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# LIMITS AND SUMS INVOLVING BETA FUNCTION

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**Abstract:** *In this paper is presented a collection of special limits and sums involving beta function with detalied solutions.*

## App. 1) If $m,n,p\in\mathbb{N}$ and $p\le n$ then holds:

$$\frac{\binom{m}{0}}{\binom{n+m}{p}}+\frac{\binom{m}{1}}{\binom{m}{p+1}}+\cdots+\frac{\binom{m}{m}}{\binom{m}{p+m}}=\frac{n+m+1}{n+1}\cdot\frac{1}{\binom{n}{n}}$$

**Solution.** We have:

$$\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{n+m}{p+k}}=(n+m+1)\sum_{k=0}^{m}\binom{m}{k}\frac{(p+k)!\,(n+m-p-k)!}{(n+m+1)!}=$$

$$(n+m+1)\sum_{k=0}^{m}\binom{m}{k}B(p+k+1,n+m-p-k+1)=$$

$$=(n+m+1)\int_0^1\left[\sum_{k=0}^{m}\binom{m}{k}t^{p+k}(1-t)^{n+m-p-k}\right]dt=$$

$$=(n+m+1)\int_0^1\left[t^p(1-t)^{n+m-p}\sum_{k=0}^{m}\binom{m}{k}\left(\frac{t}{1-t}\right)^k\right]dt=$$

$$=(n+m+1)\int_0^1 t^p(1-t)^{n+m-p}\left(1+\frac{t}{1-t}\right)^m dt=$$

$$=(n+m+1)\int_0^1 t^p(1-t)^{n-p}\,dt=(n+m+1)B(p+1,n-p+1)=$$

$$=\frac{(n+m+1)}{(n+1)\binom{n}{p}}$$

## App. 2) Find:

$$\Omega=\lim_{n\to\infty}\frac{1}{n^{\alpha}}\cdot\sum_{m=1}^{n}\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{n+m}{n+m}};\alpha\in\mathbb{R}$$

**Solution.** In Application 1), if we take $p=n$, we get:

$$\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{n+m}{n+m}}=\frac{n+m+1}{n+1};(1)$$

$$\sum_{m=1}^{n}\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{n+m}{n+m}}=\sum_{m=1}^{n}\left(1+\frac{m}{n+1}\right)=\frac{3n}{2}$$

Therefore,

$$\Omega=\lim_{n\to\infty}\frac{1}{n^{\alpha}}\cdot\sum_{m=1}^{n}\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{n+m}{n+m}}=\lim_{n\to\infty}\frac{1}{n^{\alpha}}\cdot\frac{3n}{2}=\frac{3}{2}\cdot\lim_{n\to\infty}n^{1-\alpha}=\begin{cases}0,\text{if }\alpha>1\\ \infty,\text{if }\alpha<1\end{cases}$$

**App. 3) Find:**

$$\Omega=\lim_{m\to\infty}\lim_{n\to\infty}\sum_{p=0}^{m}\left(n^{p}\cdot\sum_{k=0}^{np}\frac{\binom{pn}{k}}{\binom{(p+1)n}{p+k}}\right)^{-1}$$

**Solution.** In Application 1), if we take $m=pn$, we get:

$$\sum_{k=0}^{m}\frac{\binom{pn}{k}}{\binom{(p+1)n}{p+k}}=\frac{(p+1)n+1}{n+1}\cdot\frac{1}{\binom{n}{p}}$$

$$\lim_{n\to\infty}n^{p}\cdot\sum_{k=0}^{np}\frac{\binom{pn}{k}}{\binom{(p+1)n}{p+k}}=\lim_{n\to\infty}\frac{(p+1)n+1}{n+1}\cdot\frac{n^{p}}{\binom{n}{p}}=(p+1)!$$

Therefore,

$$\Omega=\lim_{m\to\infty}\lim_{n\to\infty}\sum_{p=0}^{m}\left(n^{p}\cdot\sum_{k=0}^{np}\frac{\binom{pn}{k}}{\binom{(p+1)n}{p+k}}\right)^{-1}=\lim_{m\to\infty}\sum_{p=0}^{m}\frac{1}{(p+1)!}=e-1$$

