R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

ABOUT A FEW SPECIAL LIMITS AND SUMS-(II)

Dedicated to Mr. Daniel Sitaru, "HAPPY BIRTHDAY"

By Florică Anastase-Romania

Abstract: *In this paper are presented few special limits and sums.*

Theorem 1. If $f,g:\mathbb{R}\to\mathbb{R}$ and $a\in\mathbb{R}$, then

$$\sum_{k=1}^{n}\frac{g(k)g(n-k)\big(a+f(k)\big)}{f(k)+2a+f(n-k)}=\frac{1}{2}\sum_{k=1}^{n-1}g(k)g(n-k)+\frac{g(0)g(n)\big(a+f(n)\big)}{f(0)+2a+f(n)}$$

Proof.

$$\text{Let: } a_n=\frac{g(k)g(n-k)\big(a+f(k)\big)}{f(k)+2a+f(n-k)},\text{then:}$$

$$a_n-a_{n-k}=\frac{g(k)g(n-k)\big(a+f(k)\big)}{f(k)+2a+f(n-k)}+\frac{g(n-k)g(k)\big(a+f(n-k)\big)}{f(n-k)+2a+f(k)}=$$

$$=g(k)g(n-k)$$

Hence,

$$2\sum_{k=1}^{n-1}a_k=\sum_{k=1}^{n-1}(a_k+a_{n-k})=\sum_{k=1}^{n-1}g(k)g(n-k)$$

and

$$\sum_{k=1}^{n}a_k=\frac{1}{2}\sum_{k=1}^{n}g(k)g(n-k)+\frac{g(0)g(n)\big(a+f(n)\big)}{f(0)+2a+f(n)}$$

Ap.1) Find:

$$\Omega=\lim_{n\to\infty}\frac{1}{n^3}\cdot\sum_{k=1}^{n}\frac{k^3(n-k)}{2k^2-2nk+n^2}$$

Solution. In Theorem 1, we consider $f(x)=x^2, g(x)=x$ and $a=0$, then it follows that

$$\sum_{k=1}^{n}\frac{k^3(n-k)}{2k^2-2nk+n^2}=\frac{n(n^2-1)}{12}$$

Therefore,

$$\Omega = \lim_{n\to\infty} \frac{1}{n^3} \cdot \sum_{k=1}^{n} \frac{k^3(n-k)}{2k^2 - 2nk + n^2} = \frac{1}{12}$$

Ap.2) Find:

$$\boldsymbol{\Omega = \lim_{n\to\infty} \frac{1}{n^3} \cdot \sum_{k=1}^{n} \sum_{k=1}^{n} \frac{k(n-k)\left(1+\pi^k\right)}{\pi^k + 2 + \pi^{n-k}}}$$

Solution. In Theorem 1, we consider $f(x) = \pi^x, g(x) = x$ and $a = 1$ it follows that

$$\sum_{k=1}^{n} \frac{k(n-k)(1+\pi^k)}{\pi^k + 2 + \pi^{n-k}} = \frac{1}{2} \sum_{k=1}^{n-1} k(n-k) = \frac{n}{2} \sum_{k=1}^{n-1} k - \frac{1}{2} \sum_{k=1}^{n-1} k^2 =$$

$$= \frac{n}{2} \sum_{k=1}^{n} k - \frac{1}{2} \sum_{k=1}^{n} k^2 - \frac{n^2}{2} + \frac{(n+1)^2}{2} =$$

$$= \frac{n^2(n+1)}{4} - \frac{n(n+1)(2n+1)}{12} - \frac{n^2}{2} + \frac{(n+1)^2}{2}$$

Therefore,

$$\Omega = \lim_{n\to\infty} \frac{1}{n^3} \cdot \sum_{k=1}^{n} \sum_{k=1}^{n} \frac{k(n-k)(1+\pi^k)}{\pi^k + 2 + \pi^{n-k}} = \frac{1}{12}$$

AP.3) Find:

$$\boldsymbol{\Omega = \lim_{n\to\infty} \frac{1}{n} \cdot \sum_{k=1}^{n} \frac{\alpha + \tan^{-1} k}{2\alpha + \tan^{-1}\left(\frac{n}{k^2 - kn + 1}\right)}}$$

Solution. In Theorem 1, we consider $f(x) = \tan^{-1} x$, $g(x) = \alpha^x, \alpha > 0$ and $a = \alpha$, then

$$\sum_{k=1}^{n} \frac{\alpha^k \cdot \alpha^{n-k}(\alpha + \tan^{-1} k)}{\tan^{-1} k + 2\alpha + \tan^{-1}(n-k)} = \sum_{k=1}^{n} \frac{\alpha^n(\alpha + \tan^{-1} k)}{2\alpha + \tan^{-1}\left(\frac{n}{k^2 - kn + 1}\right)} =$$

$$= \frac{1}{2} \sum_{k=1}^{n-1} \alpha^n + \frac{\alpha^n(\alpha + \tan^{-1} n)}{2\alpha + \tan^{-1} n} = \frac{1}{2} \alpha^n (n-1) + \frac{\alpha^{n+1} + \alpha^n \tan^{-1} n}{2\alpha + \tan^{-1} n}$$

Therefore,

$$\Omega=\lim_{n\to\infty}\frac{1}{n}\cdot\sum_{k=1}^{n}\frac{\alpha+\tan^{-1}k}{2\alpha+\tan^{-1}\left(\frac{n}{k^2-kn+1}\right)}=$$

$$=\lim_{n\to\infty}\frac{1}{n\alpha^n}\cdot\sum_{k=1}^{n}\frac{\alpha^k\cdot\alpha^{n-k}(\alpha+\tan^{-1}k)}{\tan^{-1}k+2\alpha+\tan^{-1}(n-k)}=$$

$$=\lim_{n\to\infty}\frac{1}{n\alpha^n}\left(\frac{1}{2}\alpha^n(n-1)+\frac{\alpha^n(\alpha+\tan^{-1}n)}{2\alpha+\tan^{-1}n}\right)=\frac{1}{2}$$

Theorem 2. If $f,g:\mathbb{R}\to\mathbb{R}$ and $a\in\mathbb{R}$ then

$$\sum_{k=1}^{n}\frac{\big(g(k)+g(n-k)\big)\big(a+f(k)\big)}{f(k)+2a+f(n-k)}=\sum_{k=1}^{n-1}g(k)+\frac{\big(g(0)+g(n)\big)\big(a+f(n)\big)}{f(0)+2a+f(n)}$$

Proof. Denoting:

$$a_k=\frac{\big(g(k)+g(n-k)\big)\big(a+f(k)\big)}{f(k)+2a+f(n-k)}$$

We get:

$$a_k+a_{n-k}=\frac{\big(g(k)+g(n-k)\big)\big(a+f(k)\big)}{f(k)+2a+f(n-k)}+\frac{\big(g(n-k)+g(n)\big)\big(a+f(n-k)\big)}{f(n-k)+2a+f(k)}=$$

$$=g(k)+g(n-k)$$

$$2\sum_{k=1}^{n-1}a_k=\sum_{k=1}^{n-1}(a_k+a_{n-k})=\sum_{k=1}^{n-1}\big(g(k)+g(n-k)\big)=2\sum_{k=1}^{n-1}g(k)$$

Therefore,

$$\sum_{k=1}^{n}a_k=\sum_{k=1}^{n-1}a_k+a_n=\sum_{k=1}^{n-1}g(k)+\frac{\big(g(0)+g(n)\big)\big(a+f(n)\big)}{f(0)+2a+f(n)}$$

Ap.4) Find:

$$\Omega=\lim_{n\to\infty}\frac{1}{n^a}\cdot\sum_{k=1}^{n}\frac{(n^2-2kn+2k^2)(a+k^3)}{n(n^2-2nk+3k^2)+2a},a\in\mathbb{R}$$

Solution. In Theorem 2, let $f(x)=x^3,g(x)=x^2$, then

$$\sum_{k=1}^{n}\frac{(n^2-2kn+2k^2)(a+k^3)}{n(n^2-2nk+3k^2)+2a}=\frac{n(n-1)(2n-1)}{6}+\frac{n(a+n^3)}{2a+n^3}$$

Therefore,

$$\Omega=\lim_{n\to\infty}\frac{1}{n^a}\cdot\sum_{k=1}^{n}\frac{(n^2-2kn+2k^2)(a+k^3)}{n(n^2-2nk+3k^2)+2a}=$$

$$=\lim_{n\to\infty}\frac{1}{n^a}\left(\frac{n(n-1)(2n-1)}{6}+\frac{n(a+n^3)}{2a+n^3}\right)=\begin{cases}0, \text{if } a>4\\ 1, \text{if } a=4\\ \infty, \text{if } a<4\end{cases}$$

Ap.5) Find:

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^2+1}{n^2-2nk+2k^2+2}\left(\sqrt{\frac{k}{n^3+k}}+\sqrt{\frac{n-k}{n^3+n-k}}\right)$$

Solution. In Theorem 2, let $f(x)=x^2$ and $g(x)=\sqrt{\frac{x}{n^3+x}}$ and $a=1$, then

$$\sum_{k=1}^{n}\frac{k^2+1}{n^2-2nk+2k^2+2}\left(\sqrt{\frac{k}{n^3+k}}+\sqrt{\frac{n-k}{n^3+n-k}}\right)=$$

$$=\sum_{k=1}^{n}\frac{(1+k^2)}{k^2+2+(n-k)^2}\left(\sqrt{\frac{k}{n^3+k}}+\sqrt{\frac{n-k}{n^3+n-k}}\right)=$$

$$=\sum_{k=1}^{n-1}\sqrt{\frac{k}{n^3+k}}+\frac{1+n^2}{2+n^2}\cdot\sqrt{\frac{n}{n^3+n}}=$$

$$=\sum_{k=1}^{n}\sqrt{\frac{k}{n^3+k}}+\left(\frac{1+n^2}{2+n^2}-1\right)\cdot\sqrt{\frac{n}{n^3+n}}=$$

$$=\sum_{k=1}^{n}\sqrt{\frac{k}{n^3+k}}-\frac{1}{2+n^2}\sqrt{\frac{n}{n^3+n}};(1)$$

Now, let $h(x)=\frac{x}{n^3+x}$ then $g(x)=\sqrt{h(x)}$.

$$h'(x)=\frac{n^3}{(n^3+x)^2}>0, \text{then } h-\text{increasing, so } g-\text{increasing on } [1,\infty).$$

$$g(k) \leq g(x) \leq f(k+1), \forall x \in [k, k+1]$$

$$g(k) \leq \int_k^{k+1} g(x)\,dx \leq g(k+1), \forall k \in \{1,2,\dots,n-1\}$$

$$g(1)+g(2)+\cdots+g(n-1) \leq \int_1^n g(x)\,dx \leq g(2)+g(3)+\cdots+g(n)$$

Denote: $a_n = \sum_{k=1}^{n} \sqrt{\frac{k}{n^3+k}}$, then

$$a_n - \sqrt{\frac{n}{n^3+n}} \leq \int_1^n g(x)\,dx \leq a_n - \sqrt{\frac{1}{n^3+1}}$$

On the other hand:

$$\int_1^n g(x)\,dx = \int_1^n \sqrt{\frac{x}{n^3+x}}\,dx \leq \int_1^n \frac{\sqrt{x}dx}{\sqrt{n^3+1}} = \frac{1}{\sqrt{n^3+1}} \cdot \frac{2x^{\frac{3}{2}}}{3}\Bigg|_1^n = \frac{2}{3} \cdot \frac{n^{\frac{3}{2}}-1}{\sqrt{n^3+1}} \xrightarrow{n\to\infty} \frac{2}{3}$$

$$\int_1^n g(x)\,dx \geq \int_1^n \frac{\sqrt{x}dx}{\sqrt{n^3+n}} = \frac{1}{\sqrt{n^3+n}} \cdot \frac{2}{3}\left(n^{\frac{3}{2}}-1\right) \xrightarrow{n\to\infty} \frac{2}{3}$$

Thus,

$$\lim_{n\to\infty} \int_1^n f(x)\,dx = \lim_{n\to\infty} a_n = \frac{2}{3}; (2)$$

Therefore, from (1) and (2) it followsTastați ecuația aici.

$$\Omega = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^2+1}{n^2-2nk+2k^2+2}\left(\sqrt{\frac{k}{n^3+k}} + \sqrt{\frac{n-k}{n^3+n-k}}\right) =$$

$$= \lim_{n\to\infty}\left(\sum_{k=1}^{n} \sqrt{\frac{k}{n^3+k}} - \frac{1}{2+n^2}\sqrt{\frac{n}{n^3+n}}\right) = \frac{2}{3}$$

References

[1] M. Bencze and F. Anastase- ABOUT FEW SPECIAL LIMITS AND SUMS-(I)-www.ssmrmh.ro

[2] M. Bencze- ABOUT A SPECIAL SUM-Octogon Mathematical Magazine-Vol.27, No.1/2019

[3] D. Sitaru-MATH PHENOMENON A NEW DIMENSION-2021

[4] ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro