A SIMPLE PROOF FOR BORDEN'S INEQUALITY AND APPLICATIONS

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Abstract. In this paper is presented a simple proof for Borden's inequality and a few applications.

Theorem (BORDEN'S INEQUALITY)
If $x_i, y_i > 0; i \in \overline{1,n}$ then:

$$\frac{(x_1+x_2)^{x_1+x_2}}{x_1^{x_1}\cdot x_2^{x_2}}\cdot\frac{(y_1+y_2)^{y_1+y_2}}{y_1^{y_1}\cdot y_2^{y_2}}\le\frac{(x_1+y_1+x_2+y_2)^{x_1+x_2+y_1+y_2}}{(x_1+y_1)^{x_1+y_1}\cdot(x_2+y_2)^{x_2+y_2}} \tag{1}$$

$$\frac{(x_1+x_2+x_3)^{x_1+x_2+x_3}}{x_1^{x_1}\cdot x_2^{x_2}\cdot x_3^{x_3}}\cdot\frac{(y_1+y_2+y_3)^{y_1+y_2+y_3}}{y_1^{y_1}\cdot y_2^{y_2}\cdot y_3^{y_3}}\le$$

$$\le\frac{(x_1+y_1+x_2+y_2+x_3+y_3)^{x_1+x_2+x_3+y_1+y_2+y_3}}{(x_1+y_1)^{x_1+y_1}\cdot(x_2+y_2)^{x_2+y_2}\cdot(x_3+y_3)^{x_3+y_3}} \tag{2}$$

$$\frac{(x_1+x_2+\ldots+x_n)^{x_1+x_2+\ldots+x_n}}{x_1^{x_1}\cdot x_2^{x_2}\cdot\ldots\cdot x_n^{x_n}}\cdot\frac{(y_1+y_2+\ldots+y_n)^{y_1+y_2+\ldots+y_n}}{y_1^{y_1}\cdot y_2^{y_2}\cdot\ldots\cdot y_n^{y_n}}\le$$

$$\le\frac{(x_1+y_1+x_2+y_2+\ldots+x_n+y_n)^{x_1+x_2+\ldots+x_n+y_1+y_2+\ldots+y_n}}{(x_1+y_1)^{x_1+y_1}\cdot(x_2+y_2)^{x_2+y_2}\cdot\ldots\cdot(x_n+y_n)^{x_n+y_n}} \tag{3}$$

Proof of (1).

$$\frac{(x_1+x_2)^{x_1}}{x_1^{x_1}}\cdot\frac{(y_1+y_2)^{y_1}}{y_1^{y_1}}=\left(1+\frac{x_2}{x_1}\right)^{x_1}\cdot\left(1+\frac{y_2}{y_1}\right)^{y_1}\le$$

$$\overset{\text{WEIGHTED AM-GM}}{\le}\left(\frac{x_1(1+\frac{x_2}{x_1})+y_1(1+\frac{y_2}{y_1})}{x_1+y_1}\right)^{x_1+y_1}=$$

$$=\frac{(x_1+x_2+y_1+y_2)^{x_1+y_1}}{(x_1+y_1)^{x_1+y_1}} \tag{4}$$

$$\frac{(x_1+x_2)^{x_2}}{x_2^{x_2}}\cdot\frac{(y_1+y_2)^{y_2}}{y_2^{y_2}}=\left(1+\frac{x_1}{x_2}\right)^{x_2}\cdot\left(1+\frac{y_1}{y_2}\right)^{y_2}\le$$

$$\overset{\text{WEIGHTED AM-GM}}{\le}\left(\frac{x_2(1+\frac{x_1}{x_2})^{x_2}+y_2(1+\frac{y_1}{y_2})^{y_2}}{x_2+y_2}\right)^{x_2+y_2}=$$

$$=\frac{(x_1+x_2+y_1+y_2)^{x_2+y_2}}{(x_2+y_2)^{x_2+y_2}} \tag{5}$$

By multiplying (4); (5) is obtained (1). □

Proof of (2).

$$\frac{(x_1+x_2+x_3)^{x_1}}{x_1^{x_1}}\cdot\frac{(y_1+y_2+y_3)^{y_1}}{y_1^{y_1}}=\Big(1+\frac{x_2+x_3}{x_1}\Big)^{x_1}\cdot\Big(1+\frac{y_2+y_3}{y_1}\Big)^{y_1}\le$$

$$\overset{\text{WEIGHTED AM-GM}}{\le}\Big(\frac{x_1(1+\frac{x_2+x_3}{x_1})+y_1(1+\frac{y_2+y_3}{y_1})}{x_1+y_1}\Big)^{x_1+y_1}=$$

$$=\frac{(x_1+x_2+x_3+y_1+y_2+y_3)^{x_1+y_1}}{(x_1+y_1)^{x_1+y_1}}\tag{6}$$

$$\frac{(x_1+x_2+x_3)^{x_2}}{x_2^{x_2}}\cdot\frac{(y_1+y_2+y_3)^{y_2}}{y_2^{y_2}}=\Big(1+\frac{x_1+x_3}{x_2}\Big)^{x_2}\cdot\Big(1+\frac{y_1+y_3}{y_2}\Big)^{y_2}\le$$

