**THE BEVAN POINT AND ITS SPECIAL RELATIONSHIP**

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**Abstract** In this paper we present a special relationship that gives the distance between the Bevan point and any point on the plane of the fixed triangle. We use the metric relationship of the circumcenter and geometric properties of the excentral triangle to obtain the result.

**Keywords:** Bevan Point, Excentral Triangle, Pedal Triangle, Stewart's Theorem.

# 1 Introduction

The Bevan point $V$ is the point of concurrence of the perpendiculars from the excenters to the respective sides of a triangle $ABC$. Also is the circumcenter of the excentral triangle $I_aI_bI_c$.

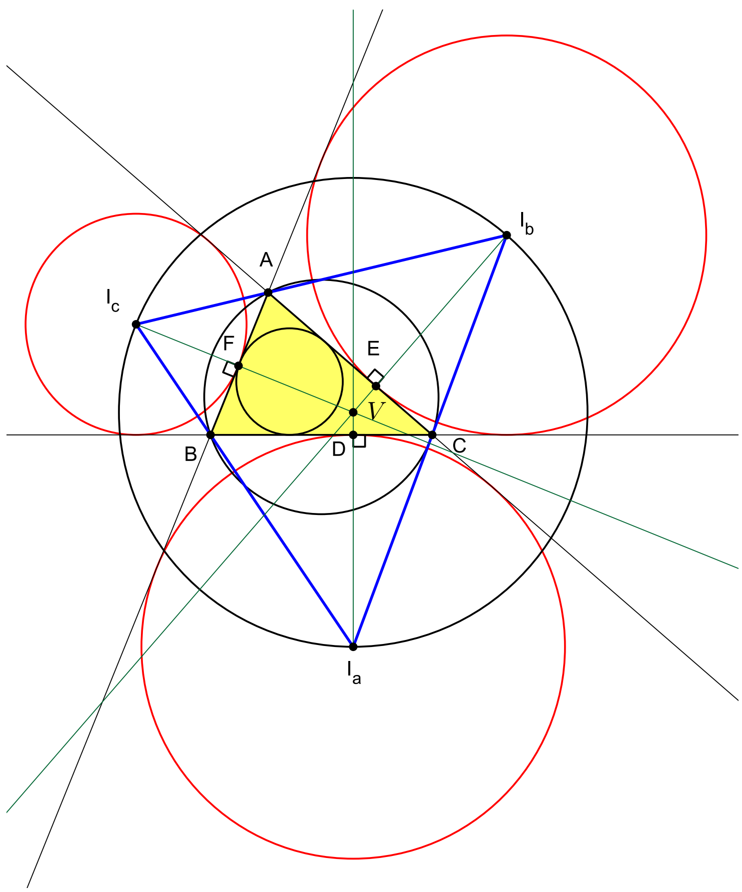


Figure 1: Bevan point

It is named in honor of Benjamin Bevan, a relatively unknown Englishman proposed the problem of proving that the circumcenter $O$ was the midpoint of the incenter $I$ and the circumcenter of the excentral triangle and that the circumradius of the excentral triangle was twice that of the radius of circle circumscribed to triangle ABC (Bevan 1806), a problem solved by John Butterworth (1806) [7].

Another interesting property is that the incenter $I$ of the triangle $ABC$ is the orthocenter of the excentral triangle, then the triangle $ABC$ is orthic of the excentral triangle [3].

In this paper, we will use the excentral triangle $I_aI_bI_c$ and we will show an identity that gives us the distance between the Bevan point of a triangle and any point on the plane that contains the triangle. Also we will use barycentric coordinates and the Conway triangle notation to show collinearity of the some classical centers.

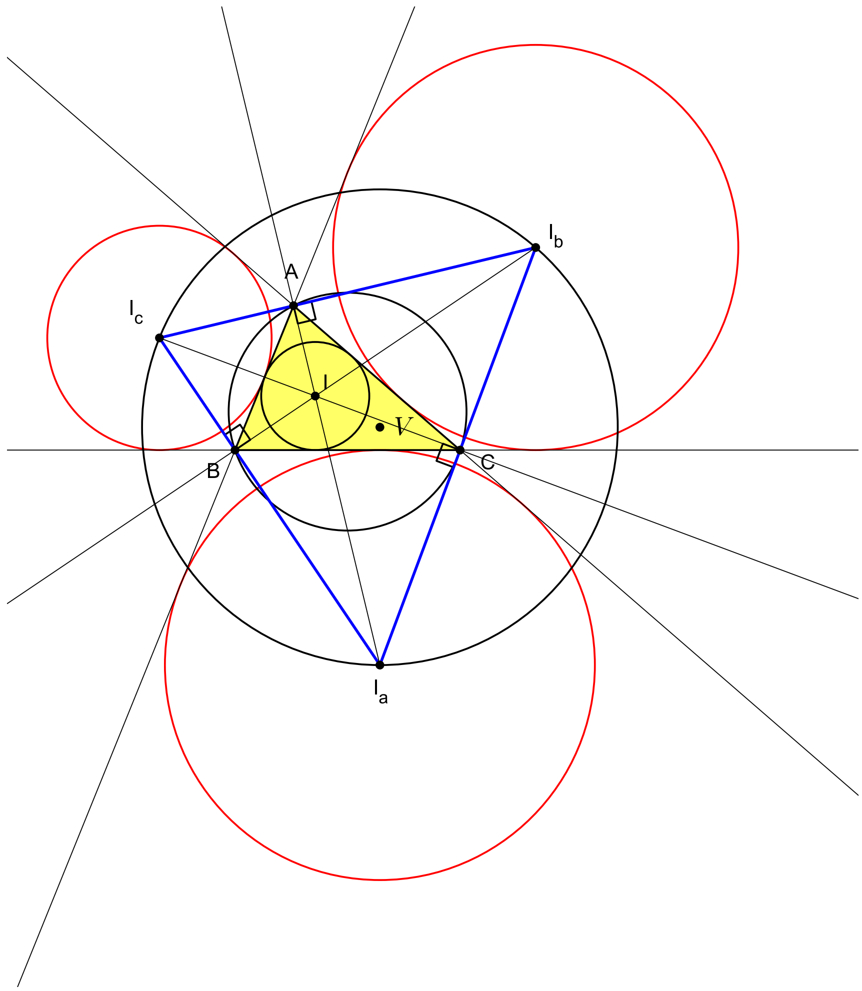


Figure 2: $ABC$ is orthic of the excentral triangle

## 2 Notation

Let $ABC$ be an acute triangle. We denote its side-lengths by $BC = a$, $AC = b$, $AB = c$, its semi perimeter by $s = \frac{1}{2}(a+b+c)$, its area by $F$, its circumradius by $R$ and inradius by $r$. Its classical centers are the Centroid $G$, the Incenter $I$, the Circumcenter $O$, the Orthocenter $H$, Nagel point $N_a$ and Spieker center $S_p$.

We list here a a few relations that hold between the elements of a triangle.

(a) $F = \sqrt{s(s-a)(s-b)(s-c)} = \frac{abc}{4R} = sr.$

(b) $\sin\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{bc}}$, $\sin\left(\frac{B}{2}\right) = \sqrt{\frac{(s-a)(s-c)}{ac}}$ and $\sin\left(\frac{C}{2}\right) = \sqrt{\frac{(s-a)(s-b)}{ab}}$

(c) $\cos\left(\frac{A}{2}\right) = \sqrt{\frac{s(s-a)}{bc}}$, $\cos\left(\frac{B}{2}\right) = \sqrt{\frac{s(s-b)}{ac}}$ and $\cos\left(\frac{C}{2}\right) = \sqrt{\frac{s(s-c)}{ab}}$

(d) $\sin(A) = \frac{2F}{bc}$, $\sin(B) = \frac{2F}{ac}$ and $\sin(C) = \frac{2F}{ab}$

Also we will use the Conway triangle notation.

Denote $S_A = \frac{1}{2}(b^2+c^2-a^2)$, $S_B = \frac{1}{2}(c^2+a^2-b^2)$ and $S_C = \frac{1}{2}(a^2+b^2-c^2)$. Let $S = 2F$, then

(e) $S^2 = S_AS_B + S_AS_C + S_BS_C = S_BS_C + a^2S_A = S_AS_C + b^2S_B = S_AS_B + c^2S_C$

(f) $2S^2 = a^2S_A + b^2S_B + c^2S_C$

The barycentric coordinates of the points which apper in our article are [4]:

| Point | Homogeneous Barycentric Coordinates | Normalized Barycentric Coordinates |
|---|---|---|
| Incenter - $I$ | $(a : b : c)$ | $\left(\frac{a}{2s}, \frac{b}{2s}, \frac{c}{2s}\right)$ |
| Circumcenter - $O$ | $(a^2S_A : b^2S_B : c^2S_C)$ | $\left(\frac{a^2S_A}{2S^2}, \frac{b^2S_B}{2S^2}, \frac{c^2S_C}{2S^2}\right)$ |
| Orthocenter - $H$ | $(S_BS_C : S_AS_C : S_AS_B)$ | $\left(\frac{S_BS_C}{S^2}, \frac{S_AS_C}{S^2}, \frac{S_AS_B}{S^2}\right)$ |
| Nagel point - $N_a$ | $(s-a : s-b : s-c)$ | $\left(\frac{s-a}{s}, \frac{s-b}{s}, \frac{s-c}{s}\right)$ |
| Spieker center - $S_p$ | $(2s-a : 2s-b : 2s-c)$ | $\left(\frac{2s-a}{4s}, \frac{2s-b}{4s}, \frac{2s-c}{4s}\right)$ |
| De Longchamps point - $L$ | $(S^2-2S_BS_C : S^2-2S_AS_C : S^2-2S_AS_B)$ | $\left(\frac{S^2-2S_BS_C}{S^2}, \frac{S^2-2S_AS_C}{S^2}, \frac{S^2-2S_AS_B}{S^2}\right)$ |
| Bevan point - $V$ | $(a^2S_A-2ar^2s : b^2S_B-2br^2s : c^2S_C-2cr^2s)$ | $\left(\frac{a^2S_A-2ar^2s}{S^2}, \frac{b^2S_B-2br^2s}{S^2}, \frac{c^2S_C-2cr^2s}{S^2}\right)$ |

## 3 Definitions

### 1. Pedal Triangle

The Pedal Triangle of a point with regard to a triangle is that triangle whose vertices are the feet of the perpendiculars from the point to the side of the given triangle.

### 2. De Longchamps Point

The de Longchamps point L is the reflection of the orthocenter H about the circumcenter O of a triangle.

### 3. Excentral Triangle

The excentral triangle of a triangle $ABC$ is the triangle $I_aI_bI_c$ with vertices corresponding to the excenters of triangle $ABC$.

The triangle $ABC$ is the pedal triangle of it's excentral triangle $I_aI_bI_c$. The angles of triangle $I_aI_bI_c$ are $90° - \dfrac{A}{2}$, $90° - \dfrac{B}{2}$, $90° - \dfrac{C}{2}$ and sides are $I_bI_c = \dfrac{a}{\sin\left(\dfrac{A}{2}\right)}$, $I_aI_c = \dfrac{b}{\sin\left(\dfrac{B}{2}\right)}$ and $I_aI_b = \dfrac{c}{\sin\left(\dfrac{C}{2}\right)}$ (see Figure 3).

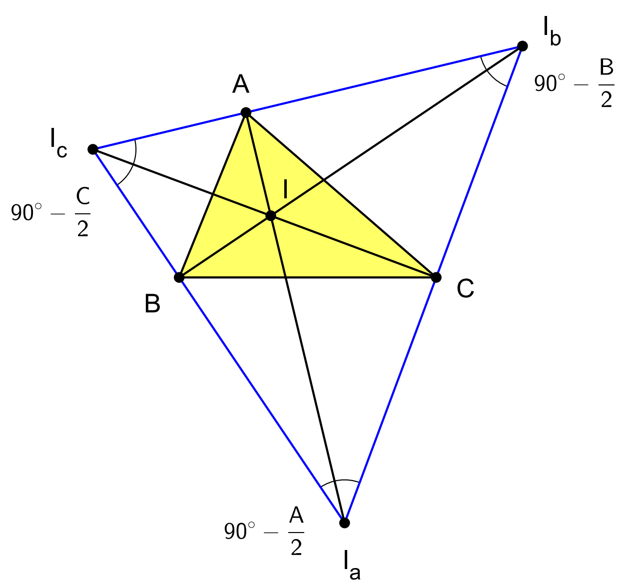


Figure 3: Angles of excentral triangle

# 4 Collinear Points

The triplet of points $(I, O, V)$, $(H, S_p, V)$ and $(N_a, V, L)$ are collinear. To prove the collinearity of the respective triples of points, we will use the homogeneous barycentric and the normalized barycentric coordinates of the respective points.

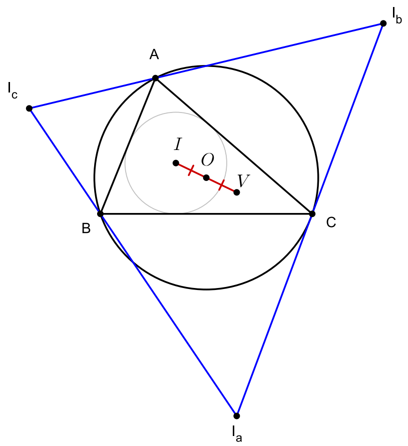


Figure 4: Incenter, Circumcenter and Bevan point

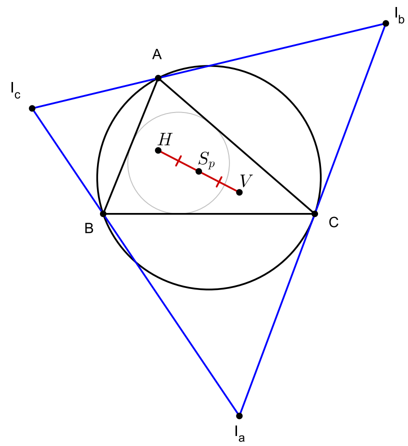


Figure 5: Orthocenter, Spiker center and Bevan point

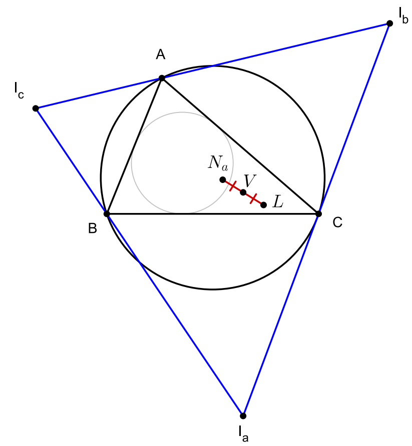


Figure 6: Nagel point, Bevan point and De Longchamps point

**Theorem 4.1** *The incenter $I$, The orthocenter $O$ and the Bevan point $V$ are collinear and*

$$IV = 2 \cdot IO$$

**Proof:** The points $(I, O, V)$ are collinear if and only if

$$\begin{vmatrix} a & b & c \\ a^2S_A - 2ar^2s & b^2S_B - 2br^2s & c^2S_C - 2cr^2s \\ a^2S_A & b^2S_B & c^2S_C \end{vmatrix} = 0 \iff \begin{vmatrix} a & b & c \\ a^2S_A & b^2S_B & c^2S_C \\ 2ar^2s & 2br^2s & 2cr^2s \end{vmatrix} = 0 \iff$$

$$\iff \begin{vmatrix} a & b & c \\ a^2S_A & b^2S_B & c^2S_C \\ a & b & c \end{vmatrix} = 0. \text{ Hence proved.}$$

The circumcenter $O$ is the midpoint of segment $IV$(see figure 4). Indeed

$$O = \frac{I + V}{2}$$

$$O = \frac{1}{2}\left[\left(\frac{a}{2s}, \frac{b}{2s}, \frac{c}{2s}\right) + \left(\frac{a^2S_A - 2ar^2s}{S^2}, \frac{b^2S_B - 2br^2s}{S^2}, \frac{c^2S_C - 2cr^2s}{S^2}\right)\right]$$

$$O = \frac{1}{2}\left(\frac{a}{2s} + \frac{a^2S_A - 2ar^2s}{S^2}, \frac{b}{2s} + \frac{b^2S_B - 2br^2s}{S^2}, \frac{c}{2s} + \frac{c^2S_C - 2cr^2s}{S^2}\right)$$

$$O = \frac{1}{2}\left(\frac{2ar^2s}{S^2} + \frac{a^2S_A - 2ar^2s}{S^2}, \frac{2br^2s}{S^2} + \frac{b^2S_B - 2br^2s}{S^2}, \frac{2cr^2s}{S^2} + \frac{c^2S_C - 2cr^2s}{S^2}\right)$$

$$O = \left(\frac{a^2S_A}{2S^2}, \frac{b^2S_B}{2S^2}, \frac{c^2S_C}{2S^2}\right)$$

Therefore,

$$IV = 2 \cdot OV$$

**Theorem 4.2** *The Orthocenter $H$, The Spiker center $S_p$ and the Bevan point $V$ are collinear and*

$$HV = 2 \cdot S_pV$$

**Proof:** The points $(H, S_p, V)$ are collinear if and only if

$$\begin{vmatrix} 2s-a & 2s-b & 2s-c \\ a^2S_A - 2ar^2s & b^2S_B - 2br^2s & c^2S_C - 2cr^2s \\ S_BS_C & S_AS_C & S_AS_B \end{vmatrix} = 0 \iff \begin{vmatrix} 2s-a & 2s-b & 2s-c \\ a^2S_A - 2ar^2s & b^2S_B - 2br^2s & c^2S_C - 2cr^2s \\ S^2 - a^2S_A & S^2 - b^2S_B & S^2 - c^2S_A \end{vmatrix} = 0 \iff$$

$$\iff \begin{vmatrix} 2s-a & 2s-b & 2s-c \\ a^2S_A - 2ar^2s & b^2S_B - 2br^2s & c^2S_C - 2cr^2s \\ S^2 - 2ar^2s & S^2 - 2br^2s & S^2 - 2cr^2s \end{vmatrix} = 0 \iff 2r^2s \begin{vmatrix} 2s-a & 2s-b & 2s-c \\ a^2S_A - 2ar^2s & b^2S_B - 2br^2s & c^2S_C - 2cr^2s \\ 2s-a & 2s-b & 2s-c \end{vmatrix} = 0.$$

Hence proved.

The Spieker center is the midpoint of segment $HV$(see figure 5). Indeed

$$S_p = \frac{H+V}{2}$$

$$S_p = \frac{1}{2}\left[\left(\frac{S_BS_C}{S^2}, \frac{S_AS_C}{S^2}, \frac{S_AS_B}{S^2}\right) + \left(\frac{a^2S_A - 2ar^2s}{S^2}, \frac{b^2S_B - 2br^2s}{S^2}, \frac{c^2S_C - 2cr^2s}{S^2}\right)\right]$$

$$S_p = \frac{1}{2}\left[\left(\frac{S^2 - a^2S_A}{S^2}, \frac{S^2 - a^2S_B}{S^2}, \frac{S^2 - a^2S_C}{S^2}\right) + \left(\frac{a^2S_A - 2ar^2s}{S^2}, \frac{b^2S_B - 2br^2s}{S^2}, \frac{c^2S_C - 2cr^2s}{S^2}\right)\right]$$

$$S_p = \frac{1}{2}\left(\frac{S^2 - 2ar^2s}{S^2}, \frac{S^2 - 2br^2s}{S^2}, \frac{S^2 - 2cr^2s}{S^2}\right)$$

$$S_p = \left(\frac{2s-a}{4s}, \frac{2s-b}{4s}, \frac{2s-c}{4s}\right)$$

Therefore,

$$HV = 2 \cdot S_pV$$

**Theorem 4.3** *The Nagel point $N_a$, The Bevan point $V$ and the De Longchamps point $L$ are collinear and*

$$N_aL = 2 \cdot VL$$

**Proof:** The points $(N_a, V, L)$ are collinear if and only if

$$\begin{vmatrix} s-a & s-b & s-c \\ a^2S_A - 2ar^2s & b^2S_B - 2br^2s & c^2S_C - 2cr^2s \\ S^2 - 2S_BS_C & S^2 - 2S_AS_C & S^2 - 2S_AS_B \end{vmatrix} = 0 \iff \begin{vmatrix} s-a & s-b & s-c \\ a^2S_A - 2ar^2s & b^2S_B - 2br^2s & c^2S_C - 2cr^2s \\ 2a^2S_A - S^2 & 2b^2S_B - S^2 & 2c^2S_C - 2S_c - S^2 \end{vmatrix} = 0 \iff$$

$$\iff \begin{vmatrix} s-a & s-b & s-c \\ a^2S_A - 2ar^2s & b^2S_B - 2br^2s & c^2S_C - 2cr^2s \\ 4ar^2s - S^2 & 2br^2s - S^2 & 2cr^2s - S^2 \end{vmatrix} = 0 \iff -4r^2s \begin{vmatrix} s-a & s-b & s-c \\ a^2S_A - 2ar^2s & b^2S_B - 2br^2s & c^2S_C - 2cr^2s \\ s-a & s-b & s-c \end{vmatrix} = 0$$

Hence proved.

The Bevan point is the midpoint of segment $N_aL$(see figure 6). Indeed

$$V = \frac{N_a + L}{2}$$

$$V = \frac{1}{2}\left[\left(\frac{s-a}{s}, \frac{s-b}{s}, \frac{s-c}{s}\right) + \left(\frac{S^2 - 2S_BS_C}{S^2}, \frac{S^2 - 2S_AS_C}{S^2}, \frac{S^2 - 2S_AS_B}{S^2}\right)\right]$$

$$V = \frac{1}{2}\left[\left(\frac{S^2 - 4ar^2s}{S2}, \frac{S^2 - 4br^2s}{S^2}, \frac{S^2 - 4cr^2s}{S^2}\right) + \left(\frac{S^2 - 2S_BS_C}{S^2}, \frac{S^2 - 2S_AS_C}{S^2}, \frac{S^2 - 2S_AS_B}{S^2}\right)\right]$$

$$V = \frac{1}{2}\left(\frac{2S^2 - 4ar^2s - 2S_BS_C}{S^2}, \frac{2S^2 - 4br^2s - 2S_AS_C}{S^2}, \frac{2S^2 - 4cr^2s - 2S_AS_B}{S^2}\right)$$

$$V = \frac{1}{2}\left(\frac{2a^2S_A - 4ar^2s}{S^2}, \frac{2b^2S_A - 4br^2s}{S^2}, \frac{2c^2S_A - 4cr^2s}{S^2}\right)$$

$$V = \left(\frac{a^2S_A - 2ar^2s}{S^2}, \frac{b^2S_A - 2br^2s}{S^2}, \frac{c^2S_A - 2cr^2s}{S^2}\right)$$

Therefore,

$$N_a L = 2 \cdot VL$$

# 5 Prepositions

**Preposition 5.1** *The excentral triangle has the area* $F_I = 2Rs$.

**Proof:**

$$F_I = \frac{1}{2} \cdot I_a I_c \cdot I_a I_b \cdot \sin(90^\circ - \frac{A}{2}) = \frac{1}{2} \cdot \frac{c}{\sin\left(\frac{C}{2}\right)} \cdot \frac{b}{\sin\left(\frac{B}{2}\right)} \cdot \cos\left(\frac{A}{2}\right).$$

$$F_I = \frac{1}{2} \cdot \frac{c}{\sqrt{\frac{(s-a)(s-b)}{ab}}} \cdot \frac{b}{\sqrt{\frac{(s-a)(s-c)}{ac}}} \cdot \sqrt{\frac{s(s-a)}{bc}} = \frac{abc}{2} \cdot \sqrt{\frac{s}{(s-a)(s-b)(s-c)}}.$$

$$F_I = \frac{abc}{2} \cdot \sqrt{\frac{s^2}{F^2}} = \frac{4RF}{2} \cdot \frac{s}{F}.$$

$$F_I = 2Rs. \tag{1}$$

**Preposition 5.2** *The circumradius of a excentral triangle is* $R_I = 2R$.

**Proof:** Using the Law of Sines in triangle $I_a I_b I_c$, we have

$$R_I = \frac{I_a I_b}{2\sin(90^\circ - \frac{C}{2})} = \frac{c}{2\sin\left(\frac{C}{2}\right)\cos\left(\frac{C}{2}\right)} = \frac{c}{\sin(C)} = \frac{2R\sin(C)}{\sin(C)}.$$

$$R_I = 2R \tag{2}$$

**Preposition 5.3** *Let* $a$, $b$ *and* $c$ *the sides of an triangle* $ABC$, *and* $s$, $r$, $R$ *and* $F$ *are, respectively, its semiperimeter, inradius, circumradius and area of that triangle, then*

**(1.1)** $ab + ac + bc = s^2 + r^2 + 4Rr$

**(1.2)** $a^2 + b^2 + c^2 = 2s^2 - 2r^2 - 8Rr$

**(1.3)** $a^3 + b^3 + c^3 = 2s^3 - 6r^2 s - 12Rrs$

**(1.4)** $a^4 + b^4 + c^4 = 2s^4 + 2r^4 - 12r^2 s^2 - 16Rrs^2 + 32R^2 r^2 + 16Rr^3$

**(1.5)** $a^5 + b^5 + c^5 = 2s^5 - 20r^2 s^3 + 10r^4 s + 60Rr^3 s - 20Rrs^3 + 80R^2 r^2 s$

**Proof:** The proofs of (1.1), (1.2) and (1.3) are available in [2] and by using (1.1), (1.2) and ($a$) we can prove (1.4). The proof of (1.5) can be obtained using (1.1), (1.2), (1.3), ($a$) and the fact what $a^5+b^5+c^5=(a^2+b^2+c^2)(a^3+b^3+c^3)-a^2b^2(a+b)-a^2c^2(a+c)-b^2c^2(b+c)$.

**Theorem 5.4** *If $O$ is the circumcenter of a triangle $ABC$ and $M$ be any point in the plane of triangle, then*

$$MO^2=\frac{R^2}{2F}\cdot(\,sin\,2A\cdot MA^2+\,sin\,2B\cdot MB^2+\,sin\,2C\cdot MC^2-2F). \qquad (3)$$

**Proof:** The proof of this Theorem can be found in [5].

**Theorem 5.5** *Let $M$ be any point in the plane of a triangle $ABC$ whith Bevan point $V$. Then:*

$$MV^2=\frac{1}{2Fr}\cdot\Big[a\big[a(s-a)-r(r+2R)\big]MA^2+b\big[b(s-b)-r(r+2R)\big]MB^2+c\big[c(s-c)-r(r+2R)\big]MC^2\Big]-2Rr. \quad (4)$$

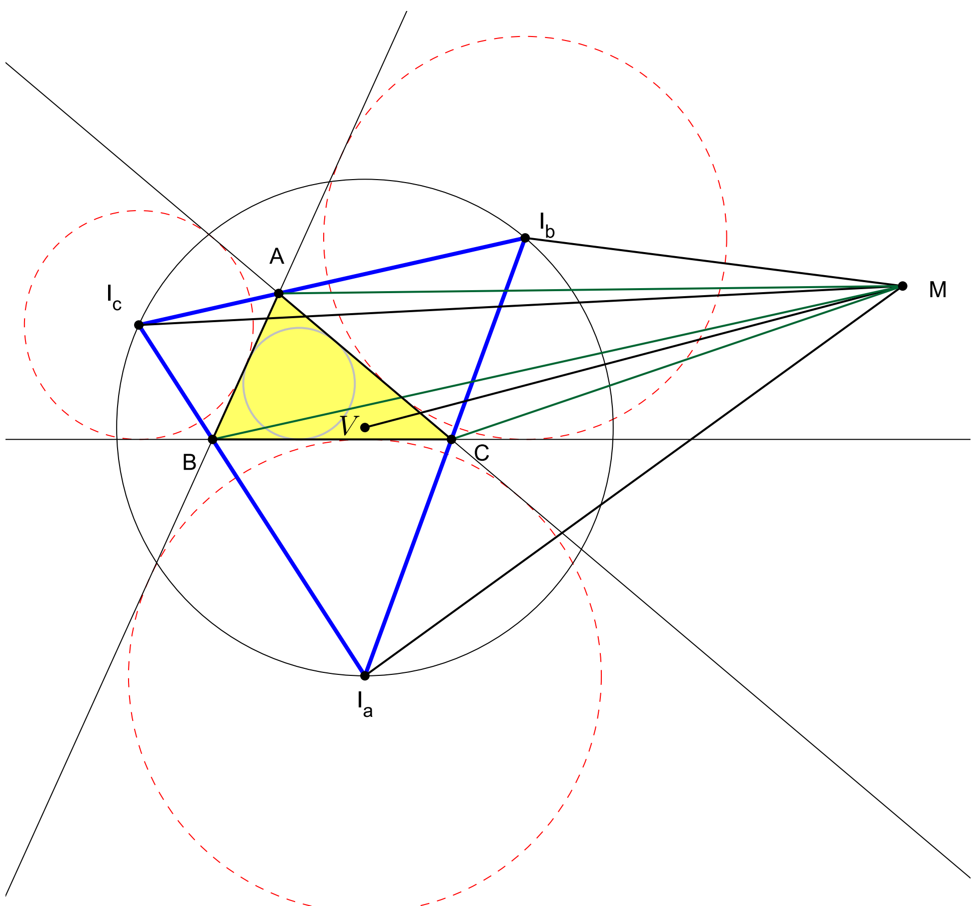


Figure 7: Bevan point

**Proof:** Using the expression (3) in the triangle $I_aI_bI_c$ we get

$$MV^2=\frac{R_I^2}{2F_I}\cdot\Big[\sin(A)\cdot MI_a^2+\sin(B)\cdot MI_b^2+\sin(C)\cdot MI_c^2-2F_I\Big] \qquad (5)$$

Applying Stewart's theorem in the triangles $MI_aI_b$, $MI_aI_c$ and $MI_bI_c$ (see Figure 7) in which cevians $MC$, $MB$, $MA$ respectively, we have

$$MI_b^2 \cdot CI_a + MI_a^2 \cdot CI_b - MC^2 \cdot I_aI_b = I_aI_b \cdot CI_b \cdot CI_a \tag{6}$$

$$MI_c^2 \cdot BI_a + MI_a^2 \cdot BI_c - MB^2 \cdot I_aI_c = I_aI_c \cdot BI_a \cdot BI_c \tag{7}$$

$$MI_b^2 \cdot AI_c + MI_c^2 \cdot AI_b - MA^2 \cdot I_bI_c = I_bI_c \cdot AI_b \cdot AI_c \tag{8}$$

Now using the Law of Sines in the triangles $BI_aC$, $AI_bC$, $AI_cB$ we can prove that

$$\frac{a}{\cos\left(\frac{A}{2}\right)} = \frac{BI_a}{\cos\left(\frac{C}{2}\right)} = \frac{CI_a}{\cos\left(\frac{B}{2}\right)} \tag{9}$$

$$\frac{b}{\cos\left(\frac{B}{2}\right)} = \frac{CI_b}{\cos\left(\frac{A}{2}\right)} = \frac{AI_b}{\cos\left(\frac{C}{2}\right)} \tag{10}$$

$$\frac{c}{\cos\left(\frac{C}{2}\right)} = \frac{AI_c}{\cos\left(\frac{B}{2}\right)} = \frac{BI_c}{\cos\left(\frac{A}{2}\right)} \tag{11}$$

Replace (9) and (10), from (6) we get,

$$a \cdot MI_b^2 \cdot \frac{\cos\left(\frac{B}{2}\right)}{\cos\left(\frac{A}{2}\right)} + b \cdot MI_a^2 \cdot \frac{\cos\left(\frac{A}{2}\right)}{\cos\left(\frac{B}{2}\right)} - c \cdot MC^2 \cdot \frac{1}{\sin\left(\frac{C}{2}\right)} = abc \cdot \frac{\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)}{\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)}$$

$$a \cdot MI_b^2 \cdot \cos^2\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right) + b \cdot MI_a^2 \cdot \cos^2\left(\frac{A}{2}\right)\sin\left(\frac{C}{2}\right) - c \cdot MC^2 \cdot \cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right) = abc \cdot \cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)$$

Using $(b)$ and $(c)$ we have

$$a \cdot MI_b^2 \cdot \frac{s(s-b)}{ac} \cdot \sqrt{\frac{(s-a)(s-b)}{ab}} + b \cdot MI_a^2 \cdot \frac{s(s-a)}{bc} \cdot \sqrt{\frac{(s-a)(s-b)}{ab}} - c \cdot MC^2 \cdot \sqrt{\frac{s(s-a)}{bc}} \cdot \sqrt{\frac{s(s-b)}{ac}} =$$
$$= abc \cdot \sqrt{\frac{s(s-a)}{bc}} \cdot \sqrt{\frac{s(s-b)}{ac}}$$

After simplifying a few steps we obtain

$$(s-b)MI_b^2 + (s-a)MI_a^2 - cMC^2 = abc \tag{12}$$

Similarly replace (9) and (11) in (7), (10) and (11) in (8) we can prove

$$(s-b)MI_b^2 + (s-c)MI_c^2 - aMA^2 = abc \tag{13}$$

$$(s-a)MI_a^2 + (s-c)MI_c^2 - bMB^2 = abc \tag{14}$$

Clearly (12), (13) and (14) represent a system of equations that can solved as a function of $MA$, $MB$ and $MC$, then

$$MI_a^2 = \frac{-aMA^2 + bMB^2 + cMC^2 + abc}{2(s-a)} \tag{15}$$

$$MI_b^2 = \frac{aMA^2 - bMB^2 + cMC^2 + abc}{2(s-b)} \tag{16}$$

$$MI_c^2 = \frac{aMA^2 + bMB^2 - cMC^2 + abc}{2(s-c)} \tag{17}$$

The relations (15), (16) and (17) represent the distance from any point $M$ in the plane of the triangle to the excenters.

Using (1), (2),($a$) and ($d$), from (5) we get,

$$MV^2 = \frac{(2R)^2}{4Rs} \cdot \left[\frac{2F}{bc} \cdot MI_a^2 + \frac{2F}{ac} \cdot MI_b^2 + \frac{2F}{ab} \cdot MI_c^2 - 4Rs\right].$$

$$MV^2 = \frac{(2R)^2}{4Rs} \cdot \left[\frac{2F}{bc} \cdot MI_a^2 + \frac{2F}{ac} \cdot MI_b^2 + \frac{2F}{ab} \cdot MI_c^2\right] - 4R^2.$$

Which implies

$$MV^2 = \frac{1}{2s} \cdot \left[aMI_a^2 + bMI_b^2 + cMI_c^2\right] - 4R^2.$$

Now using (15), (16), (17) and ($a$) we get,

$$MV^2 = \frac{1}{2s} \cdot \left[a \cdot \frac{(-aMA^2 + bMB^2 + cMC^2 + abc)}{2(s-a)} + b \cdot \frac{(aMA^2 - bMB^2 + cMC^2 + abc)}{2(s-b)} + c \cdot \frac{(aMA^2 + bMB^2 - cMC^2 + abc)}{2(s-c)}\right] - 4R^2.$$

$$\begin{aligned} MV^2 \quad &= \frac{1}{4s(s-a)(s-b)(s-c)} \cdot \Bigg[\Big(-a(s-b)(s-c) + b(s-a)(s-c) + c(s-a)(s-b)\Big)aMA^2 + \\ &+\Big(a(s-b)(s-c) - b(s-a)(s-c) + c(s-a)(s-b)\Big)bMB^2 + \\ &+\Big(a(s-b)(s-c) + b(s-a)(s-c) - c(s-a)(s-b)\Big)cMC^2 \\ &+abc\Big(a(s-b)(s-c) + b(s-a)(s-c) + c(s-a)(s-b)\Big)\Bigg] - 4R^2. \end{aligned}$$

$$
\begin{aligned}
MV^2 \quad &= \frac{1}{4F^2}\cdot\Bigg[\Big(-a(-s^2+as+bc)+b(-s^2+bs+ac)+c(-s^2+cs+ab)\Big)aMA^2+\\
&+\Big(a(-s^2+as+bc)-b(-s^2+bs+ac)+c(-s^2+cs+ab)\Big)bMB^2+\\
&+\Big(a(-s^2+as+bc)+b(-s^2+bs+ac)-c(-s^2+cs+ab)\Big)cMC^2+\\
&+\Big(a(-s^2+as+bc)+b(-s^2+bs+ac)+c(-s^2+cs+ab)\Big)\Bigg]-4R^2.
\end{aligned}
$$

$$
\begin{aligned}
MV^2 \quad &= \frac{1}{4F^2}\cdot\Bigg[\Big(s^2(a-b-c)+s(-a^2+b^2+c^2)+abc\Big)aMA^2+\\
&+\Big(s^2(b-a-c)+s(a^2-b^2+c^2)+abc\Big)bMB^2+\\
&+\Big(s^2(c-a-b)+s(a^2+b^2-c^2)+abc\Big)cMC^2+\\
&+\Big(s^2(a+b+c)+s(a^2+b^2+c^2)+3abc)\Big)\Bigg]-4R^2.
\end{aligned}
$$

$$
\begin{aligned}
MV^2 \quad &= \frac{1}{4F^2}\cdot\Bigg[\Big(-2s^2(s-a)+s(-2a^2+2s^2-2r^2-8Rr)+4Rrs\Big)aMA^2+\\
&+\Big(-2s^2(s-b)+s(-2b^2+2s^2-2r^2-8Rr)+4Rrs\Big)bMB^2+\\
&+\Big(-2s^2(s-c)+s(-2c^2+2s^2-2r^2-8Rr)+4Rrs\Big)cMC^2+\\
&+\Big(s^2(2s)+s(2s^2-2r^2-8Rr)+12Rrs\Big)\Bigg]-4R^2.
\end{aligned}
$$

After some algebraic simplifications we reached the necessary conclusion

$$
MV^2=\frac{1}{2Fr}\cdot\Big[a\big[a(s-a)-r(r+2R)\big]MA^2+b\big[b(s-b)-r(r+2R)\big]MB^2+c\big[c(s-c)-r(r+2R)\big]MC^2\Big]-2Rr.
$$

## 6 Main Result

**Corollary 6.1** *If $V$ is the Bevan Point of the triangle $ABC$, then*

$$AV^2 = \frac{4R}{a}(aR - F)$$

$$BV^2 = \frac{4R}{b}(bR - F)$$

$$CV^2 = \frac{4R}{c}(cR - F)$$

**Proof:** Using the Theorem 5.5, replacing $M$ by the $A$ and consider $AA = 0$, $AB = c$ and $AC = b$, then

$$AV^2 = \frac{1}{2Fr} \cdot \Big[a\big[a(s-a) - r(r+2R)\big]AA^2 + b\big[b(s-b) - r(r+2R)\big]AB^2 + c\big[c(s-c) - r(r+2R)\big]AC^2\Big] - 2Rr.$$

$$AV^2 = \frac{1}{2Fr} \cdot \Big[b\big[b(s-b) - r(r+2R)\big]c^2 + c\big[c(s-c) - r(r+2R)\big]b^2\Big] - 2Rr.$$

$$AV^2 = \frac{bc}{2Fr} \cdot \Big[bc(s-b) - cr(r+2R) + bc(s-c) - br(r+2R)\Big] - 2Rr.$$

$$AV^2 = \frac{bc}{2Fr} \cdot \Big[bc(2s-b-c) - r(r+2R)(b+c)\Big] - 2Rr.$$

$$AV^2 = \frac{bc}{2Fr} \cdot \Big[abc - r(r+2R)(2s-a)\Big] - 2Rr.$$

$$AV^2 = \frac{4RF}{2aRF} \cdot \Big(2aRr - 2sr^2 + ar^2\Big) - 2Rr.$$

Hence

$$AV^2 = \frac{4R}{a}(aR - F).$$

Similarly we can prove that $BV^2 = \dfrac{4R}{b}(bR - F)$ and $CV^2 = \dfrac{4R}{c}(cR - F)$.

**Corollary 6.2** *Be $G$ the centroid of the triangle $ABC$ and $V$ the Bevan point, then*

$$GV^2 = \frac{1}{9}(-5s^2 - r^2 - 4Rr + 36R^2)$$

**Proof:** In Theorem 5.5, replace $M$ by the incenter $G$. We get

$$GV^2=\frac{1}{2Fr}\cdot\Big[a\big[a(s-a)-r(r+2R)\big]GA^2+b\big[b(s-b)-r(r+2R)\big]GB^2+c\big[c(s-c)-r(r+2R)\big]GC^2\Big]-2Rr.$$

We know that $GA^2=\frac{1}{9}(2b^2+2c^2-a^2)$, $GB^2=\frac{1}{9}(2a^2+2c^2-b^2)$ and $GC^2=\frac{1}{9}(2a^2+2b^2-c^2)$, then

$$GV^2=\frac{1}{18Fr}\cdot\Big[a\big[a(s-a)-r(r+2R)\big](2b^2+2c^2-a^2)+b\big[b(s-b)-r(r+2R)\big](2a^2+2c^2-b^2)+c\big[c(s-c)-r(r+2R)\big](2a^2+2b^2-c^2)\Big]-2Rr.$$

$$GV^2=\frac{1}{18Fr}\cdot\Bigg[\Big(a^2(s-a)(2b^2+2c^2-a^2)\Big)+\Big(b^2(s-b)(2a^2+2c^2-b^2)\Big)+\Big(c^2(s-c)(2a^2+2b^2-c^2)\Big)-$$
$$-r(r+2R)\Big(a(2b^2+2c^2-a^2)+b(2a^2+2c^2-b^2)+c(2a^2+2b^2-c^2)\Big)\Bigg]-2Rr.$$

$$GV^2=\frac{1}{18Fr}\cdot\Bigg[s\Big(4(a^2b^2+a^2c^2+b^2c^2)-(a^4+b^4+c^4)\Big)-\Big(2a^2b^2(a+b)+a^2c^2(a+c)+2b^2c^2(b+c)-(a^5+b^5+c^5)\Big)+$$
$$-r(r+2R)\Big(2ab(a+b)+2ac(a+c)+2bc(b+c)-(a^3+b^3+c^3)\Big)\Bigg]-2Rr.$$

$$GV^2=\frac{1}{18Fr}\cdot\Bigg[s\Big(4(a^2b^2+a^2c^2+b^2c^2)-(a^4+b^4+c^4)\Big)-\Big(2a^2b^2(2s-c)+a^2c^2(2s-b)+2b^2c^2(2s-a)-(a^5+b^5+c^5)\Big)+$$
$$-r(r+2R)\Big(2ab(2s-c)+2ac(2s-b)+2bc(2s-a)-(a^3+b^3+c^3)\Big)\Bigg]-2Rr.$$

$$GV^2=\frac{1}{18Fr}\cdot\Bigg[s\Big(4(a^2b^2+a^2c^2+b^2c^2)-(a^4+b^4+c^4)\Big)-\Big(4s(a^2b^2+a^2c^2+b^2c^2)-2abc(ab+ac+bc)-(a^5+b^5+c^5)\Big)+$$
$$-r(r+2R)\Big(4s(ab+ac+bc)-6abc-(a^3+b^3+c^3)\Big)\Bigg]-2Rr.$$

$$GV^2=\frac{1}{18Fr}\cdot\Bigg[\Big((a^5+b^5+c^5)-s(a^4+b^4+c^4)+2abc(ab+ac+bc)\Big)-r(r+2R)\Big(4s(ab+ac+bc)-6abc-(a^3+b^3+c^3)\Big)\Bigg]-2Rr.$$

Using the expressions (1.1), (1.3), (1.4), (1.5) and $(a)$, we obtain

$$GV^2=\frac{1}{18r^2s}\cdot\Bigg[\Big((2s^5-20r^2s^3+10r^4s+60Rr^3s-20Rrs^3+80R^2r^2s)-s(2s^4+2r^4-12r^2s^2-16Rrs^2+32R^2r^2+16Rr^3\Big)+$$
$$+8Rrs(s^2+r^2+4Rr)\Big)-r(r+2R)\Big(4s(s^2+r^2+4Rr)-24Rrs-(2s^3-6r^2s-12Rrs)\Big)\Bigg]-2Rr.$$

After simplifying a few steps we obtain

$$GV^2 = \frac{1}{9}(-5s^2 - r^2 - 4Rr + 36R^2).$$

**Corollary 6.3** *Be $I$ the Incenter and $O$ the Circumcenter of the triangle $ABC$ and $V$ the Bevan point, then*

$$OV^2 = IO^2 = \frac{1}{4} \cdot IV^2 = R^2 - 2Rr$$

**Proof:** In Theorem 5.5, replace $M$ by the circumcenter $O$, and consider that $OA = OB = OC = R$. We get

$$OV^2 = \frac{R^2}{2Fr} \cdot \Big[a\big[a(s-a) - r(r+2R)\big] + b\big[b(s-b) - r(r+2R)\big] + c\big[c(s-c) - r(r+2R)\big]\Big] - 2Rr.$$

$$OV^2 = \frac{R^2}{2Fr} \cdot \Big[a^2(s-a) + b^2(s-b) + c^2(s-c) - r(r+2R)(a+b+c)\Big] - 2Rr.$$

$$OV^2 = \frac{R^2}{2Fr} \cdot \Big[s(a^2+b^2+c^2) - (a^3+b^3+c^3) - r(r+2R)(a+b+c)\Big] - 2Rr.$$

$$OV^2 = \frac{R^2}{2Fr} \cdot \Big[s(2s^2 - 2r^2 - 8Rr) - (2s^3 - 6r^2s - 12Rrs) - 2rs(r+2R)\Big] - 2Rr.$$

Further simplification gives

$$OV^2 = IO^2 = R^2 - 2Rr$$

Hence

$$IV^2 = 4R^2 - 8Rr$$

**Corollary 6.4** *Be $H$ the orthocenter, $S_p$ the Spieker center of the triangle $ABC$ and $V$, the Bevan point, then*

$$HV^2 = 4 \cdot S_pV^2 = 4 \cdot HS_p^2 = (r+4R)^2 - 3s^2$$

**Proof:** In Theorem 5.5, replace $M$ by the orthocenter $H$ and consider that $HA = 2R\cos(A)$, $HB = 2R\cos(B)$, $HC = 2R\cos(C)$. We get

$$HV^2 = \frac{1}{2Fr} \cdot \Big[a\big[a(s-a) - r(r+2R)\big]MA^2 + b\big[b(s-b) - r(r+2R)\big]MB^2 + c\big[c(s-c) - r(r+2R)\big]MC^2\Big] - 2Rr.$$

$$HV^2 = \frac{4R^2}{2Fr}\cdot\Big[a\big[a(s-a)-r(r+2R)\big]\cos^2(A)+b\big[b(s-b)-r(r+2R)\big]\cos^2(B)+c\big[c(s-c)-r(r+2R)\big]\cos^2(C)\Big]-2Rr.$$

$$HV^2 = \frac{2R^2}{Fr}\cdot\Big[a\big[a(s-a)-r(r+2R)\big](1-\sin^2(A))+b\big[b(s-b)-r(r+2R)\big](1-\sin^2(B))+c\big[c(s-c)-r(r+2R)\big](1-\sin^2(C))\Big]-2Rr.$$

Using the Law of Sine, we have

$\sin(A)=\dfrac{a}{2R}$, $\sin(B)=\dfrac{b}{2R}$ and $\sin(C)=\dfrac{c}{2R}$, then

$$\begin{aligned} HV^2 &= \frac{2R^2}{Fr}\cdot\Big[a\big[a(s-a)-r(r+2R)\big](1-\frac{a^2}{4R^2})+b\big[b(s-b)-r(r+2R)\big](1-\frac{b^2}{4R^2})+ \\ &+c\big[c(s-c)-r(r+2R)\big](1-\frac{c^2}{4R^2})\Big]-2Rr. \end{aligned}$$

$$\begin{aligned} HV^2 &= \frac{2R^2}{Fr}\cdot\Big[a^2(s-a)+b^2(s-b)+c^2(s-c)-\frac{1}{4R^2}\Big(a^4(s-a)+b^4(s-b)+c^4(s-c)\Big)- \\ &-r(r+2R)\Big((a+b+c)-\frac{1}{4R^2}(a^3+b^3+c^3)\Big)\Big]-2Rr. \end{aligned}$$

$$\begin{aligned} HV^2 &= \frac{2R^2}{Fr}\cdot\Big[s(a^2+b^2+c^2)-(a^3+b^3+c^3)-\frac{1}{4R^2}\Big(s(a^4+b^4+c^4)-(a^5+b^5+c^5)\Big)- \\ &-r(r+2R)\Big((a+b+c)-\frac{1}{4R^2}(a^3+b^3+c^3)\Big)\Big]-2Rr. \end{aligned}$$

Using the expressions (1.2), (1.3), (1.4) and (1.5) we get

$$\begin{aligned} HV^2 &= \frac{2R^2}{Fr}\cdot\Big[s(2s^2-2r^2-8Rr)-(2s^3-6r^2s-12Rrs)-\frac{1}{4R^2}\Big(s(2s^4+2r^4-12r^2s^2-16Rrs^2+32R^2r^2+16Rr^3)- \\ &-(2s^5-20r^2s^3+10r^4s+60Rr^3s-20Rrs^3+80R^2r^2s)\Big)-r(r+2R)\Big(2s-\frac{1}{4R^2}(2s^3-6r^2s-12Rrs)\Big)\Big]-2Rr. \end{aligned}$$

$$\begin{aligned} HV^2 &= \frac{2R^2}{Fr}\cdot\Big[s(2s^2-2r^2-8Rr)-(2s^3-6r^2s-12Rrs)-\frac{1}{4R^2}\Big(s(2s^4+2r^4-12r^2s^2-16Rrs^2+32R^2r^2+16Rr^3)- \\ &-(2s^5-20r^2s^3+10r^4s+60Rr^3s-20Rrs^3+80R^2r^2s)\Big)-r(r+2R)\Big(2s-\frac{1}{4R^2}(2s^3-6r^2s-12Rrs)\Big)\Big]-2Rr. \end{aligned}$$

$$HV^2 = \frac{2R^2}{Fr}\cdot\Big[4Rrs+4r^2s-\frac{1}{4R^2}\Big(8r^2s^3-8r^4s-48R^2r^2s+4Rrs^3-44Rr^3s\Big)-\frac{r(r+2R)}{4R^2}\Big(8R^2s-2s^3+6r^2s+12Rrs\Big)\Big]-2Rr.$$

$$HV^2 = \frac{1}{r^2s}\cdot(r^4s-3r^2s^3+16R^2r^2s+10Rr^3s)-2Rr.$$

Hence we get

$$HV^2 = (r + 4R)^2 - 3s^2$$

$$S_pV^2 = HS_p^2 = \frac{1}{4} \cdot \left[(r + 4R)^2 - 3s^2\right]$$

**Corollary 6.5** *Be $N_a$ the Nagel point, $L$ the De Longchamps point of the triangle $ABC$ and $V$, the Bevan point, then*

$$N_aL^2 = 4 \cdot N_aV^2 = 4 \cdot VL^2 = -4s^2 + 12r^2 + 16R^2 + 16Rr$$

**Proof:**

We know that

$$N_aA = \frac{a}{s}\sqrt{s^2 - \frac{4\Delta^2}{a(s-a)}} \Longrightarrow N_aA^2 = a^2 - \frac{4a}{s}(s-b)(s-c).$$

Similarly $N_aB^2 = b^2 - \frac{4b}{s}(s-a)(s-c)$ and $N_aC^2 = c^2 - \frac{4c}{s}(s-a)(s-b)$.

Using the Theorem 5.5, replace $M$ by Nagel point $N_a$, then

$$N_aV^2 = \frac{1}{2Fr} \cdot \Big[a\big[a(s-a) - r(r+2R)\big]N_aA^2 + b\big[b(s-b) - r(r+2R)\big]N_bB^2 + c\big[c(s-c) - r(r+2R)\big]N_cC^2\Big] - 2Rr.$$

Now

$$\begin{aligned} N_aV^2 &= \frac{1}{2Fr} \cdot \Big[a\big[a(s-a) - r(r+2R)\big]\big[a^2 - \frac{4a}{s}(s-b)(s-c)\big] + b\big[b(s-b) - r(r+2R)\big]\big[b^2 - \frac{4b}{s}(s-a)(s-c)\big] + \\ &+c\big[c(s-c) - r(r+2R)\big]\big[c^2 - \frac{4c}{s}(s-a)(s-b)\big]\Big] - 2Rr. \end{aligned}$$

$$\begin{aligned} N_aV^2 &= \frac{1}{2Fr} \cdot \Big[a^2(s-a)\big[a^2 - \frac{4a}{s}(s-b)(s-c)\big] + b^2(s-b)\big[b^2 - \frac{4b}{s}(s-a)(s-c)\big] + c^2(s-c)\big[c^2 - \frac{4c}{s}(s-a)(s-b)\big] - \\ &-r(r+2R)\big[a^3 - \frac{4a^2}{s}(s-b)(s-c) + b^3 - \frac{4b^2}{s}(s-a)(s-c) + c^3 - \frac{4c^2}{s}(s-a)(s-b)\Big] - 2Rr. \end{aligned}$$

$$\begin{aligned} N_aV^2 &= \frac{1}{2Fr} \cdot \Big[a^4(s-a) - \frac{4a^3}{s}(s-a)(s-b)(s-c) + b^4(s-b) - \frac{4b^3}{s}(s-a)(s-b)(s-c) + c^4(s-c) - \\ &-\frac{4c^3}{s}(s-a)(s-b)(s-c) - r(r+2R)\Big[a^3 - \frac{4a^2}{s}(s-b)(s-c) + b^3 - \frac{4b^2}{s}(s-a)(s-c) + c^3 - \\ &-\frac{4c^2}{s}(s-a)(s-b)\Big]\Big] - 2Rr. \end{aligned}$$

$$N_aV^2 = \frac{1}{2Fr}\cdot\Big[a^4(s-a)-\frac{4a^3}{s}(s-a)(s-b)(s-c)+b^4(s-b)-\frac{4b^3}{s}(s-a)(s-b)(s-c)+c^4(s-c)-$$
$$-\frac{4c^3}{s}(s-a)(s-b)(s-c)-r(r+2R)\Big[a^3-\frac{4a^2}{s}(s-b)(s-c)+b^3-\frac{4b^2}{s}(s-a)(s-c)+c^3-$$
$$-\frac{4c^2}{s}(s-a)(s-b)\Big]\Big]-2Rr.$$

$$N_aV^2 = \frac{1}{2Fr}\cdot\Big[-(a^5+b^5+c^5)+s(a^4+b^4+c^4)-\frac{4F^2}{s^2}(a^3+b^3+c^3)-r(r+2R)\Big[(a^3+b^3+c^3)-$$
$$-\frac{4}{s}\Big[a^2(s-b)(s-c)+b^3(s-a)(s-c)+c^3(s-a)(s-b)\Big]\Big]-2Rr.$$

$$N_aV^2 = \frac{1}{2Fr}\cdot\Big[-(a^5+b^5+c^5)+s(a^4+b^4+c^4)-4r^2(a^3+b^3+c^3)-r(r+2R)\Big[(a^3+b^3+c^3)-$$
$$-\frac{4}{s}\Big[a^2(s-b)(s-c)+b^3(s-a)(s-c)+c^3(s-a)(s-b)\Big]\Big]-2Rr.$$

Using the expressions (1.2), (1.3), (1.4) and (1.5), then

$$N_aV^2 = \frac{1}{2Fr}\cdot\Big[-(a^5+b^5+c^5)+s(a^4+b^4+c^4)-4r^2(a^3+b^3+c^3)-r(r+2R)\Big[-3(a^3+b^3+c^3)+4s(a^2+b^2+c^2)-\frac{4}{s}abc(a+b+c)\Big]\Big]-2Rr.$$

$$N_aV^2 = \frac{1}{2Fr}\cdot\Big[-(2s^5-20r^2s^3+10r^4s+60Rr^3s-20Rrs^3+80R^2r^2s)+s(2s^4+2r^4-12r^2s^2-16Rrs^2+32R^2r^2+16Rr^3)-$$
$$-4r^2(2s^3-6r^2s-12Rrs)-r(r+2R)\Big[-3(2s^3-6r^2s-12Rrs)+4s(2s^2-2r^2-8Rr)-\frac{4}{s}(4Rrs)(2s)\Big]\Big]-2Rr.$$

$$N_aV^2 = \frac{1}{2Fr}\cdot\Big[(16r^4s+4Rr^3s+4Rrs^3-48R^2r^2s)-r(r+2R)(2s^3+10r^2s-28Rrs)\Big]-2Rr.$$

$$N_aV^2 = \frac{1}{2r^2s}\cdot(-2r^2s^3+6r^4s+12Rr^3s+8R^2r^2s)-2Rr.$$

Therefore,

$$N_aV^2 = VL^2 = -s^2+3r^2+4R^2+4Rr$$

$$N_aL^2 = 4\cdot(-s^2+3r^2+4R^2+4Rr)$$

# 7 Conclusion

In this article we find a metric relationship for the Bevan point. With this relationship we can find the distance between the Bevan point and other notable centers of the triangle. The proofs presented here only require basic knowledgeof trigonometry, barycentric coordinates and no advanced knowledge of synthetic geometry and its manipulation and application.

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