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Let $8\alpha \geq 1$. In ΔABC the following relationship holds:

$$\sum_{cyc} m_a w_a \leq s^2 + \alpha[(a-b)^2 + (b-c)^2 + (c-a)^2].$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco, Solution 2 by Soumava Chakraborty-Kolkata-India

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let's\ prove\ that:\ m_a w_a \leq s(s-a) + \frac{(b-c)^2}{8}$$

$$We\ have:\ m_a w_a - s(s-a) \overset{w_a \leq \sqrt{s(s-a)}}{\leq} \sqrt{s(s-a)}.\left(m_a - \sqrt{s(s-a)}\right)$$

$$= \sqrt{s(s-a)}.\frac{{m_a}^2 - s(s-a)}{m_a + \sqrt{s(s-a)}} =$$

$$= \sqrt{s(s-a)}.\frac{(2b^2 + 2c^2 - a^2) - [(b+c)^2 - a^2]}{4\left(m_a + \sqrt{s(s-a)}\right)}$$

$$= \sqrt{s(s-a)}.\frac{(b-c)^2}{4\left(m_a + \sqrt{s(s-a)}\right)} \overset{m_a \geq \sqrt{s(s-a)}}{\leq} \sqrt{s(s-a)}.\frac{(b-c)^2}{4.2\sqrt{s(s-a)}}$$

$$Then: \quad m_a w_a - s(s-a) \le \frac{(b-c)^2}{8} \quad or \quad m_a w_a \le s(s-a) + \frac{(b-c)^2}{8} \quad (And\ analogs)$$

Therefore,

$$\sum_{cyc} m_a w_a \le s^2 + \frac{1}{8}[(a-b)^2 + (b-c)^2 + (c-a)^2] \overset{1 \le 8\alpha}{\le} s^2 + \alpha[(a-b)^2 + (b-c)^2 + (c-a)^2].$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$m_a^2 w_a^2 = \left(\frac{(b-c)^2}{4} + s(s-a)\right)\left(\frac{(b+c)^2 - (b-c)^2}{(b+c)^2}.s(s-a)\right)$$

$$= \left(\frac{(b-c)^2}{4} + s(s-a)\right)\left(s(s-a) - s(s-a).\frac{(b-c)^2}{(b+c)^2}\right)$$

$$= s^2(s-a)^2 + s(s-a).\frac{(b-c)^2}{4} - s(s-a).\left(\frac{(b-c)^2}{4} + s(s-a)\right).\frac{(b-c)^2}{(b+c)^2}$$

$$\le s^2(s-a)^2 + s(s-a).\frac{(b-c)^2}{4} + 0 \left(\because s(s-a).\left(\frac{(b-c)^2}{4} + s(s-a)\right).\frac{(b-c)^2}{(b+c)^2} \le 0\right)$$

$$\le s^2(s-a)^2 + s(s-a).\frac{(b-c)^2}{4} + \frac{(b-c)^4}{64} = \left(s(s-a) + \frac{(b-c)^2}{8}\right)^2 \Rightarrow m_a w_a \le s(s-a) + \frac{(b-c)^2}{8} \text{ and analogs} \Rightarrow \sum m_a w_a$$

$$\le s\sum(s-a) + \sum\frac{(b-c)^2}{8}$$

$$= s^2 + \frac{1}{8}[(a-b)^2 + (b-c)^2 + (c-a)^2] \overset{\frac{1}{8} \le \alpha}{\le} s^2 + \alpha[(a-b)^2 + (b-c)^2 + (c-a)^2] \text{ (QED)}$$