**App. 4) Find:**

$$\Omega=\lim_{n\to\infty}\frac{1}{n^{n\alpha}}\cdot\left\{\sum_{k=1}^{n}\left(\binom{n}{k}\cdot\left[\sum_{i=1}^{n}i^{n-k}\right]+\frac{2k}{n}\right)\right\}^{\alpha};\alpha\in\mathbb{R}$$

**Solution.**

$$(k+1)^k = k^n + \sum_{i=1}^{n-1} \binom{n}{i} \cdot k^{n-i} + 1$$

$$k = 1 \Rightarrow 2^n = 1^n + \sum_{i=1}^{n-1} \binom{n}{i} + 1$$

$$k = 2 \Rightarrow 3^n = 2^n + \sum_{i=1}^{n-1} \binom{n}{i} 2^{n-i} + 1$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$

$$k = n \Rightarrow (n+1)^n = n^n + \sum_{i=1}^{n-1} \binom{n}{i} n^{n-i} + 1$$

By adding, we get:

$$(n+1)^n = n+1 + \sum_{k=1}^{n-1} \binom{n}{i} + \sum_{i=1}^{n-1} \binom{n}{i} 2^{n-i} + \dots + \binom{n}{i} n^{n-i} =$$

$$= n+1 + \binom{n}{1}\left[\sum_{i=1}^{n} i^{n-1}\right] + \binom{n}{2}\left[\sum_{i=1}^{n} i^{n-2}\right] + \binom{n}{3}\left[\sum_{i=1}^{n} i^{n-3}\right] + \dots + \binom{n}{n-1}\left[\sum_{i=1}^{n} i\right] =$$

$$= n+1 + \sum_{k=1}^{n} \binom{n}{k}\left[\sum_{i=1}^{n} i^{n-k}\right]$$

Hence,

$$\sum_{k=1}^{n} \binom{n}{k}\left[\sum_{i=1}^{n} i^{n-k}\right] = (n+1)^n - (n+1)$$

and then

$$\sum_{k=1}^{n}\left(\binom{n}{k} \cdot \left[\sum_{i=1}^{n} i^{n-k}\right] + \frac{2k}{n}\right) = \sum_{k=1}^{n} \binom{n}{k}\left[\sum_{i=1}^{n} i^{n-k}\right] + (n+1) = (n+1)^n$$

Therefore,

$$\Omega = \lim_{n\to\infty} \frac{1}{n^{n\alpha}} \cdot \left\{\sum_{k=1}^{n}\left(\binom{n}{k} \cdot \left[\sum_{i=1}^{n} i^{n-k}\right] + \frac{2k}{n}\right)\right\}^{\alpha} = \lim_{n\to\infty} \frac{1}{n^{n\alpha}} \cdot (n+1)^{n\alpha} = e^{\alpha}.$$

**App. 5) Find:**

$$\Omega=\lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{n}\cdot\sum_{p=0}^{n}\frac{\binom{n}{p}}{\binom{n+k}{p+k}}\cdot\sum_{i=1}^{k}\sqrt[i^2]{i!},k\in\mathbb{N}^*-\text{fixed.}$$

**Solution.**

$$\frac{\binom{n}{p}}{\binom{n+k}{p+k}}=\frac{n!}{p!\,(n-p)!}\cdot\frac{(p+k)!\,(n-p)!}{(n+k)!}=$$

$$=\frac{(p+1)(p+2)\dots(p+k)}{(n+1)(n+2)\dots(n+k)};\forall p\in\{0,1,2,\dots,n\}$$

Hence,

$$\sum_{p=0}^{n}\frac{\binom{n}{p}}{\binom{n+k}{p+k}}=\frac{1}{(n+1)(n+2)\dots(n+k)}\cdot\sum_{k=1}^{n}(p+1)(p+2)\dots(p+k)=$$

$$=\frac{1\cdot2\cdot3\cdot\dots\cdot k+2\cdot3\cdot4\cdot\dots\cdot(k+1)+3\cdot4\cdot5\cdot\dots\cdot(k+2)+\dots+(n+1)(n+2)\dots(n+k)}{(n+1)(n+2)\dots(n+k)}$$

$$=\frac{\frac{(n+1)(n+2)\dots(n+k+1)}{k+1}}{(n+1)(n+2)\dots(n+k)}=\frac{n+k+1}{k+1}$$

So,

$$\Omega_k=\lim_{n\to\infty}\frac{1}{n}\cdot\sum_{p=0}^{n}\frac{\binom{n}{p}}{\binom{n+k}{p+k}}=\frac{1}{k+1}$$

