ABOUT A FEW INEQUALITIES IN TRIANGLE

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Let ΔABC be a triangle with F area, s semiperimeter and with usual notations holds:

1. If $t, u > 0$ and $t \cdot \min\{r_a, r_b, r_c\} > ur$, then:

$$\frac{tr_a - ur}{tr_a + ur} + \frac{tr_b - ur}{tr_b + ur} + \frac{tr_c - ur}{tr_c + ur} \ge \frac{3(3t-u)}{3t+u};\ (*)$$

Proof 1. We have:

$$\sum_{cyc} \frac{tr_a - ur}{tr_a + ur} = \sum_{cyc}\left(\frac{tr_a - ur}{tr_a + ur} + 1\right) - 3 = \sum_{cyc}\frac{2tr_a}{tr_a + ur} - 3 =$$

$$= 2t \cdot \sum_{cyc} \frac{r_a(s-a)}{tr_a(s-a) + ur(s-a)} - 3 = 3tF\sum_{cyc}\frac{1}{tF + ur(s-a)} - 3 \overset{Bergstrom}{\ge}$$

$$\ge 2tF \cdot \frac{9}{\sum\limits_{cyc}(tF + ur(s-a))} - 3 = \frac{18tF}{3tF + ur(s-a+s-b+s-c)} - 3 =$$

$$= \frac{18tF}{3tF + u} = \frac{3(3t-u)}{3t+u}$$

Proof 2. We have:

$$\sum_{cyc} \frac{tr_a - ur}{tr_a + ur} = \sum_{cyc}\frac{(tr_a - ur)^2}{(tr_a + ur)(tr_a + ur)} = \sum_{cyc}\frac{(tr_a - ur)^2}{t^2r_a^2 - u^2r^2} \ge$$

$$\overset{Bergstrom}{\ge} \frac{\left(\sum\limits_{cyc}(tr_a - ur)\right)^2}{\sum\limits_{cyc}(t^2r_a^2 - u^2r^2)} = \frac{\left(\sum\limits_{cyc}(tr_a(s-a)) - ur(s-a)\right)^2}{\sum\limits_{cyc}(t^2r_a^2(s-a)^2) - u^2r^2(s-a)^2)} =$$

$$= \frac{(3tF - urs)^2}{3t^2F^2 - \frac{u^2r^2((s-a)+(s-b)+(s-c))^2}{3}} = \frac{3(3t-u)^2F^2}{9t^2F^2 - u^2r^2s^2} =$$

$$= \frac{3(3t-u)^2F^2}{(9t^2 - u^2)F^2} = \frac{3(3t-u)}{3t+u}$$

2. If $t, u > 0$ and $t \cdot \min\{r_a, r_b, r_c\} > ur$, then:

$$\frac{tr_a + ur}{tr_a - ur} + \frac{tr_b + ur}{tr_b - ur} + \frac{tr_c + ur}{tr_c - ur} \ge \frac{3(3t+u)}{3t-u};\ (**)$$

Proof 1. We have:

$$\sum_{cyc} \frac{tr_a + ur}{tr_a - ur} = \sum_{cyc}\left(\frac{tr_a + ur}{tr_a - ur} + 1\right) - 3 = \sum_{cyc}\frac{tr_a}{tr_a - ur} - 3 =$$

$$= 2t \cdot \sum_{cyc} \frac{r_a(s-a)}{(tr_a - ur)(s-a)} - 3 = 2t \cdot F \sum_{cyc} \frac{1}{tF - ur(s-a)} - 3 \overset{Bergstrom}{\geq}$$

$$\geq 2t \cdot F \cdot \frac{9}{\sum_{cyc}(tF - ur(s-a))} - 3 = \frac{18tF}{3tF - ur(s-a+s-b+s-c)} - 3 =$$

$$= \frac{18t - 9t + 3u}{3t-u} \frac{3(3t+u)}{3t-u}.$$

Proof 2. We have:

$$\sum_{cyc} \frac{tr_a + ur}{tr_a - ur} = \sum_{cyc} \frac{(tr_a + ur)^2}{t^2 r_a^2 - u^2 r^2} =$$

$$= \sum_{cyc} \frac{(tr_a(s-a) + ur(s-a))^2}{t^2 r_a^2 (s-a)^2 - u^2 r^2 (s-a)^2} = \sum_{cyc} \frac{(tF + ur(s-a))^2}{t^2 F^2 - u^2 r^2 (s-a)^2} \overset{Bergstrom}{\geq}$$

$$\geq \frac{(3tF + ur(s-a+s-b+s-c))^2}{3t^2F^2 - u^2r^2((s-a)^2 + (s-b)^2 + (s-c)^2)} \geq \frac{(3tF + urs)^2}{3t^2F^2 - u^2r^2 \cdot \frac{1}{3}(s-a+s-b+s-c)^2} =$$

$$= \frac{3(3t+u)^2F^2}{9t^2F^2 - u^2r^2s^2} = \frac{3(3t+u)^2F^2}{(3t^2 - u^2)F^2} = \frac{3(3t+u)}{3t-u}$$

If $t = u = 1$ we obtain the inequality from [1], and if $t = 1, u = 2$ we obtain another inequality from [1].

REFERENCES

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