R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

O − circumcenter, I − incenter in $\Delta DABC$. Prove that :

$$\sum_{cyc}\frac{\cos\left(\widehat{OAI}\right)}{a^2+9R^2}\le\frac{1}{16r^2}.$$

Proposed by Gheorghe Molea-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Since:\ \cos x\le 1,\forall x\in R\ \ then\ \ \sum_{cyc}\frac{\cos\left(\widehat{OAI}\right)}{a^2+9R^2}\le\sum_{cyc}\frac{1}{a^2+9R^2}$$

$$By\ CBS\ inequality\ we\ have:\ (a^2+9R^2)\left(\frac{1}{a^2}+\frac{1}{R^2}\right)\ge(1+3)^2=16.$$

$$Then:\ \frac{1}{a^2+9R^2}\le\frac{1}{16}\left(\frac{1}{a^2}+\frac{1}{R^2}\right)\ \ (And\ analogs)$$

$$Thus,\quad \sum_{cyc}\frac{\cos\left(\widehat{OAI}\right)}{a^2+9R^2}\le\sum_{cyc}\frac{1}{a^2+9R^2}\le\frac{1}{16}\left(\sum_{cyc}\frac{1}{a^2}+\frac{3}{R^2}\right)$$

$$We\ know\ that:\ h_a=\frac{2sr}{a}\le\sqrt{s(s-a)}\ \ then\ \ \frac{1}{a^2}\le\frac{s-a}{4sr^2}\ \ (And\ analogs)\ \ and\ \ R\ge 2r.$$

$$Therefore,\quad \sum_{cyc}\frac{\cos\left(\widehat{OAI}\right)}{a^2+9R^2}\le\frac{1}{16}\left(\sum_{cyc}\frac{s-a}{4sr^2}+\frac{3}{(2r)^2}\right)=\frac{1}{16}\left(\frac{1}{4r^2}+\frac{3}{4r^2}\right)=\frac{1}{16r^2}.$$