A SIMPLE PROOF FOR PRESTIN'S INEQUALITY

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ABSTRACT. In this paper is presented a simple proof for Prestin's inequality and a few applications.

PRESTIN'S INEQUALITY

If $0 < |x| < \frac{\pi}{2}$ then:

$$\left|\frac{1}{\sin x} - \frac{1}{x}\right| \leq 1 - \frac{2}{\pi}$$

Proof.

We will use the well known Jordan's and Kober's inequalities:

JORDAN'S INEQUALITY

If $0 < |x| < \frac{\pi}{2}$ then:

$$\frac{2}{\pi} \leq \frac{\sin x}{x} < 1$$

Equality holds for $x = \frac{\pi}{2}$.

KOBER'S INEQUALITY

If $x \in [0, \frac{\pi}{2}]$ then:

$$1 - \frac{2}{\pi}x \leq \cos x \leq 1 - \frac{x^2}{\pi}$$

Equality holds for $x = 0$.

Observation:

If $x \in [-\frac{\pi}{2}, 0] \Rightarrow (-x) \in [0, \frac{\pi}{2}]$ then:

$$1 - \frac{2}{\pi}(-x) \leq \cos(-x) \leq 1 - \frac{(-x)^2}{\pi}$$

$$1 + \frac{2}{\pi}x \leq \cos x \leq 1 - \frac{x^2}{\pi}$$

Case 1: $x \in (0, \frac{\pi}{2}]$

Let be: $f : (0, \frac{\pi}{2}] \to \mathbb{R}; f(x) = \frac{1}{\sin x} - \frac{1}{x} - 1 + \frac{2}{\pi}$

$$f'(x) = \frac{-\cos x}{\sin^2 x} + \frac{1}{x^2} = \frac{1}{x^2 \sin^2 x}(\sin^2 x - x^2 \cos x) \geq$$

$$\overset{\text{JORDAN}}{\geq} \frac{1}{x^2 \sin^2 x}\left(\frac{4}{\pi^2}x^2 - x^2 \cos x\right) =$$

$$= \frac{1}{\sin^2 x}\left(\frac{4}{\pi^2} - \cos x\right) \overset{\text{KOBER}}{\geq}$$

$$\geq \frac{1}{\sin^2 x}\left(\frac{4}{\pi^2} - \left(\frac{2}{\pi}x - 1\right)\right) =$$

$$= \frac{1}{\sin^2 x}\left(\frac{4}{\pi^2} + 1 - \frac{2}{\pi}x\right) \geq$$

$$\left(0 < x \leq \frac{\pi}{2} \Rightarrow 0 > -\frac{2}{\pi}x > -1\right)$$

$$\geq \frac{1}{\sin^2 x}\left(\frac{4}{\pi^2}+1-1\right)=\frac{4}{\pi^2\sin^2 x}>0$$

$$f \text{ increasing } \Rightarrow \max f(x)=f\left(\frac{\pi}{2}\right)=$$

$$=1-\frac{2}{\pi}-1+\frac{2}{\pi}=0\Rightarrow f(x)\leq 0;(\forall)x\in\left(0,\frac{\pi}{2}\right]$$

$$\frac{1}{\sin x}-\frac{1}{x}-1+\frac{2}{\pi}\leq 0$$

$$\frac{1}{\sin x}-\frac{1}{x}\leq 1-\frac{2}{\pi}$$

Case 2: $x\in[-\frac{\pi}{2},0)$
Let be $g:[-\frac{\pi}{2},0)\to\mathbb{R}$

$$g(x)=\frac{1}{\sin x}-\frac{1}{x}+1-\frac{2}{\pi}$$

$$g'(x)=\frac{-\cos x}{\sin^2 x}+\frac{1}{x^2}=\frac{1}{x^2\sin^2 x}(\sin^2 x-x^2\cos x)=$$

$$=\frac{1}{(-x)^2\sin^2(-x)}(\sin^2(-x)-(-x)^2\cos(-x))\geq 0$$

True by case 1 because

$$x\in\left[-\frac{\pi}{2},0\right)\Rightarrow -x\in\left(0,\frac{\pi}{2}\right]$$

