

RMM - Inequalities Marathon 701 - 800

R M M

ROMANIAN MATHEMATICAL MAGAZINE

Founding Editor
DANIEL SITARU

Available online
www.ssmrmh.ro

ISSN-L 2501-0099



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Proposed by

Daniel Sitaru – Romania, Nguyen Van Canh-Ben Tre-Vietnam

Asmat Qatea-Afghanistan, Pavlos Trifon-Greece

Neculai Stanciu-Romania, Amrit Awasthi-India

Olimjon Jalilov-Uzbekistan, Marin Chirciu-Romania

Dang Le Gia Khanh-An Giang-Vietnam, D.M. Bătinețu-Giurgiu - Romania

Rajeev Rastogi-India, Choy Fai Lam-Hong Kong

Seyran Ibrahimov-Maasilli-Azerbaijan, Nikos Ntorvas-Greece

Florica Anastase-Romania, Srinivasa Raghava-AIRMC-India

Hikmat Mammadov-Azerbaijan, Marius Dragan – Romania

Kunihiko Chikaya-Tokyo-Japan, George Apostolopoulos-Greece



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Solutions by

Daniel Sitaru-Romania, Mohamed Amine Ben Ajiba-Tanger-Morocco

Ravi Prakash-New Delhi-India, Pavlos Trifon-Greece, Aggeliki

Papaspyropoulou-Greece, Kamel Gandouli Rezgui-Tunisia, Marian Dincă-Romania, Hikmat Mammadov-Azerbaijan, Surjeet Singhania-India

Soumitra Mandal-Chandar Nagore-India, Adrian Popa-Romania, Alex Szoros-Romania, Amrit Awasthi-India, Jamal Issah-Ghana, Florentin Vișescu-Romania, Sanong Huayrerai-Nakon Pathom-Thailand, Kunihiko Chikaya-Tokyo-Japan, Dang Le Gia Khanh-An Giang-Vietnam, George Florin Șerban-Romania, Soumava Chakraborty-Kolkata-India, Nikos Ntorvas-Greece, Nguyen Van Canh-Ben Tre-Vietnam, Michael Sterghiou-Greece, Sergey Primazon-Russia, Florică Anastase-Romania, Lazaros Zachariadis-Thessaloniki-Greece, George Titakis-Greece, Samar Das-India, Khaled Abd Imouti-Damascus-Syria, Bedri Hajrizi-Mitrