PP38308

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In all triangles ABC holds:

$$\left(\frac{r_a}{h_a}\right)^{\frac{1}{r_a}} \cdot \left(\frac{r_b}{h_b}\right)^{\frac{1}{r_b}} \cdot \left(\frac{r_c}{h_c}\right)^{\frac{1}{r_c}} \leq 1$$

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Solution by Daniel Sitaru.
Let be $f : (0, \infty) \to \mathbb{R}; f(x) = \log x$

$$f'(x) = \frac{1}{x}; f''(x) = -\frac{1}{x^2} < 0 \Rightarrow f \text{ concave}$$

By Gibbs' inequality:

$$\sum_{cyc} \frac{1}{r_a} \log\left(\frac{r_a}{h_a}\right) \leq \sum_{cyc} \frac{1}{r_a} \cdot \log\left(\frac{\sum_{cyc} \frac{1}{r_a}}{\sum_{cyc} \frac{1}{h_a}}\right) =$$

$$= \sum_{cyc} \frac{1}{r_a} \cdot \log\left(\frac{\frac{1}{r}}{\frac{1}{r}}\right) = \sum_{cyc} \frac{1}{r_a} \cdot \log 1 = 0$$

$$\sum_{cyc} \frac{1}{r_a} \log\left(\frac{r_a}{h_a}\right) \leq 0 \Rightarrow \log\left(\prod_{cyc} \left(\frac{r_a}{h_a}\right)^{\frac{1}{r_a}}\right) \leq \log 1$$

$$\Rightarrow \prod_{cyc} \left(\frac{r_a}{h_a}\right)^{\frac{1}{r_a}} \leq 1$$

Equality holds for $a = b = c$.

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