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A FEW LOGARITHMIC INEQUALITIES

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Abstract: *In this paper are presented a few logarithmic inequalities.*

Application 1. If $[a,b]\subset\mathbb{R}$, then prove that:

$$\frac{b^2-x^2}{2x}+\frac{y^2-a^2}{2y}+\log\left[\left(\frac{y}{a}\right)^y\cdot\left(\frac{b}{x}\right)^x\right]\le\frac{(b-a+y-x)(a^2+b^2)}{ab};\forall x,y\in[a,b]$$

Solution 1.

If $[a,b]\subset\mathbb{R}, x_k\in[a,b], k=\overline{1,n}$, then

$$\left(\sum_{k=1}^{n}x_k\right)\left(\sum_{k=1}^{n}\frac{1}{x_k}\right)\le\frac{(a+b)^2}{4ab}\cdot n^2,\forall n\in\mathbb{N};(Schweitzer)$$

For $n=2$ and $x_1=x, x_2=y$ we have:

$$(x+y)\left(\frac{1}{x}+\frac{1}{y}\right)\le\frac{(a+b)^2}{ab};\forall x,y\in[a,b]\Leftrightarrow$$

$$2+\frac{x}{y}+\frac{y}{x}\le\frac{(a+b)^2}{ab};\forall x,y\in[a,b];(*)$$

Integrating $(*)$ w.r.t. x from a to y, we have:

$$\int_a^y\left(2+\frac{x}{y}+\frac{y}{x}\right)dx\le\int_a^y\frac{(a+b)^2}{ab}dx\Leftrightarrow$$

$$2(y-a)+\frac{y^2-a^2}{2y}+y(\log y-\log a)\le\frac{(a+b)^2}{ab}\cdot(y-a)\Leftrightarrow$$

$$\frac{y^2-a^2}{2y}+\log\left(\frac{y}{a}\right)^y\le(y-a)\left[\frac{(a+b)^2}{ab}-2\right];\forall y\ge a;\quad(1)$$

Now, integrating $(*)$ w.r.t. y from x to b, we have:

$$\int_x^b\left(2+\frac{x}{y}+\frac{y}{x}\right)dy\le\int_x^b\frac{(a+b)^2}{ab}dy\Leftrightarrow$$

$$2(b-x)+x(\log b-\log x)+\frac{b^2-x^2}{2x}\le\frac{(a+b)^2}{ab}\cdot(b-x)\Leftrightarrow$$

$$\frac{b^2-x^2}{2x}+\log\left(\frac{b}{x}\right)^x\le(b-x)\left[\frac{(a+b)^2}{ab}-2\right];\forall x\le b;(2)$$

By adding (1) and (2), we obtain:

$$\frac{b^2-x^2}{2x}+\frac{y^2-a^2}{2y}+\log\left(\frac{b}{x}\right)^x+\log\left(\frac{y}{a}\right)^y\le\frac{(b-a+y-x)(a^2+b^2)}{ab};\forall x,y\in[a,b]\Leftrightarrow$$

$$\frac{b^2-x^2}{2x}+\frac{y^2-a^2}{2y}+\log\left(\frac{b^xy^y}{x^xa^y}\right)\le\frac{(b-a+y-x)(a^2+b^2)}{ab};\forall x,y\in[a,b]$$

Solution 2.

$$\text{Let } f(t)=(t-1)\left(t+\frac{1}{t}\right)-\frac{t^2-1}{2}-\log t\,, t>0$$

We have :

$$f'(t)=t+\frac{1}{t}+(t-1)\left(1-\frac{1}{t^2}\right)-t-\frac{1}{t}=\frac{(t-1)^2(t+1)}{t^2}\geq 0, \forall t>0.$$

Then f is increasing on $(0,\infty)$

Since $f(1)=0$ we have : $f(t)\geq 0, \forall t\geq 1$ and $f(t)\leq 0, \forall t\in(0,1]$

$$\text{Since } \frac{b}{x}\geq 1\to f\left(\frac{b}{x}\right)=\left(\frac{b}{x}-1\right)\left(\frac{b}{x}+\frac{x}{b}\right)-\frac{b^2-x^2}{2x^2}-\log\left(\frac{b}{x}\right)\geq 0 \overset{\times x}{\Leftrightarrow}$$

$$\frac{b^2-x^2}{2x}+\log\left(\frac{b}{x}\right)^x\leq (b-x)\left(\frac{b}{x}+\frac{x}{b}\right) \quad (1)$$

$$\text{And : } \frac{a}{y}\leq 1 \;\to\; f\left(\frac{a}{y}\right)=\left(\frac{a}{y}-1\right)\left(\frac{a}{y}+\frac{y}{a}\right)-\frac{a^2-y^2}{2y^2}-\log\left(\frac{a}{y}\right)\leq 0 \overset{\times(-y)}{\Leftrightarrow}$$

$$\frac{y^2-a^2}{2y}+\log\left(\frac{y}{a}\right)^y\leq (y-a)\left(\frac{a}{y}+\frac{y}{a}\right) \quad (2)$$

Now, let $g(t)=t+\frac{1}{t}, t\geq 1$. We have : $g'(t)=1-\frac{1}{t^2}\geq 0, \forall t\geq 1$ then g is increasing on $[1,\infty)$

$$\text{Since } \frac{b}{a}\geq max\left\{\frac{b}{x},\frac{y}{a}\right\}\geq 1, \quad \text{then : } g\left(\frac{b}{a}\right)\geq g\left(\frac{b}{x}\right) \text{ and } g\left(\frac{b}{a}\right)\geq g\left(\frac{y}{a}\right)$$

$$\text{Or : } \frac{b}{x}+\frac{x}{b}\leq\frac{b}{a}+\frac{a}{b}=\frac{a^2+b^2}{ab} \quad (3) \text{ and } \frac{a}{y}+\frac{y}{a}\leq\frac{b}{a}+\frac{a}{b}=\frac{a^2+b^2}{ab} \quad (4)$$

From (1), (2), (3) and (4) we obtain:

$$\frac{b^2-x^2}{2x}+\frac{y^2-a^2}{2y}+\log\left(\left(\frac{b}{x}\right)^x\left(\frac{y}{a}\right)^y\right)\leq (b-x)\cdot\frac{a^2+b^2}{ab}+(y-a)\cdot\frac{a^2+b^2}{ab}$$

Therefore,

$$\frac{b^2-x^2}{2x}+\frac{y^2-a^2}{2y}+\log\left(\frac{b^x y^y}{x^x a^y}\right)\leq\frac{(b-a+y-x)(a^2+b^2)}{ab}, \forall x,y\in[a,b].$$

Observation. Putting $y=x$ in

$$\frac{y^2-a^2}{2y}+\log\left(\frac{y}{a}\right)^y\leq (y-a)\left[\frac{(a+b)^2}{ab}-2\right]; \forall y\geq a; \quad (1)$$

we get:

$$2(x-a)+\frac{x^2-a^2}{2x}+x(\log x-\log a)\leq\frac{(a+b)^2}{ab}(x-a); (2)$$

By adding (1) and (2) it follows the proposed problem by Daniel Culea:

$$\boldsymbol{\frac{(x+y)(xy-a^2)}{2xy}+\log\left[\left(\frac{y}{a}\right)^y\cdot\left(\frac{x}{a}\right)^x\right]\leq (x+y-2a)\frac{a^2+b^2}{ab}}$$

Application 2. **If $a,b\in(1,\infty)$ then:**

$$\log\left[\left(\frac{a+b}{2a}\right)^{2a}\cdot\left(\frac{b^2+2ab}{a^2+2ab}\right)^{3ab}\right]<b-a+\log\left[\left(\frac{b}{a}\right)^{2(a+b)}\right]$$

Proof. In Kurliancik's inequality

$$\sum_{i=1}^{n}\frac{i}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_i}}<2\sum_{i=1}^{n}a_i\,,a_i>0$$

we take $a_1=x;a_2=y;a_3=z$, then

$$x+\frac{2xy}{x+y}+\frac{3xyz}{xy+yz+zx}<2(x+y+z)$$

$$1+\frac{2y}{x+y}+\frac{3yz}{x(y+z)+yz}<2\left(1+\frac{y}{x}+\frac{z}{x}\right)$$

$$\int_y^z dx+\int_y^z\frac{2y}{x+y}dx+\int_y^z\frac{3yz}{x(y+z)+yz}dx<2\int_y^z\left(1+\frac{y}{x}+\frac{z}{x}\right)dx$$

$$z-y+2y\log\left(\frac{z+y}{2y}\right)+3yz\log\left[\frac{z(y+z)+yz}{y(y+z)+yz}\right]<2(z-y)+2y\log\left(\frac{z}{y}\right)+2z\log\left(\frac{z}{y}\right)$$

$$\log\left[\left(\frac{z+y}{2y}\right)^{2y}\cdot\left(\frac{z^2+2yz}{y^2+2yz}\right)^{3yz}\right]<z-y+\log\left[\left(\frac{z}{y}\right)^{2y}\cdot\left(\frac{z}{y}\right)^{2z}\right]$$

and for $y=a,z=b$ it follows the desired inequality.

Application 3. **If $a_i>1\,,i=\overline{1,n+1}\,,n\in\mathbb{N}^*$ then prove:**

$$\sum_{cyc}\log_{a_1}^{n}(a_2a_3\cdot\ldots\cdot a_{n+1})\geq n^{n+1}$$

Solution.

$$log_{a_1}^{n}(a_2a_3\cdot\ldots\cdot a_{n+1})=\left(log_{a_1}a_2+log_{a_1}a_3+\cdots+log_{a_1}a_{n+1}\right)^n\overset{AM-GM}{\geq}$$
$$\geq n^n\cdot log_{a_1}a_2\,log_{a_1}a_3\cdot\ldots\cdot log_{a_1}a_{n+1}$$

$$\sum_{cyc}log_{a_1}^{n}(a_2a_3\cdot\ldots\cdot a_{n+1})\geq n^n\sum_{cyc}log_{a_1}a_2\,log_{a_1}a_3\cdot\ldots\cdot log_{a_1}a_{n+1}\overset{AM-GM}{\geq}$$
$$\geq n^n\cdot n\cdot\sqrt[n]{log_{a_1}a_2\,log_{a_2}a_1\cdot\ldots\cdot log_{a_n}a_{n+1}\,log_{a_{n+1}}a_n}=n^{n+1}$$

Application 4. **If $a_1,a_2,\ldots,a_n>0,n\in\mathbb{N},n>1$ then:**

$$\sum_{cyc}\log_{1+a_1a_2}\left(1+a_2^{1+a_2}\right)\left(1+a_3^{1+a_3}\right)\geq 2n$$

Solution.

$$1+a_i^{1+a_i}=1+(1+a_i-1)^{1+a_i}\overset{Bernoulli}{\geq}1+a_i^2\Rightarrow$$
$$\left(1+a_i^{1+a_i}\right)\left(1+a_j^{1+a_j}\right)\geq(1+a_i^2)(1+a_j^2)\geq(1+a_ia_j)^2\Rightarrow$$
$$\sum_{cyc}log_{1+a_1a_2}(1+a_2^{1+a_2})(1+a_3^{1+a_3})\geq 2\sum_{cyc}log_{1+a_1a_2}(1+a_2a_3)\overset{Am-Gm}{\geq}$$
$$\geq 2n\sqrt[n]{\prod_{cyc}log_{1+a_1a_2}(1+a_2a_3)}\geq 2n$$

Application 5. If $0<a\leq b<\pi$ then:

$$log\left(\frac{sinb}{sina}\right)\geq\left(1+\sqrt{\frac{a}{b}}\right)log\left(\frac{sinb}{sin(\sqrt{ab})}\right)$$

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Solution.

$$log\left(\frac{sinb}{sina}\right)\geq\left(1+\sqrt{\frac{a}{b}}\right)log\left(\frac{sinb}{sin(\sqrt{ab})}\right)$$
$$\Leftrightarrow log\left(\frac{sinb}{sina}\right)-log\left(\frac{sinb}{sin(\sqrt{ab})}\right)\geq\sqrt{\frac{a}{b}}log\left(\frac{sinb}{sin(\sqrt{ab})}\right)$$
$$\Leftrightarrow log\left(\frac{sin\sqrt{ab}}{sina}\right)\geq\sqrt{\frac{a}{b}}log\left(\frac{sinb}{sin(\sqrt{ab})}\right)$$
$$\Leftrightarrow\sqrt{b}log(sin\sqrt{ab})-\sqrt{b}log(sina)\geq\sqrt{a}log(sinb)-\sqrt{a}log(sin\sqrt{ab})$$
$$\Leftrightarrow(\sqrt{a}+\sqrt{b})log(sin\sqrt{ab})\geq\sqrt{a}log(sinb)+\sqrt{b}log(sina)$$
$$\Leftrightarrow log(sin\sqrt{ab})\geq\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}log(sinb)+\frac{\sqrt{b}}{\sqrt{a}+\sqrt{b}}log(sina);\quad(1)$$

We have:

$$\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}log(sinb)+\frac{\sqrt{b}}{\sqrt{a}+\sqrt{b}}log(sina)\overset{logt-concave}{\leq}$$
$$\leq log\left(\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}sinb+\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}sina\right)\overset{sint-concave\ (0,\pi)}{\leq}$$
$$\leq log\left(sin\left(\frac{b\sqrt{a}+a\sqrt{b}}{\sqrt{a}+\sqrt{b}}\right)\right)=log\left(sin(\sqrt{ab})\right);(2)$$

From (1),(2) it follows that:

$$log\left(\frac{sinb}{sina}\right)\geq\left(1+\sqrt{\frac{a}{b}}\right)log\left(\frac{sinb}{sin(\sqrt{ab})}\right)$$

Application 6. If $a_i>1\,,i=\overline{1,n+1}\,,n\in\mathbb{N}$ then:

$$\sum_{cyc}log_{a_1}^n(a_2a_3\cdot\ldots\cdot a_{n+1})\cdot log_{a_1a_2^2a_3^2\cdot\ldots\cdot a_{n*1}^2}a_1\geq\frac{2n^n}{3}$$

Solution.

$$\sum_{cyc}log_{a_1}^n(a_2a_3\cdot\ldots\cdot a_{n+1})\cdot log_{a_1a_2^2a_3^2\cdot\ldots\cdot a_{n*1}^2}a_1\overset{Cebâşev}{\geq}$$

$$\geq\frac{1}{n}\left(\sum_{czc}log_{a_1}^n(a_2a_3\cdot\ldots\cdot a_{n+1})\right)\left(\sum_{cyc}log_{a_1a_2^2a_3^2\cdot\ldots\cdot a_{n*1}^2}a_1\right)\quad(1)$$

$$log_{a_1}^n(a_2a_3\cdot\ldots\cdot a_{n+1})=\left(log_{a_1}a_2+log_{a_1}a_3+\cdots+log_{a_1}a_{n+1}\right)^n\overset{AM-GM}{\geq}$$
$$\geq n^n\cdot log_{a_1}a_2\,log_{a_1}a_3\cdot\ldots\cdot log_{a_1}a_{n+1}$$

$$\sum_{cyc}log_{a_1}^n(a_2a_3\cdot\ldots\cdot a_{n+1})\geq n^n\sum_{cyc}log_{a_1}a_2\,log_{a_1}a_3\cdot\ldots\cdot log_{a_1}a_{n+1}\overset{AM-GM}{\geq}$$
$$\geq n^n\cdot n\cdot\sqrt[n]{log_{a_1}a_2\,log_{a_2}a_1\cdot\ldots\cdot log_{a_n}a_{n+1}\,log_{a_{n+1}}a_n}=n^{n+1}\quad(2)$$

$$\sum_{cyc}log_{a_1a_2^2a_3^2\cdot\ldots\cdot a_{n*1}^2}a_1=\sum_{cyc}\frac{lna_1}{lna_1+2(lna_2+lna_3+\cdots+lna_{n+1}}=$$
$$=\sum_{cyc}\frac{ln^2a_1}{ln^2a_1+2(lna_2lna_3+\cdots+lna_nlna_{n+1})}\geq$$
$$\geq\frac{(lna_1+lna_2+\cdots+lna_{n+1})^2}{ln^2a_1+ln^2a_2+\cdots+ln^2a_{n+1}+4(lna_1lna_2+lna_2lna_3+\cdots+lna_{n+1}lna_1)}\geq\frac{2}{3}\Leftrightarrow$$
$$ln^2a_1+ln^2a_2+\cdots+ln^2a_{n+1}\geq lna_1lna_2+lna_2lna_3+\cdots+lna_nlna_{n+1}\quad(3)$$

From (1,2,3) we get the problem.

Application 7. If $x_i>1,\forall i=\overline{1,n};\,n\in\mathbb{N},n\geq3$ then prove:

$$\frac{logx_2}{log^2(x_1^2x_2)}+\frac{logx_3}{log^2(x_1^2x_2^2x_3)}+\cdots+\frac{logx_n}{log^2(x_1^2x_2^2\ldots x_{n-1}^2x_n)}\leq\frac{log\sqrt[4]{x_2x_3\ldots x_n}}{logx_1\cdot log(x_1x_2x_3\ldots x_n)}$$

Solution.

$$\frac{a_2}{a_1(a_1+a_2)}+\frac{a_3}{(a_1+a_2)(a_1+a_2+a_3)}=\frac{a_2+a_3}{a_1(a_1+a_2+a_3)}$$

WLOG suppose:

$$\frac{a_2}{a_1(a_1+a_2)}+\frac{a_3}{(a_1+a_2)(a_1+a_2+a_3)}+\cdots+\frac{a_n}{(a_1+a_2+\cdots+a_{n-1})(a_1+a_2+\cdots+a_n)}=\frac{a_2+a_3+\cdots+a_n}{a_1(a_1+a_2+\cdots+a_n)}\Rightarrow$$

$$\frac{a_2}{a_1(a_1+a_2)}+\frac{a_3}{(a_1+a_2)(a_1+a_2+a_3)}+\cdots+\frac{a_n}{(a_1+a_2+\cdots+a_{n-1})(a_1+a_2+\cdots+a_n)}+\frac{a_{n+1}}{(a_1+a_2+\cdots+a_n)(a_1+a_2+\cdots+a_{n+1})}=$$

$$=\frac{a_2+a_3+\cdots+a_n}{a_1(a_1+a_2+\cdots+a_n)}+\frac{a_{n+1}}{(a_1+a_2+\cdots+a_n)(a_1+a_2+\cdots+a_{n+1})}=$$

$$=\frac{(a_2+a_3+\cdots+a_n)^2+a_1(a_2+a_3+\cdots+a_n)+a_{n+1}(a_2+a_3+\cdots+a_n)+a_1a_{n+1}}{(a_1+a_2+\cdots+a_n)a_1(a_1+a_2+\cdots+a_{n+1})}=$$

$$=\frac{a_2+a_3+\cdots+a_{n+1}}{a_1(a_1+a_2+\cdots+a_{n+1})}$$

From $(x+y)^2\ge 4xy\Rightarrow\frac{1}{xy}\ge\frac{4}{(x+y)^2}$ we have:

$$\frac{a_2}{a_1(a_1+a_2)}\ge\frac{4a_2}{(2a_1+a_2)^2}$$

$$\frac{a_3}{(a_1+a_2)(a_1+a_2+a_3)}\ge\frac{4a_3}{(2a_1+2a_2+a_3)^2}$$

$$\frac{a_n}{(a_1+a_2+\cdots+a_{n-1})(a_1+a_2+\cdots+a_n)}\ge\frac{4a_n}{(2a_1+2a_2+\cdots+2a_{n-1}+a_n)^2}$$

Adding up relationships, we have:

$$\frac{4a_2}{(2a_1+a_2)^2}+\frac{4a_3}{(2a_1+2a_2+a_3)^2}+\cdots+\frac{4a_n}{(2a_1+2a_2+\cdots+2a_{n-1}+a_n)^2}\le$$

$$\le\frac{a_2}{a_1(a_1+a_2)}+\frac{a_3}{(a_1+a_2)(a_1+a_2+a_3)}+\cdots+\frac{a_n}{(a_1+a_2+\cdots+a_{n-1})(a_1+a_2+\cdots+a_n)}=\frac{a_2+a_3+\cdots+a_n}{a_1(a_1+a_2+\cdots+a_n)}\Rightarrow$$

$$\frac{a_2}{(2a_1+a_2)^2}+\frac{a_3}{(2a_1+2a_2+a_3)^2}+\cdots+\frac{a_n}{(2a_1+2a_2+\cdots+2a_{n-1}+a_n)^2}\le\frac{a_2+a_3+\cdots+a_n}{4a_1(a_1+a_2+\cdots+a_n)}$$

For: $a_1=logx_1;\ a_2=logx_2;\ldots;a_n=logx_n$ *we get:*

$$\frac{logx_2}{log^2(x_1^2x_2)}+\frac{logx_3}{log^2(x_1^2x_2^2x_3)}+\cdots+\frac{logx_n}{log^2(x_1^2x_2^2\ldots x_{n-1}^1x_n)}\le\frac{log\sqrt[4]{x_2x_3\ldots x_n}}{logx_1\cdot log(x_1x_2x_3\ldots x_n)}$$

Application 8. For $x,y,z>2$ prove:

$$\log\left(\frac{(3+x)^2(3+y)^2(3+z)^2}{8}\right)^3\le\sum_{cyc}(x^2+8)\sin\frac{\pi}{x}$$

Solution.

$$\text{If } x > 2, \text{ then } \frac{\pi}{x} < \frac{\pi}{2}, \tan\frac{\pi}{x} > \frac{\pi}{x} > \frac{3}{x},$$

$$\cos^2\frac{\pi}{x} = \frac{1}{1+\tan^2\frac{\pi}{x}} < \frac{1}{1+\left(\frac{\pi}{x}\right)^2} < \frac{1}{1+\left(\frac{3}{x}\right)^2} = \frac{x^2}{x^2+9} \Leftrightarrow \sin\frac{\pi}{x} > \frac{3}{\sqrt{x^2+9}}; (1)$$

$$\log(1+t) \le \frac{t}{\sqrt{1+t}}, \forall t \ge 0; (2)$$

$$t = x^2+8; (2) \Rightarrow \log(9+x^2) \le \frac{8+x^2}{\sqrt{9+x^2}} \Leftrightarrow \frac{\log(9+x^2)}{8+x^2} \le \frac{1}{\sqrt{9+x^2}}; (3)$$

From (1), (2), (3) it follows that:

$$\frac{\log(9+x^2)}{8+x^2} \le \frac{1}{\sqrt{9+x^2}} \le \frac{1}{3}\sin\frac{\pi}{x} \Leftrightarrow \log(9+x^2) \le \frac{1}{3}(x^2+8)\sin\frac{\pi}{x}$$

$$\log\left(\frac{(3+x)^2}{2}\right) \le \frac{1}{3}(x^2+8)\sin\frac{\pi}{x}$$

Therefore,

$$\log\left(\frac{(3+x)^2(3+y)^2(3+z)^2}{8}\right)^3 \le \sum_{cyc}(x^2+8)\sin\frac{\pi}{x}$$

Application 9. If $a, b, c > 1$, then:

$$\sum_{cyc} log_{a+b}(1+b^{b+1})(1+c^{c+1}) \ge 6(a+b)^{c-b}(b+c)^{a-c}(c+a)^{b-a}$$

Solution. From Bernoulli's inequality, we have: $\begin{cases} a^{a+1} = (1+a-1)^{a+1} \ge a^2 \\ b^{b+1} = (1+b-1)^{b+1} \ge b^2 \\ c^{c+1} = (1+c-1)^{c+1} \ge c^2 \end{cases}$

$$\begin{cases} (1+a^{a+1})(1+b^{b+1}) \ge (1+a^2)(1+b^2) \ge (a+b)^2 \\ (1+b^{b+1})(1+c^{c+1}) \ge (1+b^2)(1+c^2) \ge (b+c)^2 \\ (1+c^{c+1})(1+a^{a+1}) \ge (1+c^2)(1+a^2) \ge (c+a)^2 \end{cases}$$

$$\sum_{cyc} log_{a+b}(1+b^{b+1})(1+c^{c+1}) \ge \sum_{cyc} log_{a+b}(b+c)^2 =$$

$$= 2\sum_{cyc} log_{a+b}(b+c) \overset{Am-Gm}{\gtrapprox} 6\cdot\sqrt[3]{\prod_{cyc} log_{a+b}(b+c)} = 6 \quad (i)$$

$$\therefore x^x y^y z^z \ge x^z y^x z^y, \forall x,y,z > 1 \leftrightarrow (z-x)lnx + (x-y)lny + (y-z)lnz \le 0$$

$$If\ 1 \le x \le y \le z \to (lnx \le lny \le lnz, z-x \ge x-y) \overset{Chebyshev's}{\Rightarrow}$$

$$(z-x)lnx + (x-y)lny \le \frac{1}{2}(z-y)\,ln(xy) = (z-y)ln\sqrt{xy}$$

$$\to (z-x)lnx + (x-y)lny + (y-z)lnz \le (z-y)ln\sqrt{xy} + (y-z)lnz =$$

$$= (z-y)ln\frac{\sqrt{xy}}{z} \le 0 \therefore$$

From: $x = b+c, y = c+a, z = a+b \to (a+b)^{c-b}(b+c)^{a-c}(c+a)^{b-a} \le 1 \quad (ii)$

From $(i), (ii)$ it follows that:

$$\sum_{cyc} log_{a+b}(1+b^{b+1})(1+c^{c+1}) \ge 6(a+b)^{c-b}(b+c)^{a-c}(c+a)^{b-a}$$

Application 10. Let $a_i \in (1,\infty), i = \overline{1,n}, n \in \mathbb{N}, a_1 + a_2 + \dots + a_n = ne^4$, then:

$$\sqrt{\log a_1^{a_2} + \log a_1^{a_3} + \dots + \log a_1^{a_n}} + \dots + \sqrt{\log a_n^{a_1} + \log a_n^{a_2} + \dots + \log a_n^{a_{n-1}}} \le 9n^2$$

Solution.

$$\sqrt{log\, a_1^{a_2} + log\, a_1^{a_3} + \dots + log\, a_1^{a_n}} + \dots + \sqrt{log\, a_n^{a_1} + log\, a_n^{a_2} + \dots + log\, a_n^{a_{n-1}}} =$$

$$= \sqrt{(a_2 + a_3 + \dots + a_n)\, log\, a_1} + \dots + \sqrt{(a_1 + a_2 + \dots + a_{n-1})\, log\, a_n} \overset{CBS}{\le}$$

$$\overset{CBS}{\le} \sqrt{(a_2 + a_3 + \dots + a_n) + \dots + (a_1 + a_2 + \dots + a_{n-1})} \cdot \sqrt{\sum_{k=1}^{n} log\, a_k} =$$

$$= \sqrt{(n-1)S_n} \cdot \sqrt{log(a_1 a_2 \cdot \ldots \cdot a_n)} \le \sqrt{(n-1)S_n} \cdot \sqrt{log\left(\frac{a_1 + a_2 + \dots + a_n}{n}\right)^n} =$$

$$= \sqrt{(n-1)S_n} \cdot \sqrt{n \cdot log\left(\frac{S_n}{n}\right)} = \sqrt{(n-1)ne^4} \cdot \sqrt{4n} = 2ne^2 \cdot \sqrt{n-1} \le$$

$$\le 2ne^2 \cdot \frac{n-1+1}{2} = n^2e^2 < 9n^2$$

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