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In ΔABC the following relationship holds:

$$\sum_{cyc} \sin^2\frac{A}{2}\csc\frac{C}{2} \le \frac{1}{3}\left(\frac{4R}{r}+\frac{r}{R}-4\right).$$

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Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{Lemma}: \ \textit{if } x,y,z \ge 0 \textit{ then } 3(x^3y+y^3z+z^3x) \le (x^2+y^2+z^2)^2$$

$$\textit{Proof}: \ \textit{Using the well known inequality } (u+v+w)^2 \ge 3(uv+vw+wu)$$

$$\textit{With}: \ u = x^2+yz-xy, \ v = y^2+zx-yz, \ w = z^2+xy-zx:$$

$$(x^2+y^2+z^2)^2 = (u+v+w)^2 \ge 3(uv+vw+wu)$$

$$= 3\sum_{cyc}(x^2+yz-xy)(y^2+zx-yz) = 3(x^3y+y^3z+z^3x).$$

Now, we have : $\sum_{cyc}\sin^2\frac{A}{2}\csc\frac{C}{2}$

$$=\frac{\sin^3\frac{A}{2}\sin\frac{B}{2}+\sin^3\frac{B}{2}\sin\frac{C}{2}+\sin^3\frac{C}{2}\sin\frac{A}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}\overset{Lemma}{\overbrace{\leq}}\frac{\left(\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}\right)^2}{3\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}=$$

$$=\frac{\left(\frac{2R-r}{2R}\right)^2}{3.\frac{r}{4R}}=\frac{4R^2-4Rr+r^2}{3Rr}=\frac{1}{3}\left(\frac{4R}{r}+\frac{r}{R}-4\right).$$ *So the proof is completed.*