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TWO AMAZING STANDARD SUMS

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Prove that:

$$\log(2\sin x) = -\sum_{n=1}^{\infty} \frac{\cos(2nx)}{n} \text{ and } \sum_{n=1}^{\infty} \frac{\sin(2nx)}{n} = \frac{\pi}{2} - x$$

Proof.

$$\mathrm{Log}(1-x) = \sum_{n=0}^{\infty} \frac{(-1)^n(-x)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (-1)^n \cdot (-1) \cdot x^{n+1}}{n+1} =$$

$$= -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

Let $x \to \cos x + i\sin x$, then

$$\log(1-\cos x - i\sin x) = -\sum_{n=0}^{\infty} \frac{(\cos x + i\sin x)^{n+1}}{n+1}$$

$$\log\left(2\sin^2\frac{x}{2} - 2i\sin\frac{x}{2}\cos\frac{x}{2}\right) = -\sum_{n=0}^{\infty} \frac{\cos(n+1)x + i\sin(n+1)x}{n+1}$$

$$\log 2\sin\frac{x}{2}\left(\sin\frac{x}{2} - i\cos\frac{x}{2}\right) = -\sum_{n=0}^{\infty} \frac{\cos(n+1)x}{n+1} - i\sum_{n=0}^{\infty} \frac{\sin(n+1)x}{n+1}$$

Taking $\frac{x}{2} \to x$, it follows that:

$$\log(2\sin x) + \log(\sin x - i\cos x) = -\sum_{n=0}^{\infty} \frac{\cos 2(n+1)x}{n+1} - i\sum_{n=0}^{\infty} \frac{\sin 2(n+1)x}{n+1}$$

For $x, y > 0$, we have:

$$\log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$$

$$\log(-x+iy) = \frac{1}{2}\log(x^2+y^2) + i\left(\pi - \tan^{-1}\left(\frac{y}{x}\right)\right)$$

$$\log(-x-iy) = \frac{1}{2}\log(x^2+y^2) - i\left(\pi - \tan^{-1}\left(\frac{y}{x}\right)\right)$$

$$\log(x-iy)=\frac{1}{2}\log(x^2+y^2)-i\tan^{-1}\left(\frac{y}{x}\right)$$

Therefore,

$$\log(\sin x-i\cos x)=\frac{1}{2}\log(\sin^2 x+\cos^2 x)-i\tan^{-1}\left(\frac{\cos x}{\sin x}\right)=$$

$$=\frac{1}{2}\log 1-i\tan^{-1}\left(\tan\left(\frac{\pi}{2}-x\right)\right)=-i\left(\frac{\pi}{2}-x\right)$$

Hence,

$$\log(2\sin x)-i\left(\frac{\pi}{2}-x\right)=-\sum_{n=0}^{\infty}\frac{\cos 2(n+1)x}{n+1}-i\sum_{n=0}^{\infty}\frac{\sin 2(n+1)x}{n+1}$$

$$\log(2\sin x)=-\sum_{n=0}^{\infty}\frac{\cos 2(n+1)x}{n+1}\Rightarrow\log(2\sin x)=-\sum_{n=1}^{\infty}\frac{\cos(2nx)}{n}$$

$$\frac{\pi}{2}-x=\sum_{n=0}^{\infty}\frac{\sin 2(n+1)x}{n+1}\Rightarrow\sum_{n=1}^{\infty}\frac{\sin(2nx)}{n}=\frac{\pi}{2}-x$$

REFERENCE:

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