A SIMPLE PROOF FOR KARAMATA'S INEQUALITY AND APPLICATIONS

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Abstract. In this presented an elementary proof for Karamata's inequality (case $n = 3; n = 4$) and a few amazing applications.

Theorem 1 (Karamata)
Let be $a_1, a_2, a_3, b_1, b_2, b_3, \alpha, \beta \in \mathbb{R}$ such that:
1. $\beta > a_1 \geq a_2 \geq a_3 > \alpha; \beta > b_1 \geq b_2 \geq b_3 > \alpha$
2. $a_1 \geq b_1; a_1 + a_2 \geq b_1 + b_2$
3. $a_1 + a_2 + a_3 = b_1 + b_2 + b_3$

If $f : (\alpha, \beta) \to \mathbb{R}; f$ convex function then:

$$f(a_1) + f(a_2) + f(a_3) \geq f(b_1) + f(b_2) + f(b_3)$$

Proof.
Denote:

$$c_1 = \Delta_f(a_1, b_1) = \frac{f(b_1) - f(a_1)}{b_1 - a_1}; b_1 \neq a_1$$
$$c_2 = \Delta_f(a_2, b_2) = \frac{f(b_2) - f(a_2)}{b_2 - a_2}; b_2 \neq a_2$$
$$c_3 = \Delta_f(a_3, b_3) = \frac{f(b_3) - f(a_3)}{b_3 - a_3}; b_3 \neq a_3$$

f convex function. If $x_1 < x < x_2$ then:

$$\Delta_f(x_1, x) = \frac{f(x) - f(x_1)}{x - x_1} \leq \frac{f(x_2) - f(x)}{x - x_2} = \Delta_f(x, x_2)$$

Δ_f is simmetric in x, y:

$$\Delta_f(x, y) = \Delta_f(y, x)$$

Δ_f is increasing in first argument:

$$x_1 \leq x_2 \Rightarrow \Delta_f(x_1, x) \leq \Delta_f(x_2, x)$$
$$c_1 - c_2 = \Delta_f(a_1, b_1) - \Delta_f(a_2, b_2) =$$
$$= \Delta_f(a_1, b_1) - \Delta_f(a_2, b_1) + \Delta_f(a_2, b_1) - \Delta_f(a_2, b_2) =$$
$$= \Delta_f(a_1, b_1) - \Delta_f(a_2, b_1) + \Delta_f(b_1, a_2) - \Delta_f(b_2, a_2) \geq 0$$
$$\text{because } \Delta_f(a_1, b_1) - \Delta_f(a_2, b_1) \geq 0 \qquad (a_1 \geq a_2)$$
$$\text{and } \Delta_f(b_1, a_2) - \Delta_f(b_2, a_2) \geq 0 \qquad (b_1 \geq b_2)$$

Denote:

$$A_0 = 0; A_1 = a_1, A_2 = a_1 + a_2, A_3 = a_1 + a_2 + a_3$$
$$B_0 = 0; B_1 = b_1, B_2 = b_1 + b_2, B_3 = b_1 + b_2 + b_3$$

Let's observe that:

$$A_1 \geq B_1, A_2 \geq B_2, A_3 = B_3$$

$$f(a_1)+f(a_2)+f(a_3)-f(b_1)-f(b_2)-f(b_3)=$$

$$=\frac{f(a_1)-f(b_1)}{a_1-b_1}\cdot(a_1-b_1)+\frac{f(a_2)-f(b_2)}{a_2-b_2}\cdot(a_2-b_2)+\frac{f(a_3)-f(b_3)}{a_3-b_3}\cdot(a_3-b_3)=$$

$$=c_1(a_1-b_1)+c_2(a_2-b_2)+c_3(a_3-b_3)=$$

$$=c_1(A_1-A_0-B_1+B_0)+c_2(A_2-A_1-B_2+B_1)+c_3(A_3-A_2-B_3+B_2)=$$

$$=c_1(A_1-B_1)-c_1(A_0-B_0)+c_2(A_2-B_2)-c_2(A_1-B_1)+c_3(A_3-B_3)-c_3(A_2-B_2)=$$

$$=(c_1-c_2)(A_1-B_1)+(c_2-c_3)(A_2-B_2)+(c_3-c_2)(A_3-B_3)\geq 0$$

because $c_1-c_2, c_2-c_3, c_3-c_2\geq 0$ and

$$A_1-B_1\geq 0; A_2-B_2\geq 0; A_3-B_3=0$$

$$f(a_1)+f(a_2)+f(a_3)-f(b_1)-f(b_2)-f(b_3)\geq 0$$

$$f(a_1)+f(a_2)+f(a_3)\geq f(b_1)+f(b_2)+f(b_3)$$

□

Corollary 1
If $0<c\leq b\leq a, f:(0,\infty)\to\mathbb{R}; f$ convex function then:

$$f(2a)+f(2b)+f(2c)\geq f(a+b)+f(b+c)+f(c+a)$$

Proof.

$$2a\geq a+b; 2a+2b\geq(a+b)+(b+c);$$

$$2a+2b+2c=(a+b)+(b+c)+(c+a)$$

are obvious. By Theorem 1:

$$f(2a)+f(2b)+f(2c)\geq f(a+b)+f(b+c)+f(c+a)$$

□

Corollary 2
If $f:(0,\infty)\to\mathbb{R}; f$ convex function then in ΔABC with sides a,b,c the following relationship holds:

$$f(2a)+f(2b)+f(2c)\geq f(2s-a)+f(2s-b)+f(2s-c)$$

Proof.

$$2s=a+b+c\Rightarrow a+b=2s-c; b+c=2s-a; c+a=2s-b$$

□

We apply corollary 1.
Application 1:
If $a,b,c>0$ then:

$$\frac{1}{2a}+\frac{1}{2b}+\frac{1}{2c}\geq\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}$$

Proof.

$$\text{Let be } f:(0,\infty)\to\mathbb{R}; f(x)=\frac{1}{x}; f'(x)=\frac{-1}{x^2};$$

$$f''(x)=\frac{2}{x^3}>0 f \text{convex function. WLOG: } a\geq b\geq c$$

By corollary 1:

$$f(2a)+f(2b)+f(2c)\geq f(a+b)+f(b+c)+f(c+a)$$

$$\frac{1}{2a}+\frac{1}{2b}+\frac{1}{2b}\geq\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}$$

Equality holds for $a=b=c$. □

Application 2:
If $0<c\leq b\leq a<\pi$ then:

$$\cot(2a)+\cot(2b)+\cot(2c)\geq\cot(a+b)+\cot(b+c)+\cot(c+a)$$

Proof.

$$\text{Let be } f:(0,\pi)\to\mathbb{R}; f(x)=\cot x$$

$$f'(x)=\frac{-1}{\sin^2 x}; f''(x)=\frac{2\cos x}{\sin^4 x}>0 f \text{ convex function}$$

By corollary 1:

$$f(2a)+f(2b)+f(2c)\geq f(a+b)+f(b+c)+f(c+a)$$

$$\cot(2a)+\cot(2b)+\cot(2c)\geq\cot(a+b)+\cot(b+c)+\cot(c+a)$$

Equality holds for $a=b=c$. □

Application 3:
In ΔABC with sides a,b,c the following relationship holds:

$$e^{2s}\left(\frac{1}{e^{2a}}+\frac{1}{e^{2b}}+\frac{1}{e^{2c}}\right)\geq e^a+e^b+e^c$$

Proof.

$$\text{Let be } f:(0,\infty)\to\mathbb{R}; f(x)=e^{-x}$$

$$f'(x)=-e^{-x}; f''(x)=e^{-x}>0 f \text{ convex function}$$

By corollary 2:

$$f(2a)+f(2b)+f(2c)\geq f(2s-a)+f(2s-b)+f(2s-c)$$

$$e^{-2a}+e^{-2b}+e^{-2c}\geq e^{-(2s-a)}+e^{-(2s-b)}+e^{-(2s-c)}$$

$$\frac{1}{e^{2a}}+\frac{1}{e^{2b}}+\frac{1}{e^{2c}}\geq\frac{e^a+e^b+e^c}{e^{2s}}$$

$$e^{2s}\left(\frac{1}{e^{2a}}+\frac{1}{e^{2b}}+\frac{1}{e^{2c}}\right)\geq e^a+e^b+e^c$$

Equality holds for $a=b=c$. □

Theorem 2 (Karamata)
Let be $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, \alpha, \beta \in \mathbb{R}$ such that:
1. $\beta > a_1 \geq a_2 \geq a_3 \geq a_4 > \alpha; \beta > b_1 \geq b_2 \geq b_3 \geq b_4 > \alpha$
2. $a_1 \geq b_1; a_1 + a_2 \geq b_1 + b_2; a_1 + a_2 + a_3 \geq b_1 + b_2 + b_3$
3. $a_1 + a_2 + a_3 + a_4 = b_1 + b_2 + b_3 + b_4$
If $f : (\alpha, \beta) \to \mathbb{R}; f$ convex function then:

$$f(a_1) + f(a_2) + f(a_3) + f(a_4) \geq f(b_1) + f(b_2) + f(b_3) + f(b_4)$$

Corollary 3
If $0 < d \leq c \leq b \leq a; f : (0, \infty) \to \mathbb{R} f$ convex function then:

$$f(4a) + f(4b) + f(4c) + f(4d) \geq f(2a + b + c) + f(2b + c + d) +$$
$$+ f(2c + d + a) + f(2d + a + b)$$

Proof.

$$4a \geq 2a + b + c; 4a + 4b \geq (2a + b + c) + (2b + c + d);$$
$$4a + 4b + 4c \geq (2a + b + c) + (2b + c + d) + (2c + d + a);$$
$$4a + 4b + 4c + 4d = (2a + b + c) + (2b + c + d) + (2c + d + a) + (2d + a + b)$$

are obvious. By Theorem 2:

$$f(4a) + f(4b) + f(4c) + f(4d) \geq f(2a + b + c) +$$
$$+ f(2b + c + d) + f(2c + d + a) + f(2d + a + b)$$
$$\text{with: } a_1 = 4a; a_2 = 4b; a_3 = 4c; a_4 = 4d$$
$$b_1 = 2a + b + c; b_2 = 2b + c + d; b_3 = 2c + d + a;$$
$$b_4 = 2d + a + b$$

□

Corollary 4
If $f : (0, \infty) \to \mathbb{R}; f$ convex function then in any convex quadrilateral with sides a, b, c, d the following relationship holds:

$$f(4a) + f(4b) + f(4c) + f(4d) \geq f(2s + a - d) +$$
$$+ f(2s + b - a) + f(2s + c - b) + f(2s + d - c)$$
$$s = \frac{a + b + c + d}{2} \text{ - semiperimeter}$$

Proof.
WLOG: $d \leq c \leq b \leq a$ and by corollary 3

$$f(4a) + f(4b) + f(4c) + f(4d) \geq f(2a + b + c) +$$
$$+ f(2b + c + d) + f(2c + d + a) + f(2d + a + b) =$$
$$= f(a + b + c + d + a - d) + f(a + b + c + d + b - a) +$$
$$+ f(a + b + c + d + c - b) + f(a + b + c + d + d - c) =$$
$$= f(2s + a - d) + f(2s + b - a) + f(2s + c - b) + f(2s + d - c)$$

□

Application 4:
If $a, b, c, d > 0$ then:

$$\frac{1}{4a} + \frac{1}{4b} + \frac{1}{4c} + \frac{1}{4d} \geq \frac{1}{2a + b + c} + \frac{1}{2b + c + d} + \frac{1}{2c + d + a} + \frac{1}{2d + a + b}$$

Proof.

$$\text{Let be } f:(0,\infty)\to\mathbb{R}; f(x)=\frac{1}{x}; f'(x)=\frac{-1}{x^2};$$
$$f''(x)=\frac{2}{x^3}>0; f \text{ convex function}$$

By corollary 3:

$$f(4a)+f(4b)+f(4c)+f(4d)\geq f(2a+b+c)+$$
$$+f(2b+c+d)+f(2c+d+a)+f(2d+a+b)$$
$$\frac{1}{4a}+\frac{1}{4b}+\frac{1}{4c}+\frac{1}{4d}\geq\frac{1}{2a+b+c}+\frac{1}{2b+c+d}+\frac{1}{2c+d+a}+\frac{1}{2d+a+b}$$

Equality holds for $a=b=c=d$. □

Application 5:

In any convex quadrilateral with sides a,b,c,d the following relationship holds:

$$e^{2s}\Big(\frac{1}{4^{4a}}+\frac{1}{e^{4b}}+\frac{1}{e^{4c}}+\frac{1}{e^{4d}}\Big)\geq\frac{e^d}{e^a}+\frac{e^a}{e^b}+\frac{e^b}{e^c}+\frac{e^c}{e^d}$$

Proof.

$$\text{Let be } f:(0,\infty)\to\mathbb{R}; f(x)=e^{-x}; f'(x)=-e^{-x}$$
$$f''(x)=e^{-x}>0; f \text{ convex function}$$

By corollary 4:

$$f(4a)+f(4b)+f(4c)+f(4d)\geq f(2s+a-d)+$$
$$+f(2s+b-a)+f(2s+c-b)+f(2s+d-c)$$
$$e^{-4a}+e^{-4b}+e^{-4c}+e^{-4d}\geq e^{-(2s+a-d)}+$$
$$+e^{-(2s+b-a)}+e^{-(2s+c-b)}+e^{-(2s+d-c)}$$
$$\frac{1}{e^{4a}}+\frac{1}{e^{4b}}+\frac{1}{e^{4c}}+\frac{1}{e^{4d}}\geq\frac{e^{d-a}+e^{a-b}+e^{b-c}+e^{c-d}}{e^{2s}}$$
$$e^{2d}\Big(\frac{1}{e^{4a}}+\frac{1}{e^{4b}}+\frac{1}{e^{4c}}+\frac{1}{e^{4d}}\Big)\geq\frac{e^d}{e^a}+\frac{e^a}{e^b}+\frac{e^b}{e^c}+\frac{e^c}{e^d}$$

Equality holds for $a=b=c=d$. □

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