A SIMPLE PROOF FOR STOLARSKY'S THEOREM

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STOLARSKY'S THEOREM

Let be:

$$P(a,b,c)=2p(a^3+b^3+c^3)+q(a^2b+b^2c+c^2a+ab^2+bc^2+ca^2)+6rabc$$

where $p,q,r\in\mathbb{R}$ and a,b,c are sides in a triangle.

$$\text{If: }\begin{cases}P(1,1,1)\geq 0\\P(1,1,0)\geq 0\\P(2,1,1)\geq 0\end{cases}\quad\text{then } P(a,b,c)\geq 0$$

Proof.

Lemma

Let be:

$$P(x,y,z)=A\sum_{cyc}x^3+B\sum_{cyc}(x^2y+xy^2)+Cxyz$$

$$A,B,C\in\mathbb{R};$$

$$\text{If: }\begin{cases}P(1,1,1)\geq 0\\P(1,1,0)\geq 0\\P(1,0,0)\geq 0\end{cases}\quad\text{then } P(x,y,z)\geq 0,(\forall)x,y,z\geq 0$$

Proof of lemma:

$$\text{Let be: }\begin{cases}p=3A+6B+C=P(1,1,0)\geq 0\\2q=2A+2B=P(1,1,0)\geq 0\\r=A=P(1,0,0)\geq 0\end{cases}$$

$$A=r\Rightarrow 2q=2r+2B\Rightarrow B=q-r$$

$$P=3r+6(q-r)+C\Rightarrow C=p-6q+3r$$

$$P(x,y,z)=r\sum_{cyc}x^3+(q-r)\sum_{cyc}(x^2y+xy^2)+(p-6q+3r)xyz$$

Case I: $B\geq 0\Rightarrow q-r\geq 0\Rightarrow q\geq r$

$$P(x,y,z)=r\left(\sum_{cyc}x^3-3xyz\right)+3rxyz+$$

$$+(q-r)\sum_{cyc}(x^2y+xy^2-6xyz)+6(q-r)xyz+(p-6q+3r)xyz$$

$$P(x,y,z)=r\left(\sum_{cyc}x^3-3xyz\right)+(q-r)\sum_{cyc}(x^2y+xy^2)-6xyz)+$$

$$+xyz(3r+6q-6r+p-6q+3r)$$

$$P(x,y,z)=r\left(\sum_{cyc}x^3-3xyz\right)+(q-r)\left(\sum_{cyc}(x^2y+xy^2)-6xyz\right)+pxyz\geq$$

$$\overset{\text{AM-GM}}{\geq} r(3\sqrt[3]{x^3y^3z^3}-3xyz)+(q-r)(6\sqrt[6]{x^2y\cdot y^2z\cdot z^2x\cdot xy^2\cdot yz^2\cdot zx^2}-6xyz)+$$
$$+pxyz=$$
$$=r(3xyz-3xyz)+(q-r)(6xyz-6xyz)+pxyz=pxyz\geq 0$$

Case II: $B<0\Rightarrow q-r<0\Rightarrow q<r$

$$P(x,y,z)=r\sum_{cyc}x^3+(q-r)\sum_{cyc}(x^2y+xy^2)+(p-6q+3r)xyz=$$
$$=q\left(\sum_{cyc}x^3-3xyz\right)-q\sum_{cyc}x^3+3qxyz+$$
$$+r\left(\sum_{cyc}x^3-\sum_{cyc}x^2y-\sum_{cyc}xy^2+3xyz\right)+$$
$$+q\sum_{cyc}(x^2y+xy^2)+pxyz-6qxyz$$
$$=q\left(\sum_{cyc}x^3-3xyz\right)+(r-q)\left(\sum_{cyc}x^3-\sum x^2y-\sum xy^2+3xyz\right)+pxyz\overset{\text{SCHUR}}{\geq}$$
$$\geq q\left(\sum_{cyc}x^3-3xyz\right)+pxyz\overset{\text{AM-GM}}{\geq}$$
$$\geq q(3\sqrt[3]{x^3y^3z^3}-3xyz)+pxyz\geq pxyz\geq 0$$

Proof of main result:

$$P(1,1,1)=6p+6q+6r\geq 0\Rightarrow p+q+r\geq 0$$
$$P(1,1,0)=4p+2q\geq 0$$
$$P(2,1,1)=20p+14q+12r\geq 0$$

Let $x,y,z>0$ such that

$$a=y+z;b=z+x;c=x+y$$
$$P(a,b,c)=P(y+z,z+x,x+y)=$$
$$=2p\sum_{cyc}(y+z)^3+q+\sum_{cyc}(y+z)^2(z+x)+$$
$$+q\sum_{cyc}(y+z)(z+x)^2+6r(x+y)+(y+z)(z+x)=Q(x,y,z)$$
$$Q(1,1,1)=2p\cdot 3\cdot 8+q\cdot 3\cdot 2^3+q\cdot 3\cdot 2^3+6r\cdot 8=$$
$$=6(p+4q+4q+8r)=$$
$$=6(8p+8q+8r)=48(p+q+r)\geq 0$$
$$Q(1,1,0)=2p(8+1+1)+q(4+2+1)+q(4+2+1)+6r\cdot 2=$$
$$=2p\cdot 10+14q+12r=$$
$$=20p+14q+12r\geq 0$$
$$Q(1,0,0)=4p+2q\geq 0$$

By lemma:

$$Q(x,y,z) \geq 0; (\forall)x,y,z \geq 0$$
$$P(a,b,c) = Q(x,y,z) \geq 0$$

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References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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