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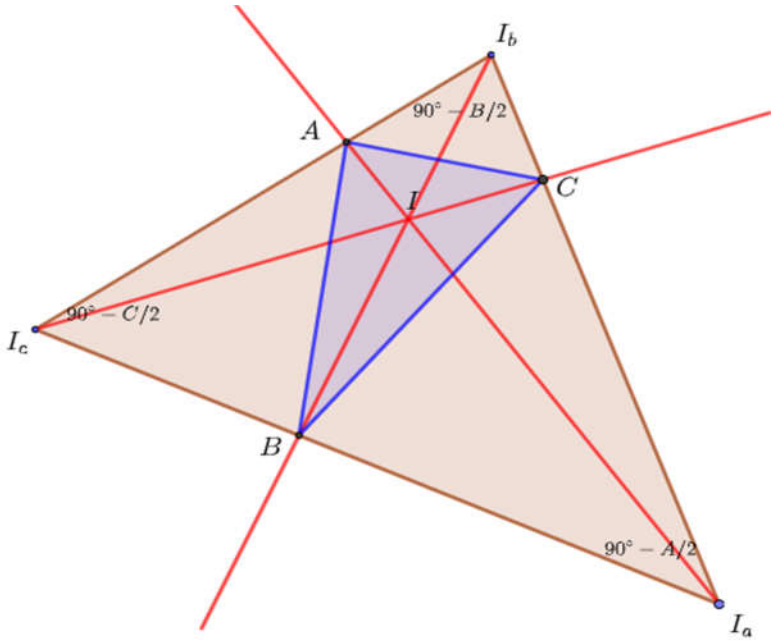

In $\Delta ABC, I_a, I_b, I_c$ —excenters, H —orthocenter and I —incenter. Prove that:

$$\frac{AH+2R}{II_a^2+I_bI_c^2}+\frac{BH+2R}{II_b^2+I_aI_c^2}+\frac{CH+2R}{II_c^2+I_aI_b^2}=\frac{4R+r}{8R^2}$$

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Solution by Jose Ferreira Queiroz-Olinda-Brazil

We know that:

$$AH=2R\cdot\cos A\,,BH=2R\cdot\cos B\,,CH=2R\cdot\cos C$$

$$II_a=2R\cdot\sin\frac{A}{2},II_b=2R\cdot\sin\frac{B}{2},II_c=2R\cdot\sin\frac{C}{2}$$

$$I_bI_c=4R\cdot\cos\frac{A}{2},I_aI_c=4R\cdot\cos\frac{B}{2},I_cI_a=4R\cdot\cos\frac{C}{2}$$

Then,

$$\frac{AH+2R}{II_a^2+I_bI_c^2}=\frac{2R\cos A+2R}{16R^2\sin^2\frac{A}{2}+16R^2\cos^2\frac{A}{2}}=\frac{2R(1+\cos A)}{16R^2}=\frac{1+\cos A}{8R}$$

Similarly,

$$\frac{BH+2R}{II_b^2+I_aI_c^2}=\frac{1+\cos B}{8R}\text{ and }\frac{CH+2R}{II_c^2+I_aI_b^2}=\frac{1+\cos C}{8R}$$

So, we get:

$$K=\frac{AH+2R}{II_a^2+I_bI_c^2}+\frac{BH+2R}{II_b^2+I_aI_c^2}+\frac{CH+2R}{II_c^2+I_aI_b^2}=\frac{\cos A+\cos B+\cos C}{8R}$$

$$K=\frac{\frac{b^2+c^2-a^2}{2bc}+\frac{a^2+c^2-b^2}{2ac}+\frac{a^2+b62-c^2}{2ab}+3}{8R}=$$

$$=\frac{ab^2+ac^2-a^3+bc^2+a^2b-b^3+a^2c+b^2c-c^3+6abc}{2abc\cdot 8R}=$$

$$=\frac{ab(a+b)+ac(a+c)+bc(b+c)-(a^3+b^3+c^3)+6abc}{2abc\cdot 8R}=$$

$$=\frac{2s(ab+bc+ca)-(a^3+b^3+c^3)+3abc}{16abcR}=$$

$$=\frac{3s(s^2+r^2+4Rr)-(2s^3-6r^2s-12Rrs)+12rRS}{64r^2RS}=$$

$$=\frac{2S^3+2R^2S+8rRs-2s^3+6r62\,s+24Rrs}{64R^2rs}=\frac{8r^2s+32Rrs}{64R^2rs}=\frac{4R+r}{8R^2}$$