R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

INEQUALITIES FOR RECURRENT SEQUENCES

By Florică Anastase-Romania

Abstract: *In this paper are presented few applications for inequalities with recurrent sequences of real numbers.*

Ap.1) Let $(a_n)_{n\geq 0}$ be sequences of real numbers strictly positive such that $a_0 = 1$and $a_{n+1} + 2a_{n+1}a_n \leq a_n$. Find $\lim\limits_{n\to\infty} a_n$.

Solution. We have:

$$a_{n+1} + 2a_{n+1}a_n \leq a_n \Leftrightarrow 2a_{n+1} + 4a_{n+1}a_n \leq 2a_n \Leftrightarrow$$

$$2a_{n+1} + 6a_{n+1}a_n \leq 2a_n + 2a_{n+1}a_n \Leftrightarrow 2a_{n+1}(1+3a_n) \leq 2a_n(1+a_{n+1}) \Leftrightarrow$$

$$\frac{2a_{n+1}}{1+a_{n+1}} \leq \frac{2a_n}{1+3a_n} \Leftrightarrow \frac{2a_{n+1}}{1+a_{n+1}} \leq \frac{\frac{2a_n}{1+a_n}}{1+\frac{2a_n}{1+a_n}}.$$

$$\text{Let } x_n = \frac{2a_n}{1+a_n}, (a_0 = 1) \Rightarrow x_0 = 1, \text{then:}$$

$$x_n > 0; \forall n \in \mathbb{N} \text{ and } x_{n+1} \leq \frac{x_n}{1+x_n}$$

$$\frac{x_{n+1}}{x_n} \leq \frac{1}{1+x_n} < 1 \Rightarrow (x_n)_{n\geq 0} \searrow \text{and from } x_n > 0; \forall n \in \mathbb{N} \Rightarrow \exists l \in \mathbb{R}$$

$$\lim_{n\to\infty} x_n = l \text{ and how } x_{n+1} \leq \frac{x_n}{1+x_n}, \text{we get: } l \leq \frac{l}{1+l} \Leftrightarrow l^2 \leq 0 \text{ but } l > 0, \text{then:}$$

$$x_n = \frac{2a_n}{1+a_n} \Leftrightarrow x_n + a_nx_n = 2a_n \Rightarrow a_n(x_n - 2) = -x_n \Rightarrow a_n = \frac{x_n}{2-x_n}$$

$$\lim_{n\to\infty} x_n = 0 \text{ and hence, } \lim_{n\to\infty} a_n = \lim_{x_n\to 0} \frac{x_n}{2-x_n} = 0$$

Ap.2) Let $(a_n)_{n\geq 0}$ be sequences of real numbers strictly positive such that $a_0 = 1 - a, a \in [0,1]$ and $(1+a)a_{n+1} + a(1-a_n) + a_{n+1}a_n \leq a_n^2$.

Find $\lim\limits_{n\to\infty} a_n^n$.

Solution. We have:

$$(1+a)a_{n+1}+a(1-a_n)+a_{n+1}a_n \le a_n^2 \Leftrightarrow$$
$$a+a_{n+1} \le aa_n - aa_{n+1} + a_n^2 - a_{n+1}a_n \Leftrightarrow$$
$$a+a_{n+1} \le (a+a_n)(a_n-a_{n+1}) \Leftrightarrow$$
$$a+a_{n+1} \le (a+a_n)(a+a_n-a-a_{n+1}) \Leftrightarrow$$
$$(a+a_{n+1})(1+a+a_n) \le (1+a_n)^2 \Leftrightarrow a+a_{n+1} \le \frac{(a+a_n)^2}{a+1+a_n}$$

Let $x_n = a+a_n$, then $x_{n+1} \le \frac{x_n^2}{1+x_n}$; $x_0 = 1$ and $x_n > 0; \forall n \in \mathbb{N}$

$\frac{x_{n+1}}{x_n} \le \frac{x_n}{1+x_n} < 1$, then $(x_n)_{n\in\mathbb{N}} \searrow$. Thus, $\exists l \in \mathbb{R}_+, l = \lim_{n\to\infty} x_n$ and from $x_{n+1} \le \frac{x_n^2}{1+x_n}$, we get: $l \le \frac{l^2}{1+l} \Leftrightarrow l \le 0$, but $l > 0$, then $l = 0$.

So, we have:

$$\lim_{n\to\infty} a_n^n = \lim_{n\to\infty}(x_n-a)^n = \lim_{n\to\infty}(-a)^n = 0; a \in [0,1]$$

Ap.3) Let $(a_n)_{n\ge0}$ be sequences of real numbers strictly positive such that

$$a_n^2 - 2a_{n+1} \le 2\lambda a_n - (\lambda+1)^2 \le 2a_{n+1}, \lambda > 0. \text{ Find } \lim_{n\to\infty} a_n.$$

Solution. We have:

$$a_n^2 - 2a_{n+1} \le 2\lambda a_n - (\lambda+1)^2 \Leftrightarrow 2a_{n+1} \ge a_n^2 - 2\lambda a_n + (\lambda+1)^2 \Leftrightarrow$$
$$2a_{n+1} \ge a_n^2 - 2\lambda a_n + \lambda^2 + 2\lambda + 1 \Leftrightarrow 2a_{n+1} \ge (a_n-\lambda)^2 + \lambda + 1 > 0.$$

Therefore, $a_n > 0; \forall n \in \mathbb{N}; (1)$

Now, we have:

$$2a_{n+1} - 2a_n \ge a_n^2 - 2\lambda a_n + (\lambda+1)^2 - 2a_n \Leftrightarrow$$
$$2(a_{n+1}-a_n) \ge (a_n-\lambda-1)^2 > 0 \Rightarrow (a_n)_{n\ge0} \nearrow; (2)$$

From (1) and (2) it follows that $(a_n)_{n\ge0}$ is convergent, then $\exists l \in \mathbb{R}_+$ such that $\lim_{n\to\infty} a_n = l$ and from $a_n^2 - 2a_{n+1} \le 2\lambda a_n - (\lambda+1)^2 \le 2a_{n+1}$, we get $l^2 - 2l \le 2l \Leftrightarrow$

$$l^2 - 4l \le 0 \Leftrightarrow l \in [0,4].$$

Ap.4) Let $(x_n)_{n\geq 1}$ be sequence of real numbers such that $x_0 = 1$ and

$$x_{n+1} \geq x_n; \forall n \in \mathbb{N}.$$

$$\text{Then: } \sum_{k=1}^{n}\left(1-\frac{x_{k-1}}{x_k}\right)\cdot\frac{1}{\sqrt[m]{x_k}} < m, \forall n \in \mathbb{N}^*, \forall m \geq 2.$$

Solution. We have:

$$\left(1-\frac{x_{k-1}}{x_k}\right)\cdot\frac{1}{\sqrt[m]{x_k}} = \frac{x_k - x_{k-1}}{x_k \cdot \sqrt[m]{x_k}} =$$

$$= \frac{\sqrt[m]{x_k} - \sqrt[m]{x_{k-1}}}{x_k \cdot \sqrt[m]{x_k}} \cdot \left(\sqrt[m]{x_k^{m-1}} + \sqrt[m]{x_k^{m-2}\cdot x_{k-1}} + \cdots + \sqrt[m]{x_k \cdot x_{k-1}^{m-2}} + \sqrt[m]{x_{k-1}^{m-1}}\right) \leq$$

$$\leq \frac{\left(\sqrt[m]{x_k} - \sqrt[m]{x_{k-1}}\right)\cdot m\sqrt[m]{x_k^{m-1}}}{x_k \cdot \sqrt[m]{x_k}} = \frac{m\left(\sqrt[m]{x_k} - \sqrt[m]{x_{k-1}}\right)}{\sqrt[m]{x_k}\cdot\sqrt[m]{x_k}} \leq \frac{m\left(\sqrt[m]{x_k} - \sqrt[m]{x_{k-1}}\right)}{\sqrt[m]{x_k}\cdot\sqrt[m]{x_{k-1}}} =$$

$$= m\left(\frac{1}{\sqrt[m]{x_{k-1}}} - \frac{1}{\sqrt[m]{x_k}}\right); \forall k \in \mathbb{N}^*$$

Therefore,

$$\sum_{k=1}^{n}\left(1-\frac{x_{k-1}}{x_k}\right)\cdot\frac{1}{\sqrt[m]{x_k}} \leq m\cdot\sum_{k=1}^{n}\left(\frac{1}{\sqrt[m]{x_{k-1}}} - \frac{1}{\sqrt[m]{x_k}}\right) = m\left(\frac{1}{\sqrt[m]{x_0}} - \frac{1}{\sqrt[m]{x_n}}\right) =$$

$$= m\left(1 - \frac{1}{\sqrt[m]{x_n}}\right) < m$$

Ap.5) Let $(a_n)_{n\geq 0}$ be sequences of real numbers strictly positive such that

$a_1 = 1$ and $(n+1)(a_n - a_{n+1}) \geq a_{n+1} + a_n a_{n+1}$. Find: $\lim\limits_{n\to\infty} a_n$.

Solution.

$$(n+1)(a_n - a_{n+1}) \geq a_{n+1} + a_n a_{n+1} \Leftrightarrow$$

$$(n+1)\cdot\frac{a_n - a_{n+1}}{a_n a_{n+1}} \geq \frac{1+a_n}{a_n} \Leftrightarrow$$

$$(n+1)\left(\frac{1}{a_{n+1}} - \frac{1}{a_n}\right) \geq 1 + \frac{1}{a_n} \Leftrightarrow$$

Let $x_n = \frac{1}{a_n}$, then $x_1 = 1$ and $(n+1)(x_{n+1} - x_n) \geq 1 + x_n; \forall n \in \mathbb{N}^*$

$$x_{n+1}-x_n \geq \frac{1}{n+1}+\frac{x_n}{n+1} > \frac{1}{n+1} > 0; \forall n \in \mathbb{N}^* \Rightarrow (x_n)_{n\geq 1} \nearrow. \text{So,}$$

$$\sum_{k=1}^{n}(x_{k+1}-x_k) > \sum_{k=1}^{n}\frac{1}{k+1}; \forall k \in \mathbb{N}^*$$

Therefore,

$$x_{n+1}-x_1 > \frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+1} \Leftrightarrow x_{n+1} > x_1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+1}; \forall n \geq 1 \Leftrightarrow$$

$$x_{n+1} > \sum_{k=1}^{n}\frac{1}{k} \Rightarrow \lim_{n\to\infty} x_n = +\infty \Rightarrow \lim_{n\to\infty} a_n = 0.$$

Ap.6) Let $(x_n)_{n\geq 1}$ be sequence of real numbers strictly positive such that

$$\boldsymbol{x_{n+1}^3 < 3 \cdot x_n - 2, \forall n \in \mathbb{N}. \text{ Find } \lim_{n\to\infty} x_n.}$$

Solution. We have:

$$x_{n+1}^3 < 3 \cdot x_n - 2, \forall n \in \mathbb{N} \Leftrightarrow 3x_n \geq x_{n+1}^3 + 2 \geq 3 \cdot \sqrt[3]{1 \cdot 1 \cdot x_{n+1}^3} = 3 \cdot x_{n+1}, \forall n \in \mathbb{N}$$

$(x_n)_{n\geq 1} \nearrow$ and $x_n \geq 0, \forall n \in \mathbb{N}$, then $\exists l \in \mathbb{R}$ such that $\lim_{n\to\infty} x_n = l$.

Hence, we have: $x^3 < 3x - 2 \Leftrightarrow x^3 - 3x + 2 < 0 \Leftrightarrow$

$$(x-1)(x^2+x-2) < 0 \Leftrightarrow (x-1)^2(x+2) < 0 \Leftrightarrow x = 1.$$

Ap.7) Let $(a_n)_{n\geq 1}$ be sequence of real numbers strictly positive such that

$$\boldsymbol{(n+2)a_{n+2} + 2n + 4 \leq n a_{n+1} + a_n. \text{ Find } \lim_{n\to\infty} a_n.}$$

Solution.

$$(n+2)a_{n+2} + 2n + 4 \leq n a_{n+1} + a_n$$

$$n^2 + 4n + 4 + (n+2)a_{n+2} \leq n^2 + n + n a_{n+1} + n + a_n$$

$$(n+2)^2 + (n+2)a_{n+2} \leq n(n+1) + n a_{n+1} + n + a_n$$

$$(n+2)(n+2+a_{n+2}) \leq n(n+1+a_{n+1}) + n + a_n$$

Let $x_n = n + a_n$, then $(n+2)x_{n+2} \leq n \cdot x_{n+1} + x_n, \forall n \in \mathbb{N}$.

$$(n+2)x_{n+2} + x_{n+1} \leq (n+1)x_{n+1} + x_n, \forall n \in \mathbb{N}.$$

Denoting $b_n = (n+1)x_{n+1} + x_n, \forall n \in \mathbb{N}$, then $b_{n+1} \leq b_n, \forall n \in \mathbb{N}$ or $(x_n)_{n\geq 1} \searrow$.

Because $b_n > 0, \forall n \in \mathbb{N}$, then $(b_n)_{n\geq 1}$ is convergent.

$$b_n = (n+1)x_{n+1} + x_n, \forall n \in \mathbb{N} \Leftrightarrow x_{n+1} + \frac{x_n}{n+1} = \frac{b_n}{n+1}, \forall n \in \mathbb{N}$$

$$\lim_{n\to\infty} \frac{b_n}{n+1} = 0. \text{So}, 0 < x_{n+1} < \frac{b_n}{n+1}, \text{then } (x_n)_{n\geq 1} \text{ is convergent.}$$

Therefore, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (n + x_n) = +\infty$.

Ap.8) Let $(a_n)_{n\geq 1}$ be sequence of real numbers strictly positive. Prove that if

$$\lim_{n\to\infty} \frac{a_{n+2}\cdot a_n}{a_{n+1}^2} = \lambda, \lambda \geq 1, \text{then } \lim_{n\to\infty} \sqrt[n(n+1)]{a_n} \geq 1.$$

Solution. We have:

$$x_n = \log\left(\sqrt[n(n+1)]{a_n}\right) = \frac{\log a_n}{n(n+1)}$$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{\log a_{n+1} - \log a_n}{2n+1} = \frac{1}{2}\cdot \lim_{n\to\infty} \frac{\log\left(\frac{a_{n+1}}{a_n}\right)}{n+1} =$$

$$= \lim_{n\to\infty} \frac{\log\left(\frac{a_{n+2}}{a_{n+1}}\right) - \log\left(\frac{a_{n+1}}{a_n}\right)}{2n} = \frac{1}{2}\cdot \lim_{n\to\infty} \log\left(\frac{a_{n+2}\cdot a_n}{a_{n+1}}\right) = \frac{1}{2}\log\lambda = \log\sqrt{\lambda}$$

Therefore, $\lim_{n\to\infty} \sqrt[n(n+1)]{a_n} = e^{\log\sqrt{\lambda}} = \sqrt{\lambda} \geq 1$.

Ap.9) Let $(x_n)_{n\geq 1}$ be an solution of the equation $f_n(x) = 0$, where

$f_n: (0,\infty) \to \mathbb{R}, f_n(x) = x^n + \log x, \forall n \in \mathbb{N}^*$. Prove that:

$$\sqrt[n]{1 - x_n} < 1 - \frac{x_n}{n} - \frac{n-1}{2n^2}\cdot x_n^2$$

Solution.

$$f_n(x) = x^n + \log x, \forall n \in \mathbb{N}^*, \text{then } f_n'(x) = nx^{n-1} + \frac{1}{x} > 0, \forall x > 0$$

f_n –strictly increasing, hence f_n is an injective function.

$$f_n(0+0) = -\infty, \lim_{x\to\infty} f_n(x) = +\infty,$$

f_n –continuous function, then $f\big((0,\infty)\big) \subset \mathbb{R}$, so f_n –is an surjection. Hence, the equation$f_n(x) = 0$ has an only solution $x_n(0,1), \forall n \in \mathbb{N}^*$.

$$\sqrt[n]{1 - x_n} < 1 - \frac{x_n}{n} - \frac{n-1}{2n^2}\cdot x_n^2 \Leftrightarrow$$

$$\frac{1}{n}\cdot\log(1-x_n)<\log\left(1-\frac{x_n}{n}-\frac{(n-1)x_n^2}{2n^2}\right)$$

$$x_n\in(0,1)\Rightarrow 1-\frac{x_n}{n}-\frac{(n-1)x_n^2}{2n^2}>0$$

Let be the function $g:[0,1]\to\mathbb{R}$,

$$g(x)=\log\left(1-\frac{x}{n}-\frac{(n-1)x^2}{2n^2}\right)-\frac{\log(1-x)}{n}$$

$$g'(x)=\frac{-\left(\frac{1}{n}+\frac{(n-1)x}{n^2}\right)}{1-\frac{x}{n}-\frac{(n-1)x^2}{2n^2}}+\frac{1}{n(1-x)}=\frac{\frac{(n-1)x^2}{n}\left(1-\frac{1}{2n}\right)}{n(1-x)\left(1-\frac{x}{n}-\frac{(n-1)x^2}{2n^2}\right)}>0$$

So, g —is an increasing function on $[0,1]$ and $g(0)=0$, then $g(x)>0,\forall x\in(0,1]$.

Ap.10) Let $(a_n)_{n\ge1}$ be sequence of real numbers such that

$$\frac{1}{8}(6a_n-a_{n-1})\le a_{n+1}\le\frac{1}{6}(5a_n-a_{n-1}),\forall n\ge1.\text{ Find }\lim_{n\to\infty}a_n.$$

Solution.

$$\frac{1}{8}(6a_n-a_{n-1})\le a_{n+1}\le\frac{1}{6}(5a_n-a_{n-1}),\forall n\ge1\Rightarrow$$

$$\frac{1}{8}(6a_n-a_{n-1})\le\frac{1}{6}(5a_n-a_{n-1}),\forall n\ge1\Leftrightarrow 2a_n\ge a_{n-1};(1),\forall n\ge1$$

$$\frac{1}{8}(6a_n-a_{n-1})\le a_{n+1},\forall n\ge1\Rightarrow-\frac{1}{8}(2a_n+a_{n-1})\le a_{n+1}-a_n;(2)$$

$$a_{n+1}\le\frac{1}{6}(5a_n-a_{n-1}),\forall n\ge1\Rightarrow a_{n+1}-a_n\le-\frac{1}{6}(a_n+a_{n-1});(3)$$

Hence,

$$-\frac{1}{8}(2a_n+a_{n-1})\le a_{n+1}-a_n\le-\frac{1}{6}(a_n+a_{n-1}),\forall n\ge1;(4)$$

Case I) $\exists m\in\mathbb{N}$ such that $a_m\ge1$, from (1), it follows that $2a_n\ge a_m\ge0,\forall n\ge m$ and from (4), it follows $a_{n+1}-a_n\le-\frac{1}{6}(a_n+a_{n-1}),\forall n\ge m$, then $(a_n)_{n\ge1}$ is convergent to 0.

Case 2) $\forall n\in\mathbb{N},a_n\ge0$ and from (1) and (2):

$a_{n+1}-a_n\ge-\frac{1}{8}(2a_n+a_{n-1})\ge0$, then $(a_n)_{n\ge1}$ is convergent to 0.

REFERENCES:

[1] ABOUT RECURRENCE RELATIONSHIPS FOR REAL NUMBER SEQUENCES-(I)-F.Anastase-www.ssmrmh.ro

[2] ABOUT RECURRENCE RELATIONSHIPS FOR REAL NUMBER SEQUENCES-(II)-F.Anastase-www.ssmrmh.ro

[3] OLYMPIC MATHEMATICAL ENERGY- M. Bencze, D. Sitaru-Studis, 2018

[4] 699 OLYMPIC MATHEMATICAL CHALLENGES- M. Bencze, D. Sitaru-Studis, 2017

[5] ȘIRURI-D.M. Bătinețu-Giurgiu, Albatros 1979

[6] OLYMPIC MATHEMATICAL POWER-M. Bencze, D. Sitaru, M. Ursărescu-Studis, 2018

[7] ANALIZĂ MATEMATICA –D.M.Bătineţu-Giurgiu & Co. –MatrixRom,2003

[8] OLIMPIC MATHEMATICAL BEAUTIES-M. Bencze, D. Sitaru, M. Ursărescu-Studis, 2018

[9] QUANTUM MATHEMATICAL POWER- M. Bencze, D. Sitaru-Studis, 2018

[10] ROMANIAN MATHEMATICAL MAGAZINE CHALLENGES 1-500-D. Sitaru, M. Ursărescu-Studis 2021

[11] ANALITYCAL PHENOMENON-D. Sitaru-Studis, 2018

[12]ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro