ABOUT FEW OUTSTANDING LIMITS

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Problem 1. Find:

$$\Omega = \lim_{n\to\infty}\left(1+\Big(\sum_{k=1}^{n-1}\frac{k}{n}\sin\frac{k\pi}{n}\Big)^{-1}\right)^{n}$$

Solution.

$$\forall n \geq 2, n \in \mathbf{N}: \sum_{k=1}^{n-1}\frac{k}{n}\sin\frac{k\pi}{n} = \frac{1}{2}\cot\frac{\pi}{2n}$$

Let be $z = \cos\frac{\pi}{n} + i\cdot\sin\frac{\pi}{n}$, then

$$\sum_{k=1}^{n-1} k\sin\frac{k\pi}{n} = Im(z+2z^2+3z^3+...+(n-1)z^{n-1}),\, z^n = -1$$

Then:

$$z+2z^2+3z^3+...+(n-1)z^{n-1} = \frac{(n-1)z^{n+1}-nz^n+z}{(z-1)^2} = \frac{(1-n)z+n+z}{(z-1)^2} =$$

$$= \frac{n-(n-2)z}{1-2\sin^2\frac{\pi}{2n}+2i\sin\frac{\pi}{2n}\cos\frac{\pi}{2n}-1} = \frac{n-(n-2)z}{-4\sin^2\frac{\pi}{2n}\Big(\cos\frac{\pi}{2n}+i\sin\frac{\pi}{2n}\Big)} =$$

$$= \frac{n-2}{4\sin^2\frac{\pi}{2n}} - \frac{n}{4\sin^2\frac{\pi}{2n}}\Big(\cos\frac{\pi}{n} - i\sin\frac{\pi}{n}\Big)$$

Hence,

$$\sum_{k=1}^{n-1} k\sin\frac{k\pi}{n} = Im\Big(\sum_{k=1}^{n-1} kz^k\Big) = \frac{n\sin\frac{\pi}{n}}{4\sin^2\frac{\pi}{n}} = \frac{n}{2}\cot\frac{\pi}{2n}$$

It follows that:

$$\sum_{k=1}^{n-1}\frac{k}{n}\sin\frac{k\pi}{n} = \frac{1}{2}\cot\frac{\pi}{2n}$$

$$\lim_{n\to\infty}\log\left(1+\Big(\sum_{k=1}^{n-1}\frac{k}{n}\sin\frac{k\pi}{n}\Big)^{-1}\right)^{n} = \lim_{n\to\infty} n\log\Big(1+2\tan\frac{\pi}{2n}\Big) =$$

$$= \lim_{n\to\infty}\frac{\log\Big(1+2\tan\frac{\pi}{2n}\Big)}{2\tan\frac{\pi}{2n}}\cdot\frac{2\tan\frac{\pi}{2n}}{\frac{\pi}{2n}}\cdot\frac{\pi}{2n}\cdot n = \pi$$

Therefore,

$$\Omega = \lim_{n\to\infty}\left(1+\Big(\sum_{k=1}^{n-1}\sin\frac{k\pi}{n}\Big)^{-1}\right)^{n} = e^{\pi}$$

Problem 2. Find:

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\sum_{k=1}^{n} \cos\frac{(n-1)k\pi}{n} \cdot \cos^{n-1}\frac{k\pi}{n}}$$

Solution. First, we prove that:

$$\sum_{k=1}^{n} \cos\frac{(n-1)k\pi}{n} \cdot \cos^{n-1}\frac{k\pi}{n} = \frac{n}{2^{n-1}}, \forall n \geq 3, n \in \mathbf{N}$$

$$(1+z)^m = \sum_{l=0}^{m} \binom{m}{1} z^l, m > 0, m \in \mathbf{N}; (1)$$

Let

$$z = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, k \in \{1, 2, ..., n\}$$

Hence,

$$1 + z = 1 + \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} = 2\cos\frac{k\pi}{n}\left(\cos\frac{k\pi}{n} + i\sin\frac{k\pi}{n}\right) =$$

$$2^m \cos^m\frac{k\pi}{n}\left(\cos\frac{mk\pi}{n} + i\sin\frac{mk\pi}{n}\right) = \sum_{l=0}^{m}\binom{m}{l}\left(\cos\frac{2lk\pi}{n} + i\sin\frac{2lk\pi}{n}\right)$$

$$2^m \cos^m\frac{k\pi}{n}\cos\frac{mk\pi}{n} = \sum_{l=0}^{m}\binom{m}{l}\cos\frac{2lk\pi}{n}, k \in \{1, 2, ..., n\}$$

$$2^m \sum_{k=1}^{n}\cos^m\frac{k\pi}{n}\cos\frac{mk\pi}{n} = \sum_{l=0}^{m}\binom{m}{l}\sum_{k=1}^{n}\cos\frac{2ik\pi}{n} =$$

$$\binom{m}{0} n + \sum_{i=1}^{n}\cos\frac{2lk\pi}{n}; (2)$$

Now,

$$\sum_{k=1}^{n} a^{k-1}\cos(k\theta) = \frac{a^{n+1}\cos(n\theta) - a^n\cos(n+1)\theta + \cos\theta - a}{a^2 - 2a\cos\theta + 1}; a = 1, \theta = \frac{2l\pi}{n}$$

Then,

$$\sum_{k=1}^{n}\cos\frac{2lk\pi}{n} = \frac{\cos(2l\pi) - \cos\frac{(n+1)2l\pi}{n} + \cos\frac{2l\pi}{n} - 1}{2 - 2\cos\frac{2l\pi}{n}}; l \in \{1, 2, ..., m\}, m < n; (3)$$

From (2),(3) it follows that:

$$2^m \sum_{k=1}^{n}\cos^m\frac{k\pi}{n}\cos\frac{mk\pi}{n} = n\binom{m}{0}$$

Hence,

$$\sum_{k=1}^{n}\cos^m\frac{k\pi}{n}\cos\frac{mk\pi}{n} = \frac{n}{2^m}$$

For $m = n-1$, it follows that:

$$\sum_{k=1}^{n}\cos^{n-1}\frac{k\pi}{n}\cos\frac{(n-1)k\pi}{n} = \frac{n}{2^{n-1}}$$

Therefore,from C-D'A we get:

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\sum_{k=1}^{n} \cos\frac{(n-1)k\pi}{n} \cdot \cos^{n-1}\frac{k\pi}{n}} = \lim_{n\to\infty} \sqrt[n]{\frac{n}{2^{n-1}}} =$$

$$= \lim_{n\to\infty} \frac{n+1}{2^n} \cdot \frac{2^{n-1}}{n} = \frac{1}{2}$$

Problem 3. If

$$\omega_n = \sum_{k=1}^{2n} \cot\frac{k\pi}{2n+1} \cdot \left(\sin\frac{2k\pi}{2n+1} + icos\frac{2k\pi}{2n+1}\right)$$

Find:

$$\Omega = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k^{\omega_n + 2-(k+n)}}$$

Solution. First, we prove that:

$$\sum_{k=1}^{2n} \cot\frac{k\pi}{2n+1} \cdot \left(\sin\frac{2k\pi}{2n+1} + icos\frac{2k\pi}{2n+1}\right) = 2n-1; \forall n \in \mathbf{N}$$

$$\sum_{k=1}^{2n} \cot\frac{k\pi}{2n+1} \cdot \left(\sin\frac{2k\pi}{2n+1} + icos\frac{2k\pi}{2n+1}\right) =$$

$$\sum_{k=1}^{n} \cot\frac{k\pi}{2n+1}\left(\sin\frac{2k\pi}{2n+1} + i\cos 2k\pi 2n+1\right) + \sum_{k=n+1}^{2n} \cot\frac{k\pi}{2n+1}\left(\sin\frac{2k\pi}{2n+1} + i\cos\frac{2k\pi}{2n+1}\right) =$$

$$= S_1 + S_2$$

Denote $2n+1-k = l$ how $k \in \{n+1, n+2, ..., 2n\}$ it follows that:
$l \in \{1, 2, ..., n-1, n\}$

$$S_2 = \sum_{l=1}^{n} \cot\frac{(2n+1-l)\pi}{2n+1}\left(\sin\frac{(4n+2-2l)\pi}{2n+1} + i\cos\frac{(4n+2-2l)\pi}{2n+1}\right) =$$

$$= \sum_{k=1}^{n} \cot\frac{l\pi}{2n+1}\left(\sin\frac{2l\pi}{2n+1} - i\cos\frac{2l\pi}{2n+1}\right)$$

$$S = S_1 + S_2 = 2\sum_{k=1}^{n} \cot\frac{2k\pi}{2n+1}\sin\frac{2k\pi}{2n+1} = 4\sum_{k=1}^{n} \cos^2\frac{k\pi}{2n+1} =$$

$$= 2\sum_{k=1}^{n}\left(1 + \cos\frac{2k\pi}{2n+1}\right) = 2n + 2\sum_{k=1}^{n} \cos\frac{2k\pi}{2n+1} = 2n-1$$

Now, because:

$$\cos\frac{[2n-(2k-1)]\pi}{2n+1} = -\cos\frac{2k\pi}{2n+1}, \forall k \in \{1, 2, ..., n\}$$

It follows that:

$$\sum_{k=1}^{n} \cos\frac{2k\pi}{2n+1} = 1 - 2\left[\cos\frac{(2n-1)\pi}{2n+1} + ... + \cos\frac{\pi}{2n+1}\right] = -\frac{1}{2}$$

Hence,

$$\Omega=\sum_{k=1}^{2n}\cot\frac{k\pi}{2n+1}\cdot\left(\sin\frac{2k\pi}{2n+1}+icos\frac{2k\pi}{2n+1}\right)=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^{n+1-k}};(1)$$

$$\frac{1}{k^{n+1-k}}=\frac{k^k}{k^{n+1}}=\frac{k^k}{(1+(k-1))^{n+1}}\le\left(\frac{k}{k-1}\right)^k\cdot\frac{1}{\binom{n+1}{k}}=$$

$$=\left(1+\frac{1}{k-1}\right)\left(1+\frac{1}{k-1}\right)^{k-1}\cdot\frac{1}{\binom{n+1}{k}}\le\frac{2e}{\binom{n+1}{k}}$$

Thus,

$$1\le\sum_{k=1}^{2n}\cot\frac{k\pi}{2n+1}\cdot\left(\sin\frac{2k\pi}{2n+1}+icos\frac{2k\pi}{2n+1}\right)\le 1+2e\sum_{k=1}^{n-1}\frac{1}{\binom{n+1}{k}}+\frac{1}{n}\le$$

$$\le 1+2e\cdot\frac{n-2}{\binom{n+1}{2}}+\frac{1}{n}$$

Therefore,

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^{\omega_n+2-(k+n)}}=1$$

Problem 4. Find:

$$\Omega=\lim_{n\to\infty}\left(\lim_{x\to\frac{\pi}{2n}}\left(\frac{\cot x}{2}\cdot\left(\sum_{k=1}^{n-1}\frac{k}{n}\sin\frac{k\pi}{n}\right)^{-1}\right)^{\frac{1}{\tan(2nx)}}\right)$$

Solution.

$$\forall n\in\mathbf{N}^*,n\ge 2:\sum_{k=1}^{n-1}\frac{k}{n}\sin\frac{k\pi}{n}=\frac{1}{2}\cot\frac{\pi}{2n}$$

Let be $z=\cos\frac{\pi}{n}+i\sin\frac{\pi}{n}$ then:

$$\sum_{k=1}^{n-1}k\sin\frac{k\pi}{n}=Im(z+2z^2+3z^3+...+(n-1)z^{n-1}),z^n=-1$$

Hence,

$$z+2z^2+3z^3+...+(n-1)z^{n-1}=\frac{(n-1)z^{n+1}-nz^n+z}{(z-1)^2}=\frac{(1-n)z+n+z}{(z-1)^2}=$$

$$=\frac{n-(n-2)z}{1-2\sin^2\frac{\pi}{2n}+2i\sin\frac{\pi}{2n}\cos\frac{\pi}{2n}-1}=\frac{n-(n-2)z}{-4\sin^2\frac{\pi}{2n}\left(\cos\frac{\pi}{n}+i\sin\frac{\pi}{n}\right)}=$$

$$=\frac{n-2}{4\sin^2\frac{\pi}{2n}}-\frac{n}{4\sin^2\frac{\pi}{2n}}\left(\cos\frac{\pi}{n}-i\sin\frac{\pi}{n}\right)$$

$$\sum_{k=1}^{n}k\sin\frac{k\pi}{n}=Im\left(\sum_{k=1}^{n-1}kz^k\right)=\frac{n\sin\frac{\pi}{n}}{4\sin^2\frac{\pi}{2n}}=\frac{n}{2}\cot\frac{\pi}{2n}$$

Thus,

$$\sum_{k=1}^{n-1}\frac{k}{n}\sin\frac{k}{n}=\frac{1}{2}\cot\frac{\pi}{2n}$$

It follows that:

$$\lim_{x\to\frac{\pi}{2n}}\left(\frac{\cot x}{2}\cdot\left(\sum_{k=1}^{n-1}\frac{k}{n}\sin\frac{k\pi}{n}\right)^{-1}\right)^{\frac{1}{\tan(2nx)}}=\lim_{x\to\frac{\pi}{2n}}\left(\frac{\cot x}{2}\cdot\left(\frac{1}{2}\cot\frac{\pi}{2n}\right)^{-1}\right)^{\frac{1}{\tan(2nx)}}=$$

$$=\lim_{x\to\frac{\pi}{2n}}\left(\frac{\cot x}{\cot\frac{\pi}{2n}}\right)^{\cot(2nx)}=$$

$$=\lim_{x\to\frac{\pi}{2n}}\left(1+\frac{\cot x-\cot\frac{\pi}{2n}}{\cot\frac{\pi}{2n}}\right)^{\cot(2nx)}=$$

$$=\exp\lim_{x\to\frac{\pi}{2n}}\cot(2nx)\cdot\frac{\sin\left(\frac{\pi}{2n}-x\right)}{\sin x\cdot\sin\frac{\pi}{2n}}=\exp\left(\frac{-1}{n\sin\frac{\pi}{n}}\right)$$

Therefore,

$$\Omega=\lim_{n\to\infty}\left(\lim_{x\to\frac{\pi}{2n}}\left(\frac{\cot x}{2}\cdot\left(\sum_{k=1}^{n-1}\frac{k}{n}\sin\frac{k\pi}{n}\right)^{-1}\right)^{\frac{1}{\tan(2nx)}}\right)=e^{-\frac{1}{\pi}}$$

Problem 5. $(a_n)_{n\geq 1}$ be a sequence of real numbers such that $0<a_k<\frac{n^2}{n^2+k^2-1}, n,k\in\mathbf{N}$ then prove:

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\cdot\sqrt[n^2+k^2-1]{\left(1+\frac{k^2}{n^2}\right)\left(1-\left(1-\frac{1}{n^2}+\frac{k^2}{n^2}\right)a_k\right)}\right)^{\frac{1}{n}}\leq\frac{\pi}{4}$$

Solution.

$$\frac{n^2}{n^2+k^2}=\frac{(n^2+k^2-1)a_k+n^2-(n^2+k^2-1)a_k}{n^2+k^2}\overbrace{\geq}^{AM-GM}$$

$$\geq\sqrt[n^2+k^2]{a_k^{n^2+k^2-1}\left(n^2-(n^2+k^2-1)a_k\right)}$$

Hence,

$$a_k^{n^2+k^2-1}(n^2-(n^2+k^2-1))a_k\leq\left(\frac{n^2}{n^2+k^2}\right)^{n^2+k^2-1}\cdot\frac{n^2}{n^2+k^2}$$

$$\frac{1}{n}\sum_{k=1}^{n}a_k\cdot\sqrt[n^2+k^2-1]{\left(1+\frac{k^2}{n^2}\right)\left(1-\left(1-\frac{1}{n^2}+\frac{k^2}{n^2}\right)a_k\right)}\leq\frac{1}{n}\cdot\sum_{k=1}^{n}\frac{1}{1+\left(\frac{k}{n}\right)^2}$$

Therefore,

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\cdot\sqrt[n^2+k^2-1]{\left(1+\frac{k^2}{n^2}\right)\left(1-\left(1-\frac{1}{n^2}+\frac{k^2}{n^2}\right)a_k\right)}\right)^{\frac{1}{n}}\leq$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+\left(\frac{k}{n}\right)^2}=\int_0^1\frac{1}{1+x^2}dx=\frac{\pi}{4}$$

Problem 6. Find:

$$\Omega=\lim_{x\to\infty}\left(\sum_{k=1}^{n}\int_{k-1}^{k}tan^{-1}\left(\frac{n(x-k)}{kx+n^2}\right)\right)$$

Solution.

$$\sum_{k=1}^{n}\int_{k-1}^{k} tan^{-1}\Big(\frac{n(x-k)}{kx+n^2}\Big) = \sum_{k=1}^{n}\int_{k-1}^{k} tan^{-1}\Big(\frac{x-k}{\frac{k}{n}x+n}\Big)dx \overbrace{=}^{x\to nx}$$

$$= n\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}} tan^{-1}\Big(\frac{x-\frac{k}{n}}{1+\frac{k}{n}x}\Big) = n\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(tan^{-1}x - tan^{-1}\left(\frac{k}{n}\right)\right)dx =$$

$$= n\bigg(\int_0^1 tan^{-1}x\,dx - \frac{1}{n}\sum_{k=1}^{n} tan^{-1}\left(\frac{k}{n}\right)\bigg)$$

Let $f:\mathbf{R}\to\mathbf{R}, f(x)=tan^{-1}x$ continuous function with $f''(x)<0, \forall x\in\mathbf{R}$ then $f'-$ decreasing and let $\frac{k-1}{n}\le x<\frac{k}{n}$. From M.V.T., $\exists c\in\left(\frac{k-1}{n};\frac{k}{n}\right)$ such that: $\frac{f(x)-f(\frac{k}{n})}{x-\frac{k}{n}} = f'(c)$

$$f'\Big(\frac{k-1}{n}\Big) > \frac{f(x)-f(\frac{k}{n})}{x-\frac{k}{n}} > f'\left(\frac{k}{n}\right)\Big|\cdot\left(x-\frac{k}{n}\right)<0$$

$$\left(x-\frac{k}{n}\right)f'\left(\frac{k-1}{n}\right) < f(x)-f\left(\frac{k}{n}\right) < \left(x-\frac{k}{n}\right)f'\left(\frac{k}{n}\right)$$

$$-\frac{1}{2n^2}\cdot f'\left(\frac{k-1}{n}\right) \le \int_{\frac{k-1}{k}}^{\frac{k}{n}}\left(tan^{-1}x - tan^{-1}\left(\frac{k}{n}\right)\right)dx \le -\frac{1}{2n^2}\cdot f'\left(\frac{k}{n}\right); n\ge 2, k\in\overline{1,n}$$

Hence,

$$-\frac{1}{2n^2}\cdot\sum_{k=1}^{n} f'\left(\frac{k-1}{n}\right) \le n\left(\int_0^1 tan^{-1}x - \frac{1}{n}\sum_{k=1}^{n} tan^{-1}\left(\frac{k}{n}\right)\right) \le -\frac{1}{2n^2}\cdot\sum_{k=1}^{n} f'\left(\frac{k}{n}\right); n\ge 2$$

Because:

$$\lim_{n\to\infty}\frac{1}{n}\cdot\sum_{k=1}^{n} f'\left(\frac{k-1}{n}\right) = \lim_{n\to\infty}\frac{1}{n}\cdot\sum_{k=1}^{n} f'\left(\frac{k}{n}\right) = \int_0^1 f'(x)\,dx = \frac{\pi}{4}$$

It follows that:

$$\Omega = \lim_{x\to\infty}\left(\sum_{k=1}^{n}\int_{k-1}^{k} tan^{-1}\Big(\frac{n(x-k)}{kx+n^2}\Big)\right) = \lim_{n\to\infty} n\left(\int_0^1 tan^{-1}x - \frac{1}{n}\cdot\sum_{k=1}^{n} tan^{-1}\left(\frac{k}{n}\right)\right) = -\frac{\pi}{8}$$

Problem 7: Find:

$$\Omega = \lim_{n\to\infty}\sum_{1\le i<j\le n}\frac{(n^2+ni+i^2)(n^2+nj+j^2)}{(n^2+i^2)(n^2+j^2)}\cdot e^{tan^{-1}\left(\frac{n(i+j)}{n^2-ij}\right)}$$

Solution.

$$\sum_{1\le i<j\le n}\frac{(n^2+ni+i^2)(n^2+nj+j^2)}{(n^2+i^2)(n^2+j^2)}\cdot e^{tan^{-1}\left(\frac{n(i+j)}{n^2-ij}\right)} =$$

$$= \sum_{1\le i<j\le n}\left(1+\frac{ni}{n^2+i^2}\right)\left(1+\frac{nj}{n^2+j^2}\right)\cdot e^{tan^{-1}\left(\frac{i}{n}\right)+tan^{-1}\left(\frac{j}{n}\right)} =$$

$$= \sum_{1\le i<j\le n}\left(1+\frac{\frac{i}{n}}{1+\left(\frac{i}{n}\right)^2}\right)e^{tan^{-1}\left(\frac{i}{n}\right)}\cdot\left(1+\frac{\frac{j}{n}}{1+\left(\frac{j}{n}\right)^2}\right)e^{tan^{-1}\left(\frac{j}{n}\right)} =$$

$$= \sum_{1\le i<j\le n} f\left(\frac{i}{n}\right) f\left(\frac{j}{n}\right) \text{ where } f:\mathbf{R}\to\mathbf{R}, f(x) = \left(1+\frac{t}{t^2+1}\right) e^{tan^{-1}t}$$

$$\sum_{1\le i<j\le n} \frac{f\left(\frac{i}{n}\right) f\left(\frac{j}{n}\right)}{\sqrt{n^4+2n^2}} \le \sum_{1\le i<j\le n} \frac{f\left(\frac{i}{n}\right) f\left(\frac{j}{n}\right)}{\sqrt{n^4+i^2+j^2}} \le \sum_{1\le i<j\le n} \frac{f\left(\frac{i}{n}\right) f\left(\frac{j}{n}\right)}{\sqrt{n^4+n^2}}$$

$$\lim_{n\to\infty} \sum_{1\le i<j\le n} \frac{f\left(\frac{i}{n}\right) f\left(\frac{j}{n}\right)}{\sqrt{n^4+2n^2}} = \lim_{n\to\infty} \frac{1}{n^2} \sum_{1\le i<j\le n} \frac{f\left(\frac{i}{n}\right) f\left(\frac{j}{n}\right)}{\sqrt{1+\frac{2}{n^2}}} = \frac{1}{2}\left(\int_0^1 f(x)dx\right)^2$$

$$\lim_{n\to\infty} \sum_{1\le i<j\le n} \frac{f\left(\frac{i}{n}\right) f\left(\frac{j}{n}\right)}{\sqrt{n^4+n^2}} = \lim_{n\to\infty} \frac{1}{n^2} \sum_{1\le i<j\le n} \frac{f\left(\frac{i}{n}\right) f\left(\frac{j}{n}\right)}{\sqrt{1+\frac{1}{n^2}}} = \frac{1}{2}\left(\int_0^1 f(x)dx\right)^2$$

Hence,

$$\lim_{n\to\infty} \sum_{1\le i<j\le n} \frac{f\left(\frac{i}{n}\right) f\left(\frac{j}{n}\right)}{\sqrt{n^4+i^2+j^2}} = \frac{1}{2}\left(\int_0^1 f(x)dx\right)^2$$

$$\int_0^1 f(x)dx = \int_0^1 \frac{x^2+x+1}{x^2+1}\cdot e^{tan^{-1}x}dx = \int_0^1 \left(1+\frac{x}{1+x^2}\right)\cdot e^{tan^{-1}x}dx =$$

$$= \int_0^1 e^{tan^{-1}x}dx + \int_0^1 \frac{x}{1+x^2}\cdot e^{tan^{-1}x}dx = e^{\frac{\pi}{4}}$$

Therefore,

$$\Omega = \lim_{n\to\infty} \sum_{1\le i<j\le n} \frac{(n^2+ni+i^2)(n^2+nj+j^2)}{(n^2+i^2)(n^2+j^2)} \cdot e^{tan^{-1}\left(\frac{n(i+j)}{n^2-ij}\right)} = \frac{1}{2}e^{\frac{\pi}{4}}$$

Problem 8:

$$\omega_n = (2n+1)\left(\sum_{k=0}^{2n} \tan\left(x+\frac{k\pi}{2n+1}\right)\right)^{-1}$$

Find:

$$\Omega = \lim_{n\to\infty}\left(\lim_{x\to\frac{\pi}{2n+1}} \left(\frac{\cot x}{\cot\frac{\pi}{2n+1}}\right)^{\omega_n}\right)$$

Solution. Roots of the equation: $z^{2n}-2z^n\cos(nx)+1=0$ are $z_{k+n}=\cos\left(x+\frac{2k\pi}{n}\right)+i\sin\left(x+\frac{2k\pi}{n}\right)$ and $z_{k+n}=\overline{z_k}=\cos\left(x+\frac{2k\pi}{n}\right)-i\sin\left(x+\frac{2k\pi}{n}\right), k=\overline{1,n-1}$

$$\prod_{k=0}^{n-1}[(z-z_k)(z-\overline{z_k})] = z^{2n}-2z^n\cos(nx)+1$$

For $z=\pm 1$, it follows that:

$$\prod_{k=0}^{n-1}\left[1-2\cos\left(x+\frac{2k\pi}{n}\right)+1\right] = 2(1-\cos(nx))$$

$$\prod_{k=0}^{n-1}\left[1+2\cos\left(x+\frac{2k\pi}{n}\right)+1\right] = 2(1+\cos(nx))$$

With substitutions: $\frac{x}{2}\to x$ and $n\to 2n+1$, we get:

$$\prod_{k=0}^{2n}\sin\left(x+\frac{k\pi}{2n+1}\right) = \frac{\sin(2n+1)x}{2^{2n}}$$

$$\prod_{k=0}^{2n} \cos\left(x + \frac{k\pi}{2n+1}\right) = (-1)^n \frac{\cos(2n+1)x}{2^{2n}}$$

Logarithmating and deriving, it follows that:

$$\sum_{k=0}^{2n} \frac{\sin\left(x + \frac{k\pi}{2n+1}\right)}{\cos\left(x + \frac{k\pi}{2n+1}\right)} = (2n+1)\tan(2n+1)x$$

Hence,

$$\omega_n = (2n+1)\left(\sum_{k=0}^{2n} \tan\left(x + \frac{k\pi}{2n+1}\right)\right)^{-1} = \cot(2n+1)x$$

$$= \lim_{x \to \frac{\pi}{2n+1}} \left(\frac{\cot x}{\cot \frac{\pi}{2n+1}}\right)^{\omega_n} = \lim_{x \to \frac{\pi}{2n+1}} \left(\frac{\cot x}{\cot \frac{\pi}{2n+1}}\right)^{\cot(2n+1)x} =$$

$$= \lim_{x \to \frac{\pi}{2n+1}} \left(1 + \frac{\cot x - \cot \frac{\pi}{2n+1}}{\cot \frac{\pi}{2n+1}}\right)^{\cot(2n+1)x} = \exp\left\{\lim_{x \to \frac{\pi}{2n+1}} \frac{\cot(2n+1)x}{\cot \frac{\pi}{2n+1}}\left(\cot x - \cot \frac{\pi}{2n+1}\right)\right\} =$$

$$= \exp\left\{\lim_{x \to \frac{\pi}{2n+1}} \frac{\cos(2n+1)x}{\sin x \cdot \cos \frac{\pi}{2n+1}} \cdot \frac{\sin\left(\frac{\pi}{2n+1} - x\right)}{\sin(2n+1)x}\right\} =$$

$$= e^{-\frac{2}{(2n+1)\sin \frac{2\pi}{2n+1}}}$$

Therefore,

$$\Omega = \lim_{n \to \infty}\left(\lim_{x \to \frac{\pi}{2n+1}} \left(\frac{\cot x}{\cot \frac{\pi}{2n+1}}\right)^{\omega_n}\right) = \exp\left\{\lim_{n \to \infty} \frac{2}{\frac{\sin \frac{2\pi}{2n+1}}{\frac{2\pi}{2n+1}} \cdot 2\pi}\right\} = e^{-\pi}$$

Problem 9: For $a, b, p, q \in \mathbf{N}$ such that $p(q-b) = a+1$. Find:

$$\Omega = \lim_{m \to \infty} \frac{1}{m}\left(\lim_{n \to \infty} \sum_{p=1}^{2^m} \sum_{i=1}^{n} i^a \sin^p\left(\frac{i^b}{n^q}\right)\right)$$

Solution.

$$a_n = \sum_{i=1}^{n} i^a \sin^p\left(\frac{i^b}{n^q}\right) = \sum_{i=1}^{n} \left(\frac{\sin\left(\frac{i^b}{n^q}\right)}{\frac{i^b}{n^q}}\right)^p \cdot \frac{i^{a+bp}}{n^{pq}} = \sum_{i=1}^{n} \left(\frac{\sin\left(\frac{i^b}{n^q}\right)}{\frac{i^b}{n^q}}\right)^p \cdot \frac{i^{a+bp}}{n^{a+bp+1}}$$

$\forall n \in \mathbf{N}, \exists \xi_n > 0$ such that:

$$1 - \xi_n \leq \left(\frac{\sin\left(\frac{i^b}{n^q}\right)}{\frac{i^b}{n^q}}\right)^p \leq 1 + \xi_n$$

$$(1-\xi_n)\sum_{i=1}^{n} \frac{i^{a+bp}}{n^{a+bp+1}} \leq a_n \leq (1+\xi_n)\sum_{i=1}^{n} \frac{i^{a+bp}}{n^{a+bp+1}}$$

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{i^{a+bp}}{n^{a+bp+1}} = \lim_{n \to \infty} \frac{1}{n}\sum_{i=1}^{n} \left(\frac{i}{n}\right)^{a+bp} = \int_0^1 x^{a+bp}dx = \frac{1}{a+bp+1}$$

Hence,

$$\lim_{n \to \infty} \sum_{i=1}^{n} i^a \sin^p\left(\frac{i^b}{n^q}\right) = \frac{1}{a+bp+1}$$

It follows that,

$$\sum_{p=1}^{2^m}\sum_{i=1}^{n} i^a \sin^p\left(\frac{i^b}{n^q}\right) = \sum_{p=1}^{2^m}\frac{1}{a+bp+1}$$

$$0 \le \frac{1}{m}\cdot\sum_{p=1}^{2^m}\frac{1}{a+bp+1} \overbrace{\le}^{AM-GM} \frac{1}{m}\cdot\sum_{p=1}^{2^m}\frac{1}{3\sqrt[3]{abp}} = \frac{1}{3\sqrt[3]{ab}}\sum_{p=1}^{2^m}\frac{1}{m\sqrt[3]{p}} = \frac{1}{3\sqrt[3]{ab}}\cdot\frac{\sum_{p=1}^{2^m}\frac{1}{\sqrt[3]{p}}}{m}$$

Now, we have:

$$\lim_{m\to\infty}\frac{1}{m}\cdot\sum_{p=1}^{2^m}\frac{1}{a+bp+1} = \lim_{m\to\infty}\frac{1}{3\sqrt[3]{ab}}\cdot\frac{\sum_{p=1}^{2^m}\frac{1}{\sqrt[3]{p}}}{m} \overbrace{=}^{L.C-S} \frac{1}{3\sqrt[3]{ab}}\cdot\lim_{m\to\infty}\frac{1}{\sqrt[3]{2^m}} = 0$$

Therefore,

$$\Omega = \lim_{m\to\infty}\frac{1}{m}\left(\lim_{n\to\infty}\sum_{p=1}^{2^m}\sum_{i=1}^{n} i^a \sin^p\left(\frac{i^b}{n^q}\right)\right) = 0$$

Problem 10.

$$(x_n)_{n\ge1}, (y_n)_{n\ge1}; x_n = \sum_{k=1}^{n}\tan^{-1}\left(1+\frac{1}{k}\right)-\frac{n\pi}{4}; y_n = \sum_{k=1}^{n}\frac{1}{\cot^{-1}(2k+1)}; H_n = 1+\frac{1}{2}+\ldots+\frac{1}{n}.$$

Find:

$$\Omega = \lim_{n\to\infty}\frac{\sum_{k=1}^{n}\Big(H_1\cdot H_2\cdot\ldots\cdot H_n\Big)^{\frac{1}{k}}}{\sum_{m=1}^{n}\left(\frac{x_m^2}{\pi}+\frac{\pi\cdot y_m^2}{4}\right)}$$

Solution. From Hardy-Carleman, we have:

$$\sum_{k=1}^{n}\Big(H_1\cdot H_2\cdot\ldots\cdot H_n\Big)^{\frac{1}{k}} < e\cdot\sum_{k=1}^{n}H_k; (1)$$

$$tan^{-1}\left(1+\frac{1}{k}\right)-\frac{\pi}{4} = \cot^{-1}(2k+1) \Rightarrow x_m = \sum_{k=1}^{m}\tan^{-1}\left(1+\frac{1}{k}\right)-\frac{m\pi}{4} = \sum_{k=1}^{m}\cot^{-1}(2k+1)$$

$$\left(\sum_{k=1}^{m}\tan^{-1}\left(1+\frac{1}{k}\right)-\frac{m\pi}{4}\right)\cdot\left(\sum_{k=1}^{m}\frac{1}{\cot^{-1}(2k+1)}\right) =$$

$$= \left(\sum_{k=1}^{m}\cot^{-1}(2k+1)\right)\cdot\left(\sum_{k=1}^{m}\frac{1}{\cot^{-1}(2k+1)}\right) \overbrace{\ge}^{C.B.S.} m^2 \Leftrightarrow$$

$$x_m\cdot y_m \ge m^2; (2)$$

$$\sum_{m=1}^{n}\left(\frac{x_m^2}{\pi}+\frac{\pi\cdot y_m^2}{4}\right) \overbrace{\ge}^{AM-GM} \sum_{m=1}^{n}x_m\cdot y_m \overbrace{\ge}^{(2)} \sum_{m=1}^{n}m^2; (3)$$

From (1),(3) it follows that:

$$0 \le \frac{\sum_{k=1}^{n}\Big(H_1\cdot H_2\cdot\ldots\cdot H_n\Big)^{\frac{1}{k}}}{\sum_{m=1}^{n}\left(\frac{x_m^2}{\pi}+\frac{\pi\cdot y_m^2}{4}\right)} \le \frac{e\cdot\sum_{k=1}^{n}H_k}{\sum_{m=1}^{n}m^2}$$

$$\lim_{n\to\infty}\frac{e\cdot\sum_{k=1}^{n}H_k}{\sum_{m=1}^{n}m^2} \overbrace{=}^{L.C-S} e\cdot\lim_{n\to\infty}\frac{H_n}{n^2} \overbrace{=}^{L.C-S} = 0$$

Therefore,

$$\Omega = \lim_{n\to\infty} \frac{\sum_{k=1}^{n} \left(H_1 \cdot H_2 \cdot ... \cdot H_n\right)^{\frac{1}{k}}}{\sum_{m=1}^{n} \left(\frac{x_m^2}{\pi} + \frac{\pi \cdot y_m^2}{4}\right)} = 0$$

REFERENCES:

[1] OLYMPIC MATHEMATICAL BEAUTIES, M.Bencze, D.Sitaru, M.Ursărescu-2020

[2] OLYMPIC MATHEMATICAL POWER, M.Bencze, D.Sitaru-2018

[3] OLYMPIC MATHEMATICAL CHALENGE, M.Bencze, D.Sitaru

[4] ANALIZĂ MATEMATICĂ, D.M. Bătineţu-Gurgiu, M. Bătineţu-Giurgiu

[5] OCTOGON MATHEMATICAL MAGAZINE

[6] ROMANIAN MATHEMATICAL MAGAZINE-Interactive Journal-www.ssmrmh.ro