R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

Prove the integrals:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sin\left(ay^2+\frac{b}{x^2}\right)\cos\left(ax^2+\frac{b}{y^2}\right)dy\,dx=\frac{\pi\cos\left(4\sqrt{ab}\right)}{2a}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sin\left(ay^2+\frac{b}{x^2}\right)\cos\left(ax^2+\frac{b}{y^2}\right)\frac{dydx}{x^2y^2}=\frac{\pi\cos\left(4\sqrt{ab}\right)}{2a}$$

$$a,b>0$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution 1 by Max Wong-Hong Kong, Solution 2 by Syed Shahabudeen-Kerala-India

Solution 1 by Max Wong-Hong Kong

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sin\left(ay^2+\frac{b}{x^2}\right)\cos\left(ax^2+\frac{b}{y^2}\right)dy\,dx=$$

$$=4\int_0^{\infty}\int_0^{\infty}\left(\sin(ay^2)\cos\left(\frac{b}{x^2}\right)+\cos(ay^2)\sin\left(\frac{b}{x^2}\right)\right)\left(\cos(ax^2)\cos\left(\frac{b}{y^2}\right)\right.$$

$$\left.-\sin(ax^2)\sin\left(\frac{b}{y^2}\right)\right)dydx=$$

$$=4\int_0^{\infty}\sin(ay^2)\cos\left(\frac{b}{y^2}\right)dy\cdot\int_0^{\infty}\cos(ax^2)\cos\left(\frac{b}{x^2}\right)dx+$$

$$+4\int_0^{\infty}\cos(ay^2)\cos\left(\frac{b}{y^2}\right)dy\cdot\int_0^{\infty}\cos(ax^2)\sin\left(\frac{b}{x^2}\right)dx-$$

$$-4\int_0^\infty \sin(ax^2)\sin\left(\frac{b}{x^2}\right)dx\cdot\int_0^\infty \cos(ay^2)\sin\left(\frac{b}{y^2}\right)dy-$$

$$-4\int_0^\infty \sin(ax^2)\cos\left(\frac{b}{x^2}\right)dx\cdot\int_0^\infty \sin(ay^2)\sin\left(\frac{b}{y^2}\right)dy=$$

$$=4I_{cc}I_{sc}+4I_{cc}I_{cs}-4I_{ss}I_{cs}-4I_{sc}I_{ss}=4(I_{sc}+I_{cs})(I_{cc}-I_{ss})$$

$$I_{sc}+I_{cs}=\int_0^\infty\left(\sin(ax^2)\cos\left(\frac{b}{x^2}\right)+\cos(ax^2)\sin\left(\frac{b}{x^2}\right)\right)dx=$$

$$=\int_0^\infty \sin\left(ax^2+\frac{b}{x^2}\right)dx=\int_0^\infty \sin\left(\left(\sqrt{a}x-\frac{\sqrt{b}}{x}\right)^2+2\sqrt{ab}\right)dx\overset{CST}{=}$$

$$=\frac{1}{\sqrt{a}}\int_0^\infty \sin\left(x^2+2\sqrt{ab}\right)dx=$$

$$=\frac{\cos\left(2\sqrt{ab}\right)}{\sqrt{a}}\int_0^\infty \sin(x^2)\,dx+\frac{\sin\left(2\sqrt{ab}\right)}{\sqrt{a}}\int_0^\infty \cos(x^2)\,dx\overset{Fresnel}{=}$$

$$=\frac{\sqrt{\pi}}{2\sqrt{2a}}\left(\cos\left(2\sqrt{ab}\right)+\sin\left(2\sqrt{ab}\right)\right)$$

Similarly,

$$I_{cc}-I_{ss}=\int_0^\infty \cos\left(ax^2+\frac{b}{x^2}\right)dx=$$

$$=\int_0^\infty \cos\left(\left(\sqrt{a}x-\frac{\sqrt{b}}{x}\right)^2+2\sqrt{ab}\right)dx\overset{CST}{=}\frac{1}{\sqrt{a}}\int_0^\infty \cos\left(x^2+2\sqrt{ab}\right)dx=$$

$$=\frac{\cos\left(2\sqrt{ab}\right)}{\sqrt{a}}\int_0^\infty \cos(x^2)\,dx-\frac{\sin\left(2\sqrt{ab}\right)}{\sqrt{a}}\int_0^\infty \sin(x^2)\,dx=$$

$$=\frac{\sqrt{\pi}}{2\sqrt{2a}}\left(\cos\left(2\sqrt{ab}\right)-\sin\left(2\sqrt{ab}\right)\right)$$

Then,

$$\int_{-\infty}^\infty\int_{-\infty}^\infty \sin\left(ay^2+\frac{b}{x^2}\right)\cos\left(ax^2+\frac{b}{y^2}\right)dy\,dx=\frac{\pi\cos\left(4\sqrt{ab}\right)}{2a}$$

Let $I(a,b)=\int_{-\infty}^\infty\int_{-\infty}^\infty \sin\left(ay^2+\frac{b}{x^2}\right)\cos\left(ax^2+\frac{b}{y^2}\right)dy\,dx=\frac{\pi\cos\left(4\sqrt{ab}\right)}{2a}$**, then**

$$\int_{-\infty}^\infty\int_{-\infty}^\infty \sin\left(ay^2+\frac{b}{x^2}\right)\cos\left(ax^2+\frac{b}{y^2}\right)\frac{dydx}{x^2y^2}=$$

$$=4\int_0^\infty\int_0^\infty \sin\left(ay^2+\frac{b}{x^2}\right)\cos\left(ax^2+\frac{b}{y^2}\right)\frac{dydx}{x^2y^2}\overset{u=\frac{1}{x};v=\frac{1}{y}}{=}$$

$$=4\int_0^\infty\int_0^\infty \sin\left(bu^2+\frac{a}{v^2}\right)\cos\left(bv^2+\frac{a}{u^2}\right)dvdu\overset{Fubini}{=}$$

$$=4\int_0^\infty\int_0^\infty \sin\left(bu^2+\frac{a}{v^2}\right)\cos\left(bv^2+\frac{a}{u^2}\right)dudv=I(b,a)=\frac{\pi\cos(4\sqrt{ab})}{2b}$$

Solution 2 by Syed Shahabudeen-Kerala-India

$$\Omega=\int_{-\infty}^\infty\int_{-\infty}^\infty \sin\left(ay^2+\frac{b}{x^2}\right)\cos\left(ax^2+\frac{b}{y^2}\right)dy\,dx$$

$$2\Omega=\int_{-\infty}^\infty\int_{-\infty}^\infty \sin\left(ay^2+\frac{b}{x^2}\right)\cos\left(ax^2+\frac{b}{y^2}\right)dy\,dx+$$

$$+\int_{-\infty}^\infty\int_{-\infty}^\infty \sin\left(ax^2+\frac{b}{y^2}\right)\cos\left(ay^2+\frac{b}{x^2}\right)dy\,dx=$$

$$=\int_{-\infty}^\infty\int_{-\infty}^\infty \sin\left(\left(ay^2+\frac{b}{x^2}\right)+\left(ax^2+\frac{b}{y^2}\right)\right)dydx=$$

$$=\int_{-\infty}^\infty\int_{-\infty}^\infty \sin\left(\left(ax^2+\frac{b}{x^2}\right)+\left(ay^2+\frac{b}{y^2}\right)\right)dydx$$

$$\Omega=\int_{-\infty}^\infty\int_{-\infty}^\infty \sin\left(ax^2+\frac{b}{x^2}\right)\cos\left(ay^2+\frac{b}{y^2}\right)dydx=$$

$$=4\int_0^\infty\int_0^\infty \sin\left(ax^2+\frac{b}{x^2}\right)\cos\left(ay^2+\frac{b}{y^2}\right)dydx=$$

$$=4\int_0^\infty \sin\left(ax^2+\frac{b}{x^2}\right)dx\int_0^\infty \cos\left(ay^2+\frac{b}{y^2}\right)dy=I_s\cdot I_c$$

$$I_s=\int_0^\infty \sin\left(\left(\sqrt{a}x-\frac{\sqrt{b}}{x}\right)^2+2\sqrt{ab}\right)dx\overset{x=\frac{\sqrt{b}}{t\sqrt{a}}}{=}$$

$$=\int_0^\infty \sin\left(\left(\sqrt{a}t-\frac{\sqrt{b}}{t}\right)^2+2\sqrt{ab}\right)\left(\frac{\sqrt{b}}{t^2\sqrt{a}}\right)dt$$

$$2I_s=\int_{-\infty}^\infty \sin\left(\left(\sqrt{a}x-\frac{\sqrt{b}}{x}\right)^2+2\sqrt{ab}\right)\left(1+\frac{\sqrt{b}}{x^2\sqrt{a}}\right)dx\overset{s=\sqrt{a}x-\frac{\sqrt{b}}{x}}{=}$$

$$I_s=\frac{1}{\sqrt{a}}\int_0^\infty \sin\left(s^2+2\sqrt{ab}\right)ds=\frac{\sqrt{\pi}}{2\sqrt{2a}}\left(\cos\left(2\sqrt{ab}\right)+\sin\left(2\sqrt{ab}\right)\right)$$

Similarly,

$$I_c=\frac{\sqrt{\pi}}{2\sqrt{2a}}\left(\cos\left(2\sqrt{ab}\right)-\sin\left(2\sqrt{ab}\right)\right)$$

$$\Omega=4\left(\frac{\sqrt{\pi}}{2\sqrt{2a}}\right)^2\left(\frac{\sqrt{\pi}}{2\sqrt{2a}}\left(\cos^2\left(2\sqrt{ab}\right)-\sin^2\left(2\sqrt{ab}\right)\right)\right)=\frac{\pi}{2a}\cos\left(4\sqrt{ab}\right)$$

$$\Phi=\int_{-\infty}^\infty\int_{-\infty}^\infty \sin\left(ay^2+\frac{b}{x^2}\right)\cos\left(ax^2+\frac{b}{y^2}\right)\frac{dydx}{x^2y^2}=$$

$$=\int_{-\infty}^\infty\int_{-\infty}^\infty \sin\left(ax^2+\frac{b}{x^2}\right)\cos\left(ay^2+\frac{b}{y^2}\right)\frac{dydx}{x^2y^2}=$$

$$=4\int_0^\infty \sin\left(ax^2+\frac{b}{x^2}\right)\frac{dx}{x^2}\int_0^\infty \cos\left(ay^2+\frac{b}{y^2}\right)\frac{dy}{y^2}=\phi_s\cdot\phi_c$$

$$\phi_s=\int_0^\infty \sin\left(ax^2+\frac{b}{x^2}\right)\frac{dx}{x^2}\overset{x=\frac{\sqrt{b}}{t\sqrt{a}}}{=}$$

$$=\sqrt{\frac{a}{b}}\int_0^\infty \sin\left(\left(\sqrt{a}x-\frac{\sqrt{b}}{x}\right)^2+2\sqrt{ab}\right)\left(\frac{\sqrt{b}}{x^2\sqrt{a}}\right)dx=$$

$$=\sqrt{\frac{a}{b}}\int_0^\infty \sin\left(\left(\sqrt{a}x-\frac{\sqrt{b}}{x}\right)^2+2\sqrt{ab}\right)dx=\sqrt{\frac{a}{b}}\int_0^\infty \sin\left(ax^2+\frac{b}{x^2}\right)dx=$$

$$=\frac{\sqrt{\pi}}{2\sqrt{2b}}\left(\cos\left(2\sqrt{ab}\right)+\sin\left(2\sqrt{ab}\right)\right)$$

Similarly,

$$\phi_c=\frac{\sqrt{\pi}}{2\sqrt{2b}}\left(\cos\left(2\sqrt{ab}\right)-\sin\left(2\sqrt{ab}\right)\right)$$

Hence,

$$\Phi=4\left(\frac{\sqrt{\pi}}{2\sqrt{2b}}\right)^2\cos\left(4\sqrt{ab}\right)=\frac{\pi\cos\left(4\sqrt{ab}\right)}{2b}$$