$$\overset{\text{WEIGHTED AM-GM}}{\le}\Big(\frac{x_2(1+\frac{x_1+x_3}{x_2})+y_2(1+\frac{y_2+y_3}{y_2})}{x_2+y_2}\Big)^{x_2+y_2}=$$

$$=\frac{(x_1+x_2+x_3+y_1+y_2+y_3)^{x_1+x_2+x_3+y_1+y_2+y_3}}{(x_2+y_2)^{x_2+y_2}}\tag{7}$$

$$\frac{(x_1+x_2+x_3)^{x_3}}{x_3^{x_3}}\cdot\frac{(y_1+y_2+y_3)^{y_3}}{y_3^{y_3}}=\Big(1+\frac{x_1+x_2}{x_3}\Big)^{x_3}\cdot\Big(1+\frac{y_1+y_2}{y_3}\Big)^{y_3}\le$$

$$\overset{\text{WEIGHTED AM-GM}}{\le}\Big(\frac{x_3(1+\frac{x_1+x_2}{x_3})+y_3(1+\frac{y_1+y_2}{y_3})}{x_3+y_3}\Big)^{x_3+y_3}=$$

$$=\frac{(x_1+x_2+x_3+y_1+y_2+y_3)^{x_1+x_2+x_3+y_1+y_2+y_3}}{(x_3+y_3)^{x_3+y_3}}\tag{8}$$

By multiplying (6); (7); (8) is obtained (2) □

Proof for (3) is similar with proof of (1); (2).

Application 1.
Let be $x_1=\sin^2 x; y_1=\cos^2 x; x_2=\sin^2 y; y_2=\cos^2 y; x,y\in(0,\frac{\pi}{2})$ in (1):

$$\frac{(\sin^2 x+\sin^2 y)^{\sin^2 x+\sin^2 y}}{(\sin^2 x)^{\sin^2 x}\cdot(\sin^2 y)^{\sin^2 y}}\cdot\frac{(\cos^2 x+\cos^2 y)^{\cos^2 x+\cos^2 y}}{(\cos^2 x)^{\cos^2 x}\cdot(\cos^2 y)^{\cos^2 y}}\le$$

$$\le\frac{(\sin^2 x+\cos^2 x+\sin^2 y+\cos^2 y)^{\sin^2 x+\cos^2 x+\sin^2 y+\cos^2 y}}{(\sin^2 x+\cos^2 x)^{\sin^2 x+\cos^2 x}\cdot(\sin^2 y+\cos^2 y)^{\sin^2 y+\cos^2 y}}=$$

$$=\frac{(1+1)^{1+1}}{1^1\cdot 1^1}=4$$

$$\frac{(\sin^2 x+\sin^2 y)^{\sin^2 x+\sin^2 y}\cdot(\cos^2 x+\cos^2 y)^{\cos^2 x+\cos^2 y}}{(\sin x)^{2\sin^2 x}\cdot(\sin y)^{2\sin^2 y}\cdot(\cos x)^{2\cos^2 x}\cdot(\cos y)^{2\cos^2 y}}\le 4$$

Equality holds for $x=y=\frac{\pi}{4}$.

Application 2.
If $a,b,c,d,e,f>0; a+b+c=d+e+f=3$ then by (2):

$$\frac{(a+b+c)^{a+b+c}}{a^ab^bc^c}\cdot\frac{(d+e+f)^{d+e+f}}{d^de^ef^f}\le$$

$$\leq \frac{(a+b+c+d+e+f)^{a+b+c+d+e+f}}{(a+d)^{a+d}\cdot(b+e)^{b+e}\cdot(c+f)^{c+f}} =$$

$$= \frac{(3+3)^{3+3}}{(a+d)^{a+d}\cdot(b+e)^{b+e}\cdot(c+f)^{c+f}}$$

$$\frac{3^3\cdot 3^3}{a^a\cdot b^b\cdot c^c\cdot d^d\cdot e^e\cdot f^f} \leq \frac{2^6\cdot 3^6}{(a+d)^{a+d}\cdot(b+e)^{b+e}\cdot(c+f)^{c+f}}$$

$$\frac{(a+d)^{a+d}\cdot(b+e)^{b+e}\cdot(c+f)^{c+f}}{a^a b^b c^c d^d e^e f^f} \leq 64$$

Equality holds for $a=b=c=d=e=f=1$.

Application 3

Let be $x_1=x; x_2=1; y_1=y; y_2=1$ in (1):

$$\frac{(x+1)^{x+1}}{x^x\cdot 1^1}\cdot\frac{(y+1)^{y+1}}{y^y\cdot 1^1} \leq \frac{(x+y+1+1)^{x+y+1+1}}{(x+y)^{x+y\cdot(1+1)^{1+1}}}$$

$$4(x+1)^{x+1}\cdot(y+1)^{y+1}\cdot(x+y)^{x+y} \leq x^x\cdot y^y\cdot(x+y+2)^{x+y+2}$$

Equality holds for $x=y=1$.

References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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