$$g \text{ increasing } \Rightarrow \min g(x)=g\left(-\frac{\pi}{2}\right)=$$

$$=-1+\frac{2}{\pi}+1-\frac{2}{\pi}=0$$

$$\Rightarrow g(x)\geq 0;(\forall)x\in\left[-\frac{\pi}{2},0\right)$$

$$\frac{1}{\sin x}-\frac{1}{x}+1-\frac{2}{\pi}\geq 0$$

$$\frac{1}{\sin x}-\frac{1}{x}\geq -\left(1-\frac{2}{\pi}\right)$$

From:

$$\frac{1}{\sin x}-\frac{1}{x}\leq 1-\frac{2}{\pi} \text{ and } \frac{1}{\sin x}-\frac{1}{x}\geq -\left(1-\frac{2}{\pi}\right)$$

we obtain:

$$\left|\frac{1}{\sin x}-\frac{1}{x}\right|\leq 1-\frac{2}{\pi}$$

□

Application 1:
If $0<|x|,|y|\leq\frac{\pi}{2}$ then:

$$\frac{1}{\sin x\sin y}+\frac{1}{xy}\leq\frac{1}{x\sin y}+\frac{1}{y\sin x}+\left(1-\frac{2}{\pi}\right)^2$$

Proof.
By Prestin's inequality:

$$\Big|\frac{1}{\sin x}-\frac{1}{x}\Big|\le 1-\frac{2}{\pi} \text{ and } \Big|\frac{1}{\sin y}-\frac{1}{y}\Big|\le 1-\frac{2}{\pi}$$

By multiplying:

$$\Big|\frac{1}{\sin x}-\frac{1}{x}\Big|\cdot\Big|\frac{1}{\sin y}-\frac{1}{y}\Big|\le\Big(1-\frac{2}{\pi}\Big)^2$$

$$\Big|\frac{1}{\sin x\sin y}-\frac{1}{x\sin y}-\frac{1}{y\sin x}+\frac{1}{xy}\Big|\le\Big(1-\frac{2}{\pi}\Big)^2$$

$$\frac{1}{\sin x\sin y}-\frac{1}{x\sin y}-\frac{1}{y\sin x}+\frac{1}{xy}\le\Big(1-\frac{2}{\pi}\Big)^2$$

$$\frac{1}{\sin x\sin y}+\frac{1}{xy}\le\frac{1}{x\sin y}+\frac{1}{y\sin x}+\Big(1-\frac{2}{\pi}\Big)^2$$

Equality holds for $x=y=\frac{\pi}{2}$. □

Application 2:
If $0<|x|\le\frac{\pi}{2}$ then:

$$\frac{1}{\sin x}+\frac{2}{\pi x}\le\frac{1}{x}+\frac{2}{\pi\sin x}+\Big(1-\frac{2}{\pi}\Big)^2$$

Proof.
In application 1 we take $y=\frac{\pi}{2}$

$$\frac{1}{\sin x\sin\frac{\pi}{2}}+\frac{1}{x\cdot\frac{\pi}{2}}\le\frac{1}{x\sin\frac{\pi}{2}}+\frac{1}{\frac{\pi}{2}\sin x}+\Big(1-\frac{2}{\pi}\Big)^2$$

$$\frac{1}{\sin x}+\frac{2}{\pi x}\le\frac{1}{x}+\frac{2}{\pi\sin x}+\Big(1-\frac{2}{\pi}\Big)^2$$

Equality holds for $x=\frac{\pi}{2}$. □

Application 3:
Prove without any software:

$$\pi^2\sqrt{2}+4<2\pi\sqrt{2}+\pi^2$$

Proof.
In application 2 we take $x=\frac{\pi}{4}$:

$$\frac{1}{\sin\frac{\pi}{4}}+\frac{2}{\pi\cdot\frac{\pi}{4}}<\frac{1}{\frac{\pi}{4}}+\frac{2}{\pi\sin\frac{\pi}{4}}+\Big(1-\frac{2}{\pi}\Big)^2$$

$$\frac{1}{\frac{\sqrt{2}}{2}}+\frac{8}{\pi^2}<\frac{4}{\pi}+\frac{2}{\pi\cdot\frac{\sqrt{2}}{2}}+1-\frac{4}{\pi}+\frac{4}{\pi^2}$$

$$\sqrt{2}+\frac{4}{\pi^2}<\frac{2\sqrt{2}}{\pi}+1$$

$$\pi^2\sqrt{2}+4<2\pi\sqrt{2}+\pi^2$$

□

References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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