Hence,

$$\Omega=\lim_{n\to\infty}\Omega_{n-1}\cdot\sum_{i=1}^{n}\sqrt[i^2]{i!}=\lim_{n\to\infty}\frac{1}{n}\cdot\sum_{i=1}^{n}\sqrt[i^2]{i}$$

Let $a_n=\sum_{i=1}^{n}\sqrt[i^2]{i}$ and $b_n=n$, we have: $\frac{a_n-a_{n-1}}{b_n-b_{n-1}}=\sqrt[n^2]{n!}$.

Let us denote $x_n=\sqrt[n^2]{([\sqrt{n}])!},n\in\mathbb{N}^*-M;M=\{k^2+2k|k\in\mathbb{N}^*\}$

Because for all $n\in\mathbb{N},\exists k\in\mathbb{N}^*$ such that $k^2\le n<k^2+2k$

$k^2<n+1<(k+1)^1\Rightarrow k\le\sqrt{n}<k+1$ and $k<\sqrt{n+1}<k+1$ which means that

$\left[\sqrt{n}\right]=\left[\sqrt{n+1}\right]=k$. Hence,

$$\lim_{n\to\infty}\sqrt[n]{x_n}=\lim_{n\to\infty}\sqrt[n]{\left(\left[\sqrt{n}\right]\right)!}=\lim_{n\to\infty}\frac{\left(\left[\sqrt{n+1}\right]\right)!}{\left(\left[\sqrt{n}\right]\right)!}=1$$

Therefore,

$$\Omega=\lim_{n\to\infty}\Omega_{n-1}\cdot\sum_{i=1}^{n}\sqrt[i^2]{i!}=\lim_{n\to\infty}\frac{1}{n}\cdot\sum_{i=1}^{n}\sqrt[i^2]{i}\overset{C-S}{=}1$$

**App. 6) Let $(a_n)_{n\ge1}$ be a sequence of real numbers such that $a_n\le n;\forall n\ge1$**

$$\text{and }\sum_{k=1}^{n-1}\cos\frac{\pi a_k}{n}=0;\forall n\ge2.\text{ Find:}$$

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{2}\cdot\sum_{k=0}^{2n}\frac{\binom{2n}{k}}{\binom{4n}{2k}}\right)^{a_{2n+1}}$$

**Solution.** Is true that $a_1=1$. For $\cos\frac{\pi a_1}{3}+\cos\frac{\pi a_2}{3}=0$, we get $a_2=2$.

Suppose that $a_k=k;\forall k=\overline{1,n-1}$ and from hypothesis, we get:

$$\cos\frac{\pi a_n}{n+1}=-\sum_{k=1}^{n-1}\cos\frac{\pi k}{n+1}$$

Let be the number $z=\cos\frac{\pi}{n+1}+i\sin\frac{\pi}{n+1}$.

$$z+z^2+z^3+\cdots+z^n=\frac{z-z^{n+1}}{1-z}=\frac{1+z}{1-z}$$

$$z\cdot\bar{z}=1\Rightarrow\overline{\left(\frac{1+z}{1-z}\right)}=-\frac{1+z}{1-z}\Rightarrow Re\left(\frac{1+z}{1-z}\right)=0\Rightarrow\sum_{k=1}^{n}\cos\frac{\pi k}{n+1}=0$$

Hence, $\cos\frac{\pi a_k}{n+1}=\cos\frac{\pi n}{n+1}$ and from $a_n\le n$, it follows that $a_n=n;\forall n\ge2$.

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{2}\cdot\sum_{k=0}^{2n}(-1)^k\frac{\binom{2n}{k}}{\binom{4n}{2k}}\right)^{2n+1};(1)$$

$$S_n=\sum_{k=0}^{2n}\frac{\binom{2n}{k}}{\binom{4n}{2k}}=(4n+1)\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\frac{(2k)!\,(4n-2k)!}{(4n+1)!}=$$

$$= (4n+1)\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\int_0^1 x^{2k}(1-x)^{4n-2k}\,dx =$$

$$= (4n+1)\int_0^1\left[(1-x)^{4n}\sum_{k=0}^{2n}\binom{2n}{2k}\left(\frac{-x^2}{(1-x)^2}\right)^k\right]dx =$$

$$= (4n+1)\int_0^1 (1-x)^{4n}\left(1-\frac{x^2}{(1-x)^2}\right)^{2n}dx =$$

$$= (4n+1)\int_0^1 (1-2x)^{2n}\,dx = \frac{4n+1}{2n+1};(2)$$

From (1) and (2), it follows that:

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{2}\cdot\frac{4n+1}{2n+1}\right)^{2n+1} = \lim_{n\to\infty}\left(1+\frac{4n+1}{2(2n+1)}-1\right)^{2n+1} =$$

$$= \lim_{n\to\infty}\left[\left(1-\frac{1}{2(2n+1)}\right)^{-2(2n+1)}\right]^{\frac{-1}{2}} = \frac{1}{\sqrt{e}}$$

**App. 7) For $m,n\in\mathbb{N}$ find:**

$$\Omega_m = \sum_{n=1}^{m}\left(\sum_{k=0}^{2n}(-1)^k\frac{\binom{4n}{2k}}{\binom{2n}{k}}\right)^{-1}$$

**Solution.**

$$\text{Let } S_n = \sum_{k=0}^{2n}(-1)^k\frac{\binom{4n}{2k}}{\binom{2n}{k}}, \text{ we have:}$$

$$S_n = (2n+1)\sum_{k=0}^{2n}(-1)^k\binom{4n}{2k}\frac{k!\,(2n-k)!}{(2n+1)!}$$

$$= (2n+1)\sum_{k=0}^{2n}(-1)^k\binom{4n}{2k}\int_0^1 x^k(1-x)^{2n-k}\,dx =$$

$$= (2n+1)\int_0^1\left[\sum_{k=0}^{2n}\binom{4n}{2k}(-1)^k(1-x)^{2n-k}\right]dx =$$

$$= \frac{2n+1}{2}\int_0^1\left[\left(\sqrt{1-x}+i\sqrt{x}\right)^{4n}+\left(\sqrt{1-x}-i\sqrt{x}\right)^{4n}\right]dx$$

Because $\sqrt{1-x}\pm i\sqrt{x}=\cos\left(\tan^{-1}\sqrt{\frac{x}{1-x}}\right)\pm i\sin\left(\tan^{-1}\sqrt{\frac{x}{1-x}}\right)$, we get:

$$S_n=(2n+1)\int_0^1\cos\left(4n\cdot\tan^{-1}\sqrt{\frac{x}{1-x}}\right)dx\overset{\tan^{-1}\sqrt{\frac{x}{1-x}}=t}{=}$$

$$=(2n+1)\int_0^{\frac{\pi}{2}}\cos(4nt)\sin(2t)\,dt=$$

$$=\frac{2n+1}{2}\int_0^1[\sin(4n+2)t-\sin(4n-2)t]\,dt=$$

$$=\frac{2n+1}{2}\left(\frac{2}{4n+2}-\frac{2}{4n-2}\right)=-\frac{1}{2n-1};$$

$$\sum_{n=1}^{m}\left(\sum_{k=0}^{2n}(-1)^k\frac{\binom{4n}{2k}}{\binom{2n}{k}}\right)^{-1}=-\sum_{n=1}^{m}(2n-1)=-\left(\frac{2m(m+1)}{2}-m\right)=-m^2$$

**App. 8) Find:**

$$\boldsymbol{\Omega_n=\sum_{n=1}^{\infty}\frac{1}{n^2\binom{2n}{n}}}$$

**Solution.** We have:

$$\Omega=\sum_{n=1}^{\infty}\frac{1}{n^2\binom{2n}{n}}=\sum_{n=1}^{\infty}\frac{1}{n^2\cdot\frac{(2n)!}{n!\cdot n!}}=\sum_{n=1}^{\infty}\frac{n!\cdot n!}{n\cdot n\cdot(2n)!}=\sum_{n=1}^{\infty}\frac{(n-1)!\,(n-1)!}{(2n)!}=$$

$$=\sum_{n=1}^{\infty}\frac{\Gamma(n)\cdot\Gamma(n)}{2n\cdot\Gamma(2n)}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}B(n,n)=\frac{1}{2}\sum_{n=1}^{\infty}\int_0^1 t^{n-1}dt\int_0^1 x^{n-1}(1-x)^{n-1}\,dx=$$

$$=\frac{1}{2}\int_0^1\int_0^1\sum_{n=0}^{\infty}t^n\cdot x^n(1-x)^n\,dtdx=\frac{1}{2}\int_0^1\int_0^1\sum_{n=0}^{\infty}\big(tx(1-x)\big)^n dtdx=$$

$$=\frac{1}{2}\int_0^1\int_0^1\frac{dtdx}{1-tx(1-x)}=-\frac{1}{2}\int_0^1\frac{\log(1-x+x^2)}{x(1-x)}dx=$$

$$=-\frac{1}{2}\left(\int_0^1\frac{\log(1-x+x^2)}{x}dx+\int_0^1\frac{\log(1-x+x^2)}{1-x}dx\right)\overset{y=1-x}{=}$$

$$=-\frac{1}{2}\left(\int_0^1\frac{\log(1-x+x^2)}{x}dx+\int_0^1\frac{\log(1-y+y^2)}{y}dy\right)=$$

$$=-\int_0^1 \frac{\log(1-x+x^2)}{x}dx=$$

$$=-\int_0^1 \frac{\log\left(\frac{1+x^3}{1+x}\right)}{x}dx=-\int_0^1 \frac{\log(1+x^3)}{x}dx+\int_0^1 \log\frac{1+x}{x}dx=$$

$$=-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^1 x^{3n-1}dx-Li_2(-1)=-\sum_{n=1}^{\infty}\frac{(-1)^n x^{3n}}{3n^2}\Bigg|_0^1-Li_2(-1)=$$

$$=\frac{1}{3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}-Li_2(-1)=\frac{1}{3}Li_2(-1)-Li_2(-1)=-\frac{2}{3}Li_2(-1)=\frac{\pi^2}{18}$$

**App. 9) Find:**

$$\Omega_n=\sum_{n=0}^{\infty}\frac{1}{\binom{2n}{n}}$$

**Solution.** We have:

$$\Omega=\sum_{n=0}^{\infty}\frac{1}{\binom{2n}{n}}=\sum_{n=0}^{\infty}\frac{n!\cdot n!}{(2n)!}=1+\sum_{n=1}^{\infty}\frac{n(n-1)!\cdot n(n-1)!}{2n(2n-1)!}=$$

$$=1+\sum_{n=1}^{\infty}\frac{n^2\cdot\Gamma^2(n)}{2n\cdot\Gamma(2n)}=1+\frac{1}{2}\sum_{n=1}^{\infty}nB(n,n)=$$

$$=1+\frac{1}{2}\sum_{n=1}^{\infty}n\int_0^1 x^{n-1}(1-x)^{n-1}dx=1+\frac{1}{2}\frac{\partial}{\partial t}\sum_{n=1}^{\infty}t^n\int_0^1 x^{n-1}(1-x)^{n-1}dx\Bigg|_{t=1}=$$

$$=1+\frac{1}{2}\frac{\partial}{\partial t}\int_0^1 t\sum_{n=1}^{\infty}\big(tx(1-x)\big)^{n-1}dx\Bigg|_{t=1}=1+\frac{1}{2}\frac{\partial}{\partial t}\int_0^1\frac{t}{1-tx(1-x)}dx\Bigg|_{t=1}=$$

$$=1+\frac{1}{2}\frac{\partial}{\partial t}t\int_0^1\frac{dx}{1-tx+tx^2}\Bigg|_{t=1}=1+\frac{\partial}{\partial t}\frac{1}{2}\int_0^1\frac{dx}{x^2-x+\frac{1}{t}}\Bigg|_{t=1}=$$

$$=1+\frac{\partial}{\partial t}\frac{1}{2}\int_0^1\frac{dx}{\left(x-\frac{1}{2}\right)^2+\frac{4-t}{4t}}\Bigg|_{t=1}=1+\frac{\partial}{\partial t}\frac{1}{2}\sqrt{\frac{4t}{4-t}}\cdot\tan^{-1}\frac{x-\frac{1}{2}}{\sqrt{\frac{4-t}{4t}}}\Bigg|_0^1\Bigg|_{t=1}=$$

$$= 1 + \frac{\partial}{\partial t}\sqrt{\frac{4t}{4-t}} \cdot \tan^{-1}\frac{1}{2}\sqrt{\frac{4t}{4-t}}\Bigg|_{t=1} =$$

$$1 + \frac{\frac{4(4-t)+4t}{(4-t)^2}}{2\sqrt{\frac{4t}{4-t}}}\tan^{-1}\frac{1}{2}\sqrt{\frac{4t}{4-t}} + \sqrt{\frac{4t}{4-t}} \cdot \frac{\frac{1}{2}\frac{4(4-t)+4t}{2\sqrt{\frac{4t}{4-t}}}}{1+\frac{t}{4-t}}\Bigg|_{t=1} =$$

$$= 1 + \frac{\frac{16}{9}}{2\sqrt{\frac{4}{3}}}\tan^{-1}\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} \cdot \frac{\frac{16}{4}\sqrt{3}}{\frac{4}{3}} = 7 + \frac{2\pi\sqrt{3}}{27}$$

**App. 10) For $n \in \mathbb{N}, n$ –even number, find:**

$$\Omega_n = \sum_{k=1}^{3n-1} \frac{(-1)^{k-1} \cdot k}{\binom{3n}{k}}$$

**Solution.**

$$\Omega_n = \sum_{k=1}^{3n-1} \frac{(-1)^{k-1} \cdot k}{\binom{3n}{k}} = \sum_{k=1}^{3n-1} \frac{(-1)^{k-1} \cdot k}{\frac{(3n)!}{k!(3n-k)!}} = \sum_{k=1}^{3n-1} \frac{(-1)^{k-1}(3n-k)! \cdot k \cdot k!}{(3n)!} =$$

$$= \sum_{k=1}^{3n-1} \frac{(-1)^{k-1} \cdot k\Gamma(k+1) \cdot \Gamma(3n-k+1)}{\Gamma(3n+1)} = \sum_{k=1}^{3n-1} \frac{(-1)^{k-1} \cdot k\Gamma(k+1)\Gamma(3n-k+1)}{\frac{\Gamma(3n+2)}{3n+1}} =$$

$$= (3n+1)\sum_{k=1}^{3n-1} (-1)^{k-1} \cdot kB(k+1; 3n-k+1) =$$

$$= (3n+1)\sum_{k=1}^{3n-1} (-1)^{k-1} k \int_0^1 t^k (1-t)^{3n-k} dt =$$

$$= (3n+1)\frac{\partial}{\partial u}\sum_{k=1}^{3n-1} u^k \int_0^1 t^k (1-t)^{3n-k} dt\Bigg|_{u=-1} =$$

$$= (3n+1)\frac{\partial}{\partial u}\left(u\int_0^1 t(1-t)^{3n-1}\sum_{k=1}^{3n-1}\frac{u^{k-1}t^{k-1}}{(1-t)^{k-1}}dt\right)\Bigg|_{u=-1} =$$

$$= (3n+1)\frac{\partial}{\partial u}\left(u\int_0^1 t(1-t)^{3n-1}\cdot\frac{\left(\frac{ut}{1-t}\right)^{3n-1}-1}{\frac{ut}{1-t}-1}dt\right)\Bigg|_{u=-1} =$$

$$= (3n+1)\frac{\partial}{\partial u}\int_0^1 \frac{u^{3n}t^{3n}(1-t)-ut(1-t)^{3n}}{t(u+1)-1}dt\Bigg|_{u=-1} =$$

$$= (3n+1)\int_0^1 \frac{(3nu^{3n-1}t^{3n}(1-t)-t(1-t)^{3n})(t(u+1)-1)-u^{3n}t^{3n+1}(1-t)-ut^2(1-t)^{3n}}{(t(u+1)-1)^2}dt\ \Big|_{u=-1}$$

$$\overset{u=-1}{=} (3n+1)\int_0^1 [-3nt^{3n}(1-t)+t(1-t)^{3n}-t^{3n+1}(1-t)+t^2(1-t)^{3n}]dt =$$

$$= (3n+1)[-3nB(3n+1;2)+B(2;3n+1)-B(3n+2;2)+B(3;3n+1)] =$$

$$= (3n+1)\left(-3n\cdot\frac{(3n)!}{(3n+2)!}+\frac{(3n)!}{(3n+2)!}+\frac{(3n+1)!}{(3n+3)!}+\frac{2(3n)!}{(3n+3)!}\right) =$$

$$= \frac{-9n^2+9n+4}{(3n+2)(3n+3)}$$

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