

**Abstract:** In this paper are presented few special techniques for recurrent limits.

**Ap.1) Find:**

$$\Omega_n = \lim_{x \rightarrow 0} \frac{1 - \sqrt{1+x^2} \cdot \sqrt[3]{1+x^2} \cdot \dots \cdot \sqrt[n]{1+x^2}}{x^2}, n \in \mathbb{N}$$

**Solution.** Let be  $f_k(x) = \sqrt[k]{1+x^2}, k \in \overline{2, n}, n \in \mathbb{N}$ , then

$$\begin{aligned} f'_k(x) &= \frac{2x}{k \sqrt[k]{(1+x^2)^{k-1}}} \\ \Omega_n &= \lim_{x \rightarrow 0} \frac{1 - \sqrt{1+x^2} \cdot \sqrt[3]{1+x^2} \cdot \dots \cdot \sqrt[n]{1+x^2}}{x^2} = \\ &= \lim_{x \rightarrow 0} \frac{1 - \prod_{k=2}^n f_k(x)}{x^2} = \lim_{x \rightarrow 0} \frac{(1 - \prod_{k=2}^n f_k(x))'}{2x} = \\ &= - \lim_{x \rightarrow 0} \frac{(\prod_{k=2}^n f_k(x))'}{2x} = - \frac{1}{2} \lim_{x \rightarrow 0} \left[ \prod_{k=2}^n f_k(x) \cdot \sum_{k=2}^n \left( \frac{f'_k(x)}{f_k(x)} \cdot \frac{1}{x} \right) \right] = \\ &= - \frac{1}{2} \sum_{k=2}^n \lim_{x \rightarrow 0} \frac{f'_k(x)}{x} = - \frac{1}{2} \sum_{k=2}^n \frac{2}{k \sqrt[k]{(1+x)^{k-1}}} = - \sum_{k=1}^n \frac{1}{k} \end{aligned}$$

**Ap.2) Find:**

$$\Omega_n = \lim_{x \rightarrow \infty} \frac{1 - \cos x \cdot \cos 2x \cdot \dots \cdot \cos nx}{x^2}, n \in \mathbb{N}^*$$

**Solution.** For  $n = 1$ , we have:

$$\Omega_1 = \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{x}{2}}{x^2} = \frac{1}{2} \lim_{x \rightarrow 0} \left( \frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2 = \frac{1}{2}$$

Now, we have:

$$\Omega_n - \Omega_{n-1} = \lim_{x \rightarrow 0} \frac{(1 - \cos nx) \cos x \cdot \cos 2x \cdot \dots \cdot \cos(n-1)x}{x^2} =$$

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$$= \lim_{x \rightarrow 0} \frac{1 - \cos nx}{x^2} = \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{nx}{2}}{x^2} = \frac{n^2}{2} \lim_{x \rightarrow 0} \left( \frac{\sin \frac{nx}{2}}{\frac{nx}{2}} \right)^2 = \frac{n^2}{2}$$

Hence,  $\Omega_n - \Omega_{n-1} = \frac{n^2}{2}$ . Therefore,

$$\begin{cases} \Omega_2 - \Omega_1 = \frac{2^2}{2} \\ \Omega_3 - \Omega_2 = \frac{3^2}{2} \\ \dots \dots \dots \dots \dots \dots \\ \Omega_n - \Omega_{n-1} = \frac{n^2}{2} \end{cases}$$

By adding, we get:

$$\Omega_n = \Omega_1 + \frac{2^2 + 3^2 + \dots + n^2}{2} = \frac{n(n+1)(2n+1)}{12}$$

We'll prove using the mathematical induction.

$$P(1): \Omega_1 = \frac{1 \cdot 2 \cdot 3}{12} = \frac{1}{2}$$

We consider  $P(n-1): \Omega_{n-1} = \frac{(n-1)n(2n-1)}{12}$  to be true. Thus, from  $\Omega_n - \Omega_{n-1} = \frac{n^2}{2}$

$$\begin{aligned} \text{we get: } \Omega_n &= \Omega_{n-1} + \frac{n^2}{2} = \frac{(n-1)n(2n-1)}{12} + \frac{n^2}{2} = \frac{n(2n^2 + 3n + 1)}{12} = \\ &= \frac{n(n+1)(2n+1)}{12} \end{aligned}$$

Therefore,

$$\Omega_n = \frac{n(n+1)(2n+1)}{12}$$

### Ap.3) Find:

$$\Omega_n = \lim_{x \rightarrow 0} \frac{n! x^n - \sin x \cdot \sin 2x \cdot \dots \cdot \sin nx}{x^{n+2}}, n \in \mathbb{N}^*$$

**Solution.** For  $n = 1$  we have:

$$\Omega_1 = \lim_{x \rightarrow 0} \frac{x - \sin x}{x^3} = \frac{1}{6}$$

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$$\begin{aligned}\Omega_n - n\Omega_{n-1} &= \lim_{x \rightarrow 0} \frac{nx \cdot \sin x \cdot \dots \cdot \sin(n-1)x - \sin x \cdot \sin 2x \cdot \dots \cdot \sin nx}{x^{n+2}} = \\ &= \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \frac{\sin 2x}{x} \cdot \dots \cdot \frac{\sin(n-1)x}{x} \cdot \lim_{x \rightarrow 0} \frac{nx - \sin nx}{x^3} = \\ &= (n-1)! \lim_{x \rightarrow 0} \frac{n - n \cos nx}{3n^2} = n! \lim_{x \rightarrow 0} \frac{n \sin nx}{6n} = n! \cdot \frac{n^2}{6}\end{aligned}$$

$$\text{Hence, } \Omega_n = n\Omega_{n-1} + n! \cdot \frac{n^2}{6}$$

$$\text{Let } P(n): \Omega_n = \frac{n(2n+1)}{36} (n+1)!$$

$$P(1): \Omega_1 = \frac{1 \cdot 3 \cdot 2!}{36} = \frac{1}{6}$$

Suppose that  $P(n): \Omega(n-1) = \frac{(n-1)(2n-1)n!}{36}$ , then

$$\Omega_n = n \frac{(n-1)(2n-1)n!}{36} + \frac{n^2 n!}{6} = \frac{n! n}{36} [(n-1)(2n-1) + 6n] = \frac{n! n(n+1)(2n+1)}{36}$$

**Ap.4) Find:**

$$\Omega_n = \lim_{x \rightarrow 0} \frac{n! x^n - \tan x \cdot \tan 2x \cdot \dots \cdot \tan nx}{x^{n+2}}, n \in \mathbb{N}^*$$

**Solution.** Let us denote:

$$f_k(x) = \frac{\tan kx}{kx}; \lim_{x \rightarrow 0} f_k(x) = 1 \Rightarrow \lim_{x \rightarrow 0} \prod_{k=1}^n f_k(x) = 1$$

$$\begin{aligned}\Omega_n &= \lim_{x \rightarrow 0} n! x^n \cdot \frac{1 - \prod_{k=1}^n \frac{\tan kx}{kx}}{x^{n+2}} = -n! \cdot \lim_{x \rightarrow 0} \frac{\prod_{k=1}^n \frac{\tan kx}{kx} - 1}{x^2} = \\ &= -n! \lim_{x \rightarrow 0} \frac{e^{\log \prod_{k=1}^n f_k(x)} - 1}{\log \prod_{k=1}^n f_k(x)} \cdot \frac{\log \prod_{k=1}^n f_k(x)}{x^2}\end{aligned}$$

$$\text{But: } \lim_{x \rightarrow 0} \frac{\log \prod_{k=1}^n f_k(x)}{x^2} = \lim_{x \rightarrow 0} \sum_{k=1}^n \frac{\log f_k(x)}{x^2} = \sum_{k=1}^n \frac{k^2}{3}, \text{ where}$$

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\log f_k(x)}{x^2} &= \lim_{x \rightarrow 0} \frac{\log(1 + f_k(x) - 1)}{f_k(x) - 1} \cdot \frac{f_k(x) - 1}{x^2} = \lim_{x \rightarrow 0} \frac{\tan kx - kx}{kx^3} = \\ &= k^2 \cdot \lim_{x \rightarrow 0} \frac{\tan kx - kx}{(kx)^3} = \frac{k}{3}, \text{ where}\end{aligned}$$

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$$\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3} = \frac{1}{3}$$

Therefore,

$$\Omega_n = \lim_{x \rightarrow 0} \frac{n! x^n - \tan x \cdot \tan 2x \cdot \dots \cdot \tan nx}{x^{n+2}} = -\frac{n!}{3} \sum_{k=1}^n k^2 = -n! \cdot \frac{n(n+1)(2n+1)}{18}$$

**Ap.5) Find:**

$$\Omega_n = \lim_{x \rightarrow 0} \frac{n! x^n - \sin^{-1} x \cdot \sin^{-1} 2x \cdot \dots \cdot \sin^{-1} nx}{x^{n+2}}, n \in \mathbb{N}^*$$

**Ap.6) Find:**

$$\Omega_n = \lim_{x \rightarrow 0} \frac{n! x^n - \tan^{-1} x \cdot \tan^{-1} 2x \cdot \dots \cdot \tan^{-1} nx}{x^{n+2}}, n \in \mathbb{N}^*$$

**Ap.7) Find:**

$$\Omega_n = \lim_{x \rightarrow 0} \frac{n! x^n - \log(1+x) \cdot \log(1+2x) \cdot \dots \cdot \log(1+nx)}{x^{n+1}}, n \in \mathbb{N}^*$$

**Solution.** Let us denote:

$$f_k(x) = \frac{\log(1+kx)}{kx}, \text{ then } \lim_{x \rightarrow 0} f_k(x) = 1 \text{ and } \lim_{x \rightarrow 0} \prod_{k=1}^n f_k(x) = 1$$

$$\Omega_n = \lim_{x \rightarrow 0} \frac{n! x^n}{x^{n+1}} \left( 1 - \prod_{k=1}^n \frac{\log(1+kx)}{kx} \right) = -n! \cdot \lim_{x \rightarrow 0} \frac{e^{\log \prod_{k=1}^n f_k(x)} - 1}{x} =$$

$$= -n! \cdot \lim_{x \rightarrow 0} \frac{e^{\log \prod_{k=1}^n f_k(x)} - 1}{\log \prod_{k=1}^n f_k(x)} \cdot \frac{\log \prod_{k=1}^n f_k(x)}{x} =$$

$$= -n! \cdot \lim_{x \rightarrow 0} \sum_{k=1}^n \frac{\log f_k(x)}{x}$$

$$\text{But: } \lim_{x \rightarrow 0} \frac{\log f_k(x)}{x} = \lim_{x \rightarrow 0} \frac{\log(1+f_k(x)-1)}{f_k(x)-1} \cdot \frac{f_k(x)-1}{x} = \lim_{x \rightarrow 0} \frac{\frac{\log(1+kx)}{kx} - 1}{x} =$$

$$= k \cdot \lim_{x \rightarrow 0} \frac{\log kx - kx}{(kx)^2} = -\frac{k}{2}$$

Therefore,

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$$\Omega_n = \lim_{x \rightarrow 0} \frac{n! x^n - \log(1+x) \cdot \log(1+2x) \cdot \dots \cdot \log(1+nx)}{x^{n+1}} = (-n!) \sum_{k=1}^n \left(-\frac{k}{2}\right)$$

$$= n! \frac{n(n+1)}{4}$$

**Ap.8) Find:**

$$\Omega_n = \lim_{x \rightarrow 0} \frac{n! x^n - (e^x - 1)(e^{2x} - 1) \cdot \dots \cdot (e^{nx} - 1)}{x^{n+1}}, n \in \mathbb{N}^*$$

**Ap.9) Find:**

$$\Omega_n = \lim_{x \rightarrow 0} \frac{n! \tan x \cdot \tan \frac{x}{2} \cdot \dots \cdot \tan \frac{x}{n} - x^n}{x^{n+2}}, n \in \mathbb{N}$$

**Ap.10) Find:**

$$\Omega_n = \lim_{x \rightarrow 0} \frac{1 - \cosh x \cdot \cosh^2 2x \cdot \dots \cdot \cosh^n nx}{x^2}, n \in \mathbb{N}^*$$

**Solution.** We have:

$$-\frac{\prod_{k=1}^n \cosh^k kx - 1}{x^2} = -\frac{e^{\log \prod_{k=1}^n \cosh^k kx} - 1}{x^2} =$$

$$= -\frac{e^{\sum_{k=1}^n k \log \cosh kx} - 1}{\sum_{k=1}^n k \log \cosh kx} \cdot \frac{\sum_{k=1}^n k \log \cosh kx}{x^2}$$

$$\lim_{x \rightarrow 0} \frac{k \log \cosh kx}{x^2} = \lim_{x \rightarrow 0} \frac{k \log(1 + \cosh kx - 1)}{\cosh kx - 1} \cdot \frac{\cosh kx - 1}{k^2 x^2} = \frac{k}{2} \cdot k^2 = \frac{k^3}{2}$$

$$\text{Because: } \lim_{x \rightarrow 0} \frac{\cosh x - 1}{x^2} = \frac{1}{2}$$

Therefore,

$$\Omega_n = \lim_{x \rightarrow 0} \frac{1 - \cosh x \cdot \cosh^2 2x \cdot \dots \cdot \cosh^n nx}{x^2} = -\frac{n^2(n+1)^2}{12}$$

**Ap.11) Find:**

$$\Omega_n = \lim_{x \rightarrow 0} \frac{n! x^n - \sinh x \cdot \sinh 2x \cdot \dots \cdot \sinh nx}{x^{n+2}}$$

**Ap.12) Find:**

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$$\Omega_n = \lim_{x \rightarrow 0} \frac{\cosh x \cdot \cosh 2x \cdot \cosh 2^2 x \cdot \dots \cdot \cosh 2^n x - e^{x^2}}{x^2}, n \in \mathbb{N}$$

**Ap.13) Find:**

$$\Omega_n = \lim_{x \rightarrow 0} \frac{a_1 a_2 \cdot \dots \cdot a_n x^n - \text{trig } a_1 x \cdot \text{trig } a_2 x \cdot \dots \cdot \text{trig } a_n x}{x^{n+2}}, a_i \in \mathbb{R}^*,$$

$$i = \overline{1, n}, \text{ where trig} \in \{\sin, \tan, \sinh, \tanh\}$$

**Solution.** Let us denote  $\text{trig } a_k x = f(a_k x)$

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{\text{trig } t}{t} &= \lim_{t \rightarrow 0} \frac{f(t)}{t} = 1. \text{ We have:} \\ \lim_{x \rightarrow 0} \frac{a_1 a_2 \cdot \dots \cdot a_n x^n \left(1 - \frac{f(a_1 x)}{a_1 x} \cdot \dots \cdot \frac{f(a_n x)}{a_n x}\right)}{x^{n+2}} &= -a_1 a_2 \cdot \dots \cdot a_n \lim_{x \rightarrow 0} \frac{e^{\log \prod_{k=1}^n \frac{f(a_k x)}{a_k x}} - 1}{x^2} = \\ &= -a_1 a_2 \cdot \dots \cdot a_n \lim_{x \rightarrow 0} \frac{e^{\log \prod_{k=1}^n \frac{f(a_k x)}{a_k x}} - 1}{\log \prod_{k=1}^n \frac{f(a_k x)}{a_k x}} \cdot \frac{\log \prod_{k=1}^n \frac{f(a_k x)}{a_k x}}{x^2} \\ \lim_{x \rightarrow 0} \frac{\log \prod_{k=1}^n \frac{f(a_k x)}{a_k x}}{x^2} &= \lim_{x \rightarrow 0} \frac{\sum_{k=1}^n \log \frac{f(a_k x)}{a_k x}}{x^2} = \\ &= \lim_{x \rightarrow 0} \sum_{k=1}^n \frac{\log \left(1 + \frac{f(a_k x)}{a_k x} - 1\right)}{\frac{f(a_k x)}{a_k x} - 1} \cdot \frac{f(a_k x) - a_k x}{(a_k x)^3} \cdot a_k^2 = \begin{cases} \frac{1}{6} \sum_{k=1}^n a_k^2, f \in \{\sinh, \sin\} \\ \frac{1}{3} \sum_{k=1}^n a_k^2, f \in \{\tan, \tanh\} \end{cases} \end{aligned}$$

**Ap.14) Find:**

$$\Omega_n = \lim_{n \rightarrow \infty} \frac{\left(\frac{\pi}{2}\right)^n - \cos^{-1} x \cdot \cos^{-1} 2x \cdot \dots \cdot \cos^{-1} nx}{x}, n \in \mathbb{N}^*$$

**Solution.** We have:

$$\Omega_n = \frac{\pi}{2} \Omega_{n-1} + \lim_{x \rightarrow 0} \frac{\frac{\pi}{2} - \cos^{-1} nx}{x} \cdot \prod_{k=1}^{n-1} \cos^{-1}(n-1)x =$$

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$$\begin{aligned}
 &= \frac{\pi}{2} \Omega_{n-1} + \left(\frac{\pi}{2}\right)^{n-1} \cdot \lim_{y \rightarrow \frac{\pi}{2}} \frac{\frac{\pi}{2} - y}{\cos y} = \frac{\pi}{2} \Omega_{n-1} + \lim_{y \rightarrow \frac{\pi}{2}} \frac{\frac{\pi}{2} - t}{\sin\left(\frac{\pi}{2} - y\right)} \cdot \left(\frac{\pi}{2}\right)^{n-1} \cdot n = \\
 &= \frac{\pi}{2} \Omega_{n-1} + n \left(\frac{\pi}{2}\right)^{n-1}
 \end{aligned}$$

Hence,

$$\left(\frac{2}{\pi}\right)^n \Omega_n = \Omega_{n-1} \left(\frac{2}{\pi}\right)^{n-1} + \frac{2n}{\pi}. \text{ Denote } \omega_n = \left(\frac{2}{\pi}\right)^n \Omega_n, \forall n \in \mathbb{N}^*, \text{ then}$$

$$\omega_n = \omega_{n-1} + \frac{2n}{\pi}, \forall n \geq 2. \text{ Hence, } \sum_{k=2}^n \omega_k = \sum_{k=1}^{n-1} \omega_k + \frac{2}{\pi} \sum_{k=2}^n k$$

$$\Rightarrow \omega_n = \omega_1 + \frac{2}{\pi} (2 + 3 + \dots + n); (1)$$

$$\omega_1 = \frac{2}{\pi} \cdot \lim_{x \rightarrow 0} \frac{\frac{\pi}{2} - \cos^{-1} x}{x} = \frac{2}{\pi} \cdot \lim_{y \rightarrow \frac{\pi}{2}} \frac{\frac{\pi}{2} - y}{\cos y} = \frac{2}{\pi}; (2)$$

$$(1) + (2) \Rightarrow \omega_n = \frac{2}{\pi} \cdot \frac{n(n+1)}{2} \Rightarrow \Omega_n = \left(\frac{\pi}{2}\right)^{n-1} - \frac{n(n+1)}{2}$$

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