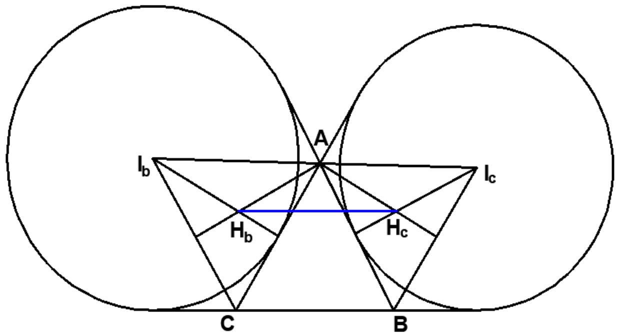
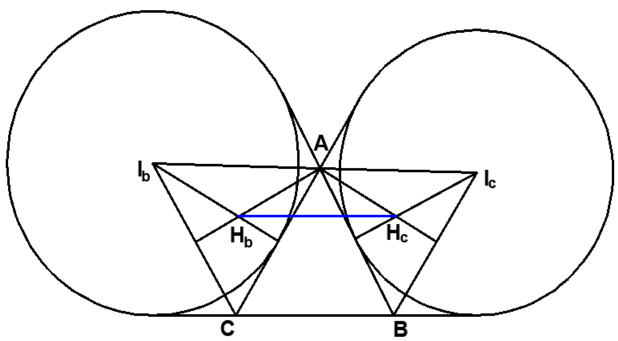


I_a, I_b, I_c –excenters in ΔABC, H_a, H_b, H_c –orthocenters in $\Delta BI_aC, \Delta CI_bA, \Delta AI_cB$. Prove that :

$$H_aH_b^{\ 2} + H_bH_c^{\ 2} + H_cH_a^{\ 2} \leq 9R^2.$$

Proposed by Eldeniz Hesenov-Georgia

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco



$$We\ have:\ R_{\Delta CI_bA} = \frac{CA}{2\sin \widehat{CI_bA}}\ with\ \widehat{CI_bA} = \pi - \frac{\pi - A}{2} - \frac{\pi - C}{2}$$

$$= \frac{\pi - B}{2}\ then\ R_{\Delta CI_bA} = \frac{b}{2\sin\left(\frac{\pi - B}{2}\right)} = \frac{b}{2\cos\frac{B}{2}} = 2R\sin\frac{B}{2}$$

$$Then:\ AH_b = 2R_{\Delta CI_bA}.\cos\widehat{I_bAC} = 2.2R\sin\frac{B}{2}.\cos\left(\frac{\pi - A}{2}\right)$$

$$= 4R\sin\frac{A}{2}\sin\frac{B}{2}.\ Similarly, we\ have:\ AH_c = 4R\sin\frac{A}{2}\sin\frac{C}{2}$$

By the Law of Cosines in ΔAH_bH_c, we have :

$$H_bH_c^{\ 2} = AH_b^{\ 2} + AH_c^{\ 2} - 2.AH_b.AH_c.\cos\widehat{H_bAH_c}$$

$$With:\ \widehat{H_bAH_c} = A + \widehat{H_bAC} + \widehat{BAH_c} = A + \frac{C}{2} + \frac{B}{2} = \frac{\pi + A}{2}$$

$$Then:$$

$$H_bH_c^{\ 2} = \left(4R\sin\frac{A}{2}\sin\frac{B}{2}\right)^2 + \left(4R\sin\frac{A}{2}\sin\frac{C}{2}\right)^2 - 2\left(4R\sin\frac{A}{2}\sin\frac{B}{2}\right)\left(4R\sin\frac{A}{2}\sin\frac{C}{2}\right)\cos\left(\frac{\pi + A}{2}\right) =$$

$$= \left(4R\sin\frac{A}{2}\right)^2\left(\sin^2\frac{B}{2} + \sin^2\frac{C}{2} + 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right) = \left(4R\sin\frac{A}{2}\right)^2\left(1 - \sin^2\frac{A}{2}\right)$$

$$= \left(4R\cos\frac{A}{2}\sin\frac{A}{2}\right)^2 = a^2$$

$Then:\ H_bH_c = a\ (And\ analogs).\ Therefore,$

$$H_aH_b^{\ 2} + H_bH_c^{\ 2} + H_cH_a^{\ 2} = a^2 + b^2 + c^2 \overset{Leibniz}{\leq} 9R^2.$$