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Find all real numbers $x_1, x_2, \dots, x_n$ such that

$$\frac{1}{n}\sum_{i=1}^{n} x_i = \frac{\sum_{i=1}^{n} x_i^4}{\sum_{i=1}^{n} x_i^3} = 1$$

Proposed by Neculai Stanciu-Romania

Solution 1 by Adrian Popa-Romania, Solution 2 by Ravi Prakash-New Delhi-India

Solution 1 by Adrian Popa-Romania

$$\frac{\sum_{i=1}^{n} x_i^4}{\sum_{i=1}^{n} x_i^3} = \frac{\sum_{i=1}^{n} x_i^3 \cdot x_i}{\sum_{i=1}^{n} x_i^3} \overset{Chebishev}{\geq} \frac{\frac{1}{n}\left(\sum_{i=1}^{n} x_i^3\right)\left(\sum_{i=1}^{n} x_i\right)}{\sum_{i=1}^{n} x_i^3} = \frac{1}{n}\sum_{i=1}^{n} x_i = 1$$

Equality holds for $x_1 = x_2 = \cdots = x_n$.

Solution 2 by Ravi Prakash-New Delhi-India

$$Let: x_i = 1 + t_i, i = \overline{1,n}, then\ \frac{1}{n}\sum_{i=1}^{n} x_i = 1\ we\ get\ \sum_{i=1}^{n}(1 + t_i) = n$$

$$Hence, n + \sum_{i=1}^{n} t_i = n\ or\ \sum_{i=1}^{n} t_i = 0$$

$$\text{Next,}\quad \frac{\sum_{i=1}^{n} x_i^4}{\sum_{i=1}^{n} x_i^3} = 1 \Rightarrow \sum_{i=1}^{n}(1+t_i)^4 = \sum_{i=1}^{n}(1+t_i)^3$$

$$\sum_{i=1}^{n}(1+t_i)^3(1+t_i-1) = 0 \Rightarrow \sum_{i=1}^{n}(1+t_i)^3 t_i = 0$$

$$\sum_{i=1}^{n}\left(t_i + 3t_i^2 + 3t_i^3 + t_i^4\right) = 0 \Rightarrow \sum_{i=1}^{n} t_i + \sum_{i=1}^{n} t_i^2\left(t_i^2 + 3t_i + 3\right) = 0$$

$$\sum_{i=1}^{n} t_i^2\left[\left(t_i + \frac{3}{2}\right)^2 + \frac{3}{4}\right] = 0,\ \text{thus},\ t_i^2\left[\left(t_i + \frac{3}{2}\right)^2 + \frac{3}{4}\right] = 0 \text{ for } i = 1,2,\dots,n.$$

$$t_i = 0 \text{ for } i = 1,2,\dots,n.\ \text{Thus},\ x_1 = x_2 = \cdots = x_n = 1.$$