FAMOUS INEQUALITIES REDESIGNED IN THE TRIANGLE WITH SIDES m_a, m_b, m_c

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Abstract. For a given triangle ABC with sides a, b, c we denote the medians as m_a, m_b, m_c. In a new triangle $A'B'C'$ with sides m_a, m_b, m_c we will redesign the form of a few famous inequalities.

Preliminaries:

We will use the following notations:

For $\Delta ABC : a, b, c$ - sides; s - semiperimeter; F - area; R - circumradii; r - inradii.

For $\Delta A'B'C' : a', b', c'$ - sides; s' - semiperimeter; F' - area; R' - circumradii; r' - inradii.

It is clear that:

$$a' = m_a; b' = m_b; c' = m_c; s' = \frac{m_a + m_b + m_c}{2}$$

Theorem: In ΔABC the following relationships holds:

$$F' = \frac{3F}{4}; R' = \frac{m_a m_b m_c}{3F}; r' = \frac{3F}{2(m_a + m_b + m_c)}$$

Proof.

$$2m_b^2 m_c^2 - m_a^4 = 2 \cdot \frac{2(a^2 + c^2) - b^2}{4} \cdot \frac{2(a^2 + b^2) - c^2}{4} - \frac{(2(b^2 + c^2) - a^2)^2}{16} =$$

$$= \frac{1}{16}((4a^2 + 4c^2 - 2b^2)(2a^2 + 2b^2 - c^2) - (2b^2 + 2c^2 - a^2)^2) =$$

$$= \frac{1}{16}(7a^4 - 8b^4 - 8c^4 + 8a^2b^2 + 8b^2c^2 + 8c^2a^2)$$

$$\frac{1}{16}\sum_{cyc}(2m_b^2 m_c^2 - m_a^4) = \frac{1}{16}\sum_{cyc}(7 - 8 - 8)a^4 + \frac{1}{16}\sum_{cyc}(8 + 8 + 2)a^2b^2 =$$

$$= -\frac{9}{16}\sum_{cyc} a^4 + \frac{18}{16}\sum_{cyc} a^2b^2 \tag{1}$$

$$F'^2 = \frac{1}{8}\sum_{cyc}(a')^2(b')^2 - \frac{1}{16}\sum_{cyc}(a')^4 =$$

$$= \frac{1}{16}\sum_{cyc}(2m_b^2 m_c^2 - m_a^4) =$$

$$\overset{(1)}{=} -\frac{9}{16}\sum_{cyc} a^4 + \frac{18}{16}\sum_{cyc} a^2b^2$$

$$16F'^2 = 9\left(\sum_{cyc} a^2b^2 - \sum_{cyc} a^4\right)$$

$$16F'^2 = 9F^2$$
$$F'^2 = \frac{9F^2}{16}$$
$$F' = \frac{3F}{4}$$
$$R' = \frac{a'b'c'}{4F'} = \frac{m_a m_b m_c}{4 \cdot \frac{3F}{4}} = \frac{m_a m_b m_c}{3F}$$
$$r' = \frac{F'}{s'} = \frac{\frac{3F}{4}}{\frac{m_a+m_b+m_c}{2}} = \frac{3F}{2(m_a+m_b+m_c)}$$

MITRINOVIC'S INEQUALITY

In $\Delta A'B'C'$ holds:

$$3\sqrt{3}r' \leq s'; s' \leq \frac{3\sqrt{3}}{2}R'$$

Redesigned:

$$s' \geq 3\sqrt{3}r' \Rightarrow \frac{m_a+m_b+m_c}{2} \geq 3\sqrt{3} \cdot \frac{3F}{2(m_a+m_b+m_c)}$$
$$(m_a+m_b+m_c)^2 \geq 9\sqrt{3}F$$
$$s' \leq \frac{3\sqrt{3}}{2}R' \Rightarrow \frac{m_a+m_b+m_c}{2} \leq \frac{3\sqrt{3}}{2} \cdot \frac{m_a m_b m_c}{3F}$$
$$2(m_a+m_b+m_c)F \leq \sqrt{3}m_a m_b m_c$$

STEINIG'S INEQUALITY

In $\Delta A'B'C'$ holds:

$$\frac{1}{2R'r'} \leq \frac{1}{(a')^2} + \frac{1}{(b')^2} + \frac{1}{(c')^2} \leq \frac{1}{4(r')^2}$$

Redesigned:

$$\frac{1}{m_a^2} + \frac{1}{m_b^2} + \frac{1}{m_c^2} \geq \frac{2}{2 \cdot \frac{m_a m_b m_c}{3F} \cdot \frac{3F}{2(m_a+m_b+m_c)}} =$$
$$= \frac{m_a+m_b+m_c}{m_a m_b m_c}$$
$$\frac{1}{m_a^2} + \frac{1}{m_b^2} + \frac{1}{m_c^2} \geq \frac{m_a+m_b+m_c}{m_a m_b m_c}$$
$$\frac{1}{m_a^2} + \frac{1}{m_b^2} + \frac{1}{m_c^2} \leq \frac{1}{4 \cdot \frac{9F^2}{4(m_a+m_b+m_c)^2}} = \frac{(m_a+m_b+m_c)^2}{9F^2}$$
$$\frac{1}{m_a^2} + \frac{1}{m_b^2} + \frac{1}{m_c^2} \leq \frac{(m_a+m_b+m_c)^2}{9F^2}$$

EULER'S INEQUALITY

In $\Delta A'B'C'$ holds:

$$R' \geq 2r'$$

Redesigned:

$$\frac{m_a m_b m_c}{3F} \geq 2 \cdot \frac{3F}{2(m_a+m_b+m_c)}$$
$$(m_a+m_b+m_c)m_a m_b m_c \geq 9F^2$$

BĂNDILĂ'S INEQUALITY

In $\Delta A'B'C'$ holds:

$$\frac{b'}{c'}+\frac{c'}{b'}\leq\frac{R'}{r'}$$

Redesigned:

$$\frac{m_b}{m_c}+\frac{m_c}{m_b}\leq\frac{\frac{m_a m_b m_c}{3F}}{\frac{3F}{2(m_a+m_b+m_c)}}=\frac{2m_a m_b m_c(m_a+m_b+m_c)}{9F^2}$$

$$\frac{m_b}{m_c}+\frac{m_c}{m_b}\leq\frac{2m_a m_b m_c(m_a+m_b+m_c)}{9F^2}$$

IONESCU-WEITZENBOCK'S INEQUALITY

In $\Delta A'B'C'$ holds:

$$(a')^2+(b')^2+(c')^2\geq 4\sqrt{3}F'$$

Redesigned:

$$m_a^2+m_b^2+m_c^2\geq 4\sqrt{3}\cdot\frac{3F}{4}=3\sqrt{3}F$$

$$m_a^2+m_b^2+m_c^2\geq 3\sqrt{3}F$$

NEUBERG'S INEQUALITY

In $\Delta A'B'C'$ holds:

$$36(r')^2\leq(a')^2+(b')^2+(c')^2\leq 9(R')^2$$

Redesigned:

$$m_a^2+m_b^2+m_c^2\geq 36\cdot\Big(\frac{3F}{2(m_a+m_b+m_c)}\Big)^2=$$

$$=36\cdot\frac{9F^2}{4(m_a+m_b+m_c)^2}=\frac{81F^2}{(m_a+m_b+m_c)^2}$$

$$(m_a^2+m_b^2+m_c^2)(m_a+m_b+m_c)^2\geq 81F^2$$

$$m_a^2+m_b^2+m_c^2\leq 9\cdot\Big(\frac{m_a m_b m_c}{3F}\Big)^2=$$

$$=9\cdot\frac{m_a^2 m_b^2 m_c^2}{9F^2}=\frac{m_a^2 m_b^2 m_c^2}{F^2}$$

$$m_a^2+m_b^2+m_c^2\leq\frac{m_a^2 m_b^2 m_c^2}{F^2}$$

LEUENBERGER'S INEQUALITY

In $\Delta A'B'C'$ holds:

$$36(r')^2\leq a'b'+b'c'+c'a'\leq 9(R')^2$$

Redesigned:

$$m_a m_b+m_b m_c+m_c m_a\geq 36\cdot\Big(\frac{3F}{2(m_a+m_b+m_c)}\Big)^2=$$

$$=36\cdot\frac{9F^2}{4(m_a+m_b+m_c)^2}=\frac{81F^2}{(m_a+m_b+m_c)^2}$$

$$(m_a m_b+m_b m_c+m_c m_a)(m_a+m_b+m_c)^2\geq 81F^2$$

$$m_a m_b+m_b m_c+m_c m_a\leq 9\cdot\Big(\frac{m_a m_b m_c}{3F}\Big)^2=$$

$$=9\cdot\frac{m_a^2 m_b^2 m_c^2}{9F^2}=\frac{m_a^2 m_b^2 m_c^2}{F^2}$$

$$m_a m_b + m_b m_c + m_c m_a \le \frac{m_a^2 m_b^2 m_c^2}{F^2}$$

GORDON'S INEQUALITY
In $\Delta A'B'C'$ holds:

$$a'b' + b'c' + c'a' \ge 4\sqrt{3}F'$$

Redesigned:

$$m_a m_b + m_b m_c + m_c m_a \ge 4\sqrt{3} \cdot \frac{m_a m_b m_c}{3F}$$

$$m_a m_b + m_b m_c + m_c m_a \ge \frac{4\sqrt{3} m_a m_b m_c}{3F}$$

GOLDNER'S INEQUALITY
In $\Delta A'B'C'$ holds:

$$(a')^4 + (b')^4 + (c')^4 \ge 16(F')^2$$

Redesigned:

$$m_a^4 + m_b^4 + m_c^4 \ge 16 \cdot \left(\frac{3F}{4}\right)^2 = 16 \cdot \frac{9F^2}{16}$$

$$m_a^4 + m_b^4 + m_c^4 \ge 9F^2$$

WALKER'S INEQUALITY
In $\Delta A'B'C'$ holds:

$$3\left(\frac{(a')^2}{(b')^2} + \frac{(b')^2}{(c')^2} + \frac{(c')^2}{(a')^2}\right) \ge ((a')^2 + (b')^2 + (c')^2)\left(\frac{1}{(a')^2} + \frac{1}{(b')^2} + \frac{1}{(c')^2}\right)$$

Redesigned:

$$3\left(\frac{m_a^2}{m_b^2} + \frac{m_b^2}{m_c^2} + \frac{m_c^2}{m_a^2}\right) \ge (m_a^2 + m_b^2 + m_c^2)\left(\frac{1}{m_a^2} + \frac{1}{m_b^2} + \frac{1}{m_c^2}\right)$$

HADWIGER-FINSLER'S INEQUALITY
In $\Delta A'B'C'$ holds:

$$(a')^2 + (b')^2 + (c')^2 \ge 4\sqrt{3}F' + (a' - b')^2 + (b' - c')^2 + (c' - a')^2$$

Redesigned:

$$m_a^2 + m_b^2 + m_c^2 \ge 4\sqrt{3} \cdot \frac{3F}{4} + (m_a - m_b)^2 + (m_b - m_c)^2 + (m_c - m_a)^2$$

$$m_a^2 + m_b^2 + m_c^2 \ge 3\sqrt{3}F + (m_a - m_b)^2 + (m_b - m_c)^2 + (m_c - m_a)^2$$

CURRY'S INEQUALITY
In $\Delta A'B'C'$ holds:

$$\frac{9a'b'c'}{a' + b' + c'} \ge 4\sqrt{3}F'$$

Redesigned:

$$\frac{9m_a m_b m_c}{m_a + m_b + m_c} \ge 4\sqrt{3} \cdot \frac{m_a m_b m_c}{3F}$$

$$\frac{9}{m_a + m_b + m_c} \ge \frac{4\sqrt{3}}{3F}$$

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