# Interesting central binomial coefficient sums

Vuk Stojiljković

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**Abstract**

In this short article we will find generating functions and sums for some series associated with central binomial coefficients.

**Introduction**

Central binomial coefficients have been researched by many , they also find a lot of applications in combinatorics, probability, computer science. The first person writing more in detail about their generating function being Lehmer [1]. In this paper we will prove some equalities surrounding them. Let us define binomial coefficients as follows We will need the following lemma.

**Lemma 0.1.** *Let $a_n$ be a sequence, then the following holds*

$$\sum_{n=0}^{\infty} a_{2n} = \frac{1}{2}\sum_{n=0}^{\infty} a_n + \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n a_n$$

*Lets write out the rigth hand side*

*Proof.*

$$a_0 + a_1 + a_2 + a_3 + ... + (a_0 - a_1 + a_2 - a_3 + ...) = 2a_0 + 2a_2 + 2a_4 + ....$$

$$\sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty}(-1)^n a_n = 2\sum_{n=0}^{\infty} a_{2n} \backslash \frac{1}{2}$$

$$\frac{1}{2}\sum_{n=0}^{\infty} a_n + \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n a_n = \sum_{n=0}^{\infty} a_{2n}$$

The proof is complete. □

**Theorem 0.2.** *The following equality holds:*

$$\sum_{n=0}^{\infty} \frac{\binom{4n}{2n}\binom{2n}{n}}{2^{6n}(2n+1)} = \frac{2}{\pi}\left(\ln(1+\sqrt{2}) - \ln(\sqrt{2}-1)\right)$$

*Proof.* Let us consider a function

$$(1+x)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n} x^n}{2^{2n}} \backslash \int_0^x dx$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n} x^n}{2^{2n}(n+1)} = \frac{2\sqrt{x+1}-2}{x} \tag{1}$$

Back to our problem, let us do a substitution $x \to -x$ in (1) , then we get

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n} x^n}{2^{2n}(n+1)} = \frac{2\sqrt{-x+1}-2}{-x} = \frac{2-2\sqrt{1-x}}{x} \tag{2}$$

Adding (1) and (2) and using the Lemma we get

$$\sum_{n=0}^{\infty} \frac{\binom{4n}{2n} x^{2n}}{2^{4n}(2n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n} x^n}{2^{2n}(n+1)} + \sum_{n=0}^{\infty} \frac{\binom{2n}{n} x^n}{2^{2n}(n+1)} = \frac{2\sqrt{x+1}-2\sqrt{1-x}}{2x}$$

Introducing a subsitution $x \to \sin(x)$ and integrating from 0 to $\frac{\pi}{2}$ and using the fact

$$\int_0^{\frac{\pi}{2}} \sin^{2n}(x) dx = \frac{\pi}{2} \frac{\binom{2n}{n}}{2^{2n}}$$

we get

$$\sum_{n=0}^{\infty} \frac{\binom{4n}{2n}}{2^{4n}(2n+1)} \cdot \frac{\pi}{2} \frac{\binom{2n}{n}}{2^{2n}} = \int_0^{\frac{\pi}{2}} \frac{2\sqrt{\sin(x)+1}-2\sqrt{1-\sin(x)}}{2\sin(x)} dx$$

Which can be solved by using a sub $\sin(x) = k => dx = \frac{dk}{\cos x}$

$$= \int_0^1 \frac{\sqrt{1+k}-\sqrt{1-k}}{k\sqrt{1-k^2}} dk$$

This can further be solved in the following way, first introduce a substitution $\sqrt{1-k} = k$ then $k = \sqrt{2}\sin(v)$ which evaluates to

$$= \Big(-\ln(|\sqrt{t+1}-1|)+\ln(\sqrt{t+1}+1)-\ln(\sqrt{1-t}+1)+\ln(|\sqrt{1-t}-1|)\Big)\Big|_0^1 = \dots \ln(1+\sqrt{2})-\ln(\sqrt{2}-1)$$

Which gives us

$$\sum_{n=0}^{\infty} \frac{\binom{4n}{2n}}{2^{4n}(2n+1)} \cdot \frac{\pi}{2} \frac{\binom{2n}{n}}{2^{2n}} = \ln(1+\sqrt{2}) - \ln(\sqrt{2}-1)$$

Which when cleared out of the constants, becomes

$$\sum_{n=0}^{\infty} \frac{\binom{4n}{2n}\binom{2n}{n}}{2^{6n}(2n+1)} = \frac{2}{\pi}\Big(\ln(1+\sqrt{2}) - \ln(\sqrt{2}-1)\Big)$$

The proof is complete. □

Using another representation of sin($x$) integral,

$$\int_0^{\frac{\pi}{2}} \sin^{2n-1}(x)dx = \frac{4^n}{\binom{2n}{n}} \cdot \frac{1}{2n}$$

We can get a different representation of the sum we derived in the last theorem.
At the point of evaluating the integral we will use the identity given here. The result doesnt change on the right hand side but the one on the left hand side does. We get

$$\sum_{n=0}^{\infty} \frac{\binom{4n}{2n} 4^{n+\frac{1}{2}}}{2^{4n}(2n+1)2(n+\frac{1}{2})\binom{2n+1}{n+\frac{1}{2}}} = \sum_{n=0}^{\infty} \frac{\binom{4n}{2n}\binom{2n}{n}}{2^{6n}(2n+1)} \frac{\pi}{2}$$

$$\sum_{n=0}^{\infty} \frac{\binom{4n}{2n}}{2^{2n}(2n+1)(n+\frac{1}{2})\binom{2n+1}{n+\frac{1}{2}}} = \sum_{n=0}^{\infty} \frac{\binom{4n}{2n}\binom{2n}{n}}{2^{6n}(2n+1)} \frac{\pi}{2}$$

**Theorem 0.3.** *The following equality holds for* $|x| < 4$.

$$\sum_{n=1}^{\infty} \frac{x^{2n}}{\binom{4n}{2n}} = \frac{2\sqrt{x}(x+4)\arctan\left(\frac{-\sqrt{x}}{2\sqrt{1-\frac{x}{4}}}\right)}{2\sqrt{1-\frac{x}{4}}(x^2-16)} + \frac{2\sqrt{x}(x-4)\arctan\left(\frac{\sqrt{x}}{2\sqrt{-1-\frac{x}{4}}}\right)}{2\sqrt{-1-\frac{x}{4}}(x^2-16)} - \frac{4x^2-6x^2+x^2+8}{(x^2-16)(-1)} + \frac{8}{(x^2-16)(-1)}$$

*Proof.* Let us observe the beta integral

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx.$$

We proceed as following

$$B(2n+1,2n) = \int_0^1 t^{2n}(1-t)^{2n-1}dt = \frac{\Gamma(2n+1)\Gamma(2n)}{\Gamma(4n+1)} = \frac{(2n)!(2n-1)!}{(4n)!} \cdot \frac{(2n)}{(2n)} = \frac{1}{\binom{4n}{2n} \cdot 2n}$$

We end up with

$$\frac{1}{\binom{4n}{2n} \cdot 2n} = B(2n+1,2n) \backslash \cdot x^{2n} \backslash \cdot \sum_{n=1}^{\infty}$$

Which we multiply by $x^{2n}$ and sum it from 1 to $\infty$.

$$\sum_{n=1}^{\infty} \frac{x^{2n}}{\binom{4n}{2n}} = \sum_{n=1}^{\infty} 2nB(2n+1,2n)x^{2n} = 2\sum_{n=1}^{\infty} \int_0^1 nx^{2n}t^{2n}(1-t)^{2n-1}dt = 2\int_0^1 tx \sum_{n=1}^{\infty} nt^{2n-1}x^{2n-1}(1-t)^{2n-1}dt =$$

Here we apply the formula

$$\sum_{n=1}^{\infty} nt^{2n-1} = \frac{t}{(t^2-1)^2}, |t| < 1$$

In our case $t = tx(1-t), 0 \leqslant t \leqslant 1$. The maximum of $t(1-t)$ is at $t = \frac{1}{2}$. Therefore we get

$$\left|\frac{x}{4}\right| < 1 => |x| < 4.$$

In our case, we get

$$= 2\int_0^1 tx \cdot \frac{tx(1-t)}{\left(\left(xt(1-t)\right)^2 - 1\right)^2} dt =$$

$$= \left( -\frac{\sqrt{x}\,(x+4)\arctan\left(\frac{\sqrt{x}(2t-1)}{2\sqrt{1-\frac{x}{4}}}\right)}{2\sqrt{1-\frac{x}{4}}\,(x^2-16)} + \frac{\sqrt{x}\,(x-4)\arctan\left(\frac{\sqrt{x}(2t-1)}{2\sqrt{-\frac{x}{4}-1}}\right)}{2\sqrt{-\frac{x}{4}-1}\,(x^2-16)} - \frac{4x^2t^3 - 6x^2t^2 + x^2t + 8}{(x^2-16)\left(x^2\,(t-1)^2\,t^2 - 1\right)} \right)\Bigg|_0^1 =$$

$$= \frac{2\sqrt{x}(x+4)\arctan\left(\frac{-\sqrt{x}}{2\sqrt{1-\frac{x}{4}}}\right)}{2\sqrt{1-\frac{x}{4}}(x^2-16)} + \frac{2\sqrt{x}(x-4)\arctan\left(\frac{\sqrt{x}}{2\sqrt{-1-\frac{x}{4}}}\right)}{2\sqrt{-1-\frac{x}{4}}(x^2-16)} - \frac{4x^2 - 6x^2 + x^2 + 8}{(x^2-16)(-1)} + \frac{8}{(x^2-16)(-1)}$$

What we can observe is that the domains cross in a real sense, so we will consider the negative square root, for which we can see will also come in pair with an imaginary part in arctan so that they will cancel themselves out. In the end, we get a real value for any $|x| < 4$.
The proof is complete. $\square$

Now we will give an example of the previously derived generating function.
**Example**
Let us consider the previously derived generating function

$$\sum_{n=1}^{\infty} \frac{x^{2n}}{\binom{4n}{2n}} = \frac{2\sqrt{x}(x+4)\arctan\left(\frac{-\sqrt{x}}{2\sqrt{1-\frac{x}{4}}}\right)}{2\sqrt{1-\frac{x}{4}}(x^2-16)} + \frac{2\sqrt{x}(x-4)\arctan\left(\frac{\sqrt{x}}{2\sqrt{-1-\frac{x}{4}}}\right)}{2\sqrt{-1-\frac{x}{4}}(x^2-16)} - \frac{4x^2 - 6x^2 + x^2 + 8}{(x^2-16)(-1)} + \frac{8}{(x^2-16)(-1)}.$$

Setting $x = \pi$ we get

$$\sum_{n=1}^{\infty} \frac{\pi^{2n}}{\binom{4n}{2n}} = \frac{2\csc^{-1}(\frac{2}{\sqrt{\pi}})}{(4-\pi)^{\frac{3}{2}}} - \frac{2\sqrt{\pi}\tanh^{-1}(\sqrt{\frac{\pi}{4+\pi}})}{(4+\pi)^{\frac{3}{2}}} + \frac{\pi^2}{16-\pi^2} \sim 6.316.$$

## References


[1] Lehmer D.H., *Interesting series involving the central binomial coefficient*, Amer. Math. Monthly 92 (1985), 449–457.



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Web:http://www.ssmrmh.ro
The Author: This article is published with open access vuk.stojiljkovic999@gmail.com
University of Novi Sad
Novi Sad, Serbia