holds:

$$\frac{8(a+b+c)abc\sqrt{(a+b+c)abc}}{(a+b)(b+c)(c+a)} \ge 27 \cdot 2Rr \cdot \sqrt{2Rr}$$
$$\frac{(a+b+c)abc\sqrt{(a+b+c)abc}}{(a+b)(b+c)(c+a)} \ge \frac{27Rr\sqrt{2Rr}}{4}$$

4. SANTALO'S INEQUALITY
In $\Delta A'B'C'$ the following relationship holds:

$$(m_a')^2 + (m_b')^2 + (m_c')^2 \geq 3\sqrt{3}F'$$

Redesigned: In ΔABC holds:

$$\sum_{cyc}(m_a')^2 = \frac{3}{4}(a+b+c)^2 + \frac{3}{4}(a^2+b^2+c^2)$$

$$\frac{3}{4}(a+b+c)^2 + \frac{3}{4}(a^2+b^2+c^2) \geq 3\sqrt{3}\cdot\sqrt{(a+b+c)abc}$$

$$\frac{1}{4}(a+b+c)^2 + \frac{1}{4}(a^2+b^2+c^2) \geq \sqrt{3abc(a+b+c)}$$

$$(a+b+c)^2 + a^2 + b^2 + c^2 \geq 4\sqrt{3abc(a+b+c)}$$

5. GORDON'S INEQUALITY
In $\Delta A'B'C'$ the following relationship holds:

$$a'b' + b'c' + c'a' \geq 4\sqrt{3}F'$$

Redesigned: In ΔABC holds:

$$(a+b)(a+c) + (b+c)(b+a) + (c+a)(c+b) \geq 4\sqrt{3abc(a+b+c)}$$

6. GOLDNER'S INEQUALITY - I
In $\Delta A'B'C'$ the following relationship holds:

$$(a')^4 + (b')^4 + (c')^4 \geq 16(F')^2$$

Redesigned: In ΔABC holds:

$$(a+b)^4 + (b+c)^4 + (c+a)^2 \geq 16abc(a+b+c)$$

7. GOLDNER'S INEQUALITY - II
In $\Delta A'B'C'$ the following relationship holds:

$$(a')^2(b')^2 + (b')^2(c')^2 + (c')^2(a')^2 \geq 16(F')^2$$

Redesigned: In ΔABC holds:

$$(a+b)^2(b+c)^2 + (b+c)^2(c+a)^2 + (c+a)^2(a+b)^2 \geq 16abc(a+b+c)$$

8. BĂNDILĂ'S INEQUALITY
In $\Delta A'B'C'$ the following relationship holds:

$$\max\left(\frac{a'}{b'} + \frac{b'}{a'}, \frac{b'}{c'} + \frac{c'}{b'}, \frac{c'}{a'} + \frac{a'}{c'}\right) \leq \frac{R'}{r'}$$

Redesigned: In ΔABC holds:

$$\max\left(\frac{a+b}{a+c} + \frac{a+c}{a+b}, \frac{b+c}{b+a} + \frac{b+a}{b+c}, \frac{c+a}{c+b} + \frac{c+b}{c+a}\right) \leq$$

$$\leq \frac{\frac{(a+b)(b+c)(c+a)}{8s\sqrt{2Rr}}}{\sqrt{2Rr}} = \frac{(a+b)(b+c)(c+a)}{16sRr} = \frac{(a+b)(b+c)(c+a)}{16RF}$$

$$\max\left(\frac{a+b}{a+c}+\frac{a+c}{a+b}, \frac{b+c}{b+a}+\frac{b+a}{b+c}, \frac{c+a}{c+b}+\frac{c+b}{c+a}\right) \leq \frac{(a+b)(b+c)(c+a)}{16RF}$$

9. LEIBNIZ'S INEQUALITY

In $\Delta A'B'C'$ the following relationship holds:

$$(a')^2+(b')^2+(c')^2 \leq 9(R')^2$$

Redesigned: In ΔABC holds:

$$(a+b)^2+(b+c)^2+(c+a)^2 \leq 9\left(\frac{(a+b)(b+c)(c+a)}{8s\sqrt{2Rr}}\right)^2 =$$

$$= \frac{9(a+b)^2(b+c)^2(c+a)^2}{128s^2Rr} = \frac{9(a+b)^2(b+c)^2(c+a)^2}{128RFs}$$

$$(a+b)^2+(b+c)^2+(c+a)^2 \leq \frac{9(a+b)^2(b+c)^2(c+a)^2}{128RFs}$$

10. EULER'S INEQUALITY

In $\Delta A'B'C'$ the following relationship holds:

$$R' \geq 2r'$$

Redesigned: In ΔABC holds:

$$\frac{(a+b)(b+c)(c+a)}{8s\sqrt{2Rr}} \geq 2\sqrt{2Rr}$$

$$(a+b)(b+c)(c+a) \geq 32sRr$$

$$(a+b)(b+c)(c+a) \geq 32RF$$

11. STEINIG'S INEQUALITY

In $\Delta A'B'C'$ the following relationship holds:

$$\frac{1}{a'}+\frac{1}{b'}+\frac{1}{c'} \leq \frac{\sqrt{3}}{2r'}$$

Redesigned: In ΔABC holds:

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} \leq \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2Rr}} = \frac{1}{2}\sqrt{\frac{3}{2Rr}} = \frac{1}{4}\sqrt{\frac{6}{Rr}}$$

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} \leq \frac{1}{4}\sqrt{\frac{6}{Rr}}$$

12. LEUENBERGER'S INEQUALITY

In $\Delta A'B'C'$ the following relationship holds:

$$\frac{1}{(R')^2} \leq \frac{1}{a'b'}+\frac{1}{b'c'}+\frac{1}{c'a'} \leq \frac{1}{4(r')^2}$$

Redesigned: In ΔABC holds:

$$\frac{128s^2Rr}{(a+b)^2(b+c)^2(c+a)^2} \leq \frac{1}{(a+b)^2(a+c)^2}+\frac{1}{(b+c)^2(b+a)^2}+\frac{1}{(c+a)^2(c+b)^2} \leq \frac{1}{4\cdot 2Rr}$$

$$\frac{128RFs}{(a+b)^2(b+c)^2(c+a)^2} \leq \frac{1}{(a+b)^2(a+c)^2}+\frac{1}{(b+c)^2(b+a)^2}+\frac{1}{(c+a)^2(c+b)^2} \leq \frac{1}{8Rr}$$

13. KLAMKIN'S INEQUALITY - I
In $\Delta A'B'C'$ the following relationship holds:

$$4(r')^2 \leq \frac{a'b'c'}{a'+b'+c'} \leq (R')^2$$

Redesigned in ΔABC:

$$4 \cdot 2Rr \leq \frac{(a+b)(b+c)(c+a)}{a+b+b+c+c+a} \leq \frac{(a+b)^2(b+c)^2(c+a)^2}{128s^2Rr}$$

$$8Rr \leq \frac{(a+b)(b+c)(c+a)}{2(a+b+c)} \leq \frac{(a+b)^2(b+c)^2(c+a)^2}{128RFs}$$

$$16Rr \leq \frac{(a+b)(b+c)(c+a)}{a+b+c} \leq \frac{(a+b)^2(b+c)^2(c+a)^2}{64RFs}$$

14. KLAMKIN'S INEQUALITY - II
In $\Delta A'B'C'$ the following relationship holds:

$$9r' \leq r'_a + r'_b + r'_c \leq \frac{9R'}{2}$$

Redesigned in ΔABC:

$$9\sqrt{2Rr} \leq \frac{(ab+bc+ca)\sqrt{2Rr}}{s^2+r^2+2Rr} \leq \frac{9(a+b)(b+c)(c+a)}{16s\sqrt{2Rr}}$$

$$9 \leq \frac{ab+bc+ca}{s^2+r^2+2Rr} \leq \frac{9(a+b)(b+c)(c+a)}{32RF}$$

15. BOKOV'S INEQUALITY
In $\Delta A'B'C'$ the following relationship holds:

$$(r'_a)^2 + (r'_b)^2 + (r'_c)^2 \geq (s')^2$$

Redesigned: In ΔABC holds:

$$\frac{(F')^2}{(s'-a')^2} + \frac{(F')^2}{(s'-b')^2} + \frac{(F')^2}{(s'-c')^2} \geq (a+b+c)^2$$

$$(a+b+c)abc\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) \geq (a+b+c)^2$$

$$abc\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) \geq a+b+c$$

$$\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c} \geq a+b+c$$

16. JANIC'S INEQUALITY
In $\Delta A'B'C'$ the following relationship holds:

$$\frac{(a')^2}{r'_b r'_c} + \frac{(b')^2}{r'_c r'_a} + \frac{(c')^2}{r'_a r'_b} \geq 4$$

Redesigned: In ΔABC holds:

$$\frac{(b+c)^2}{\frac{F'}{s'-b'} \cdot \frac{F'}{s'-c'}} + \frac{(c+a)^2}{\frac{F'}{s'-c'} \cdot \frac{F'}{s'-a'}} + \frac{(a+b)^2}{\frac{F'}{s'-a'} \cdot \frac{F'}{s'-b'}} \geq 4$$

$$\frac{(b+c)^2}{\frac{(a+b+c)abc}{bc}}+\frac{(c+a)^2}{\frac{(a+b+c)abc}{ca}}+\frac{(a+b)^2}{\frac{(a+b+c)abc}{ab}}\geq 4$$

$$\frac{(b+c)^2}{a}+\frac{(c+a)^2}{b}+\frac{(a+b)^2}{c}\geq 4(a+b+c)$$

17. TRIGONOMETRIC INEQUALITY - I
In $\Delta A'B'C'$ the following relationship holds:

$$\sin\frac{A'}{2}+\sin\frac{B'}{2}+\sin\frac{C'}{2}\leq\frac{3}{2}$$

Redesigned: In ΔABC holds:

$$\sqrt{\frac{bc}{(a+b)(a+c)}}+\sqrt{\frac{ca}{(b+c)(b+a)}}+\sqrt{\frac{ab}{(c+a)(c+b)}}\leq\frac{3}{2}$$

18. TRIGONOMETRIC INEQUALITY - II
In $\Delta A'B'C'$ the following relationship holds:

$$\cos\frac{A'}{2}+\cos\frac{B'}{2}+\cos\frac{C'}{2}\leq\frac{3\sqrt{3}}{2}$$

Redesigned: In ΔABC holds:

$$\sqrt{\frac{(a+b+c)a}{(a+b)(a+c)}}+\sqrt{\frac{(a+b+c)b}{(b+a)(b+c)}}+\sqrt{\frac{(a+b+c)c}{(c+a)(c+b)}}\leq\frac{3\sqrt{3}}{2}$$

$$\sqrt{\frac{a}{(a+b)(a+c)}}+\sqrt{\frac{b}{(b+a)(b+c)}}+\sqrt{\frac{c}{(c+a)(c+b)}}\leq\frac{3\sqrt{3}}{2\sqrt{a+b+c}}$$

19. TRIGONOMETRIC INEQUALITY - III
In $\Delta A'B'C'$ the following relationship holds:

$$\sin\frac{A'}{2}\sin\frac{B'}{2}+\sin\frac{B'}{2}\sin\frac{C'}{2}+\sin\frac{C'}{2}\sin\frac{A'}{2}\leq\frac{3}{4}$$

Redesigned: In ΔABC holds:

$$\sqrt{\frac{bc}{(a+b)(a+c)}\cdot\frac{ca}{(b+c)(b+a)}}+\sqrt{\frac{ca}{(b+c)(b+a)}\cdot\frac{ab}{(c+a)(c+b)}}+\sqrt{\frac{ab}{(c+a)(c+b)}\cdot\frac{bc}{(a+b)(a+c)}}\leq\frac{3}{4}$$

$$\frac{c}{a+b}\sqrt{\frac{abc}{(a+c)(b+c)}}+\frac{a}{b+c}\sqrt{\frac{abc}{(b+a)(c+a)}}+\frac{b}{c+a}\sqrt{\frac{abc}{(c+b)(a+b)}}\leq\frac{3}{4}$$

$$\frac{c}{(a+b)\sqrt{(a+c)(b+c)}}+\frac{a}{(b+c)\sqrt{(b+a)(c+a)}}+\frac{b}{(c+a)\sqrt{(c+b)(c+a)}}\leq\frac{3}{4\sqrt{abc}}$$

20. TRIGONOMETRIC INEQUALITY - IV
In $\Delta A'B'C'$ the following relationship holds:

$$\cos\frac{A'}{2}\cos\frac{B'}{2}+\cos\frac{B'}{2}\cos\frac{C'}{2}+\cos\frac{C'}{2}\cos\frac{A'}{2}\leq\frac{9}{4}$$

Redesigned: In ΔABC holds:

$$\sqrt{\frac{(a+b+c)a}{(a+b)(a+c)}}\cdot\sqrt{\frac{(a+b+c)b}{(b+c)(b+a)}}+\sqrt{\frac{(a+b+c)b}{(b+a)(b+c)}}\sqrt{\frac{(a+b+c)c}{(c+a)(c+b)}}+$$

$$+\sqrt{\frac{(a+b+c)c}{(c+a)(c+b)}}\cdot\sqrt{\frac{(a+b+c)a}{(a+b)(a+c)}}\le\frac{9}{4}$$

$$\frac{1}{a+b}\sqrt{\frac{ab}{(c+a)(c+b)}}+\frac{1}{b+c}\sqrt{\frac{bc}{(a+b)(a+c)}}+\frac{1}{c+a}\sqrt{\frac{ca}{(b+c)(b+a)}}\le\frac{9}{4(a+b+c)}$$

21. TRIGONOMETRIC INEQUALITY - V

In $\Delta A'B'C'$ the following relationship holds:

$$\sec^2\left(\frac{A'}{2}\right)+\sec^2\left(\frac{B'}{2}\right)+\sec^2\left(\frac{C'}{2}\right)\ge 4$$

Redesigned: In ΔABC holds:

$$\frac{(a+b)(a+c)}{a(a+b+c)}+\frac{(b+a)(b+c)}{b(a+b+c)}+\frac{(c+a)(c+b)}{c(a+b+c)}\ge 4$$

$$\left(1+\frac{b}{a}\right)\left(1+\frac{c}{a}\right)+\left(1+\frac{a}{b}\right)\left(1+\frac{c}{b}\right)+\left(1+\frac{a}{c}\right)\left(1+\frac{b}{c}\right)\ge 4(a+b+c)$$

22. TRIGONOMETRIC INEQUALITY - VI

In $\Delta A'B'C'$ the following relationship holds:

$$\csc^2\left(\frac{A'}{2}\right)+\csc^2\left(\frac{B'}{2}\right)+\csc^2\left(\frac{C'}{2}\right)\ge 12$$

Redesigned: In ΔABC holds:

$$\frac{(a+b)(a+c)}{bc}+\frac{(b+c)(b+a)}{ca}+\frac{(c+a)(c+b)}{ab}\ge 12$$

$$a(a+b)(a+c)+b(b+c)(b+a)+c(c+a)(c+b)\ge 12abc$$

References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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