R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Let $a, b, c \geq 0$ such that $a+b+c=2(ab+bc+ca)$. Prove that:

$$(a+1)\sqrt{a+(bc)^2}+(b+1)\sqrt{b+(ac)^2}+(c+1)\sqrt{c+(ab)^2}+3abc \geq 5(ab+bc+ca)$$

Proposed by Phan Ngoc Chau, Binh Dinh-Ho Chi Minh-Vietnam

Solution by Nguyen Van Canh-BenTre-Vietnam

By B. C. S Inequality we have:

$$(a+1)\sqrt{a+(bc)^2}=\sqrt{(a+1)^2(a+(bc)^2)}=\sqrt{(a^2+2a+1)(a+(bc)^2)}$$

$$=\sqrt{(a^2-2a+1+4a)(a+(bc)^2)}$$

$$=\sqrt{\left((1-a)^2+\left(2\sqrt{a}\right)^2\right)\left((bc)^2+\left(\sqrt{a}\right)^2\right)} \geq (1-a)bc+2a;\ (1)$$

Similary:

$$(b+1)\sqrt{b+(ac)^2} \geq (1-b)ac+2b;\ (2)$$

$$(c+1)\sqrt{c+(ab)^2} \geq (1-c)ab+2c;\ (3)$$

$(1)+(2)+(3) \Rightarrow$

$$(a+1)\sqrt{a+(bc)^2}+(b+1)\sqrt{b+(ac)^2}+(c+1)\sqrt{c+(ab)^2}$$

$$\geq bc+ac+ab+2(a+b+c)-3abc$$

$$=bc+ac+ab+4(ab+bc+ca)-3abc=5(ab+bc+ca)-3abc;$$

$$\Rightarrow (a+1)\sqrt{a+(bc)^2}+(b+1)\sqrt{b+(ac)^2}+(c+1)\sqrt{c+(ab)^2}+3abc$$
$$\geq 5(ab+bc+ca)$$

Proved. Equality $\Leftrightarrow (a,b,c)\in\{(0,0,0),(1,1,0),(0,1,1),(1,0,1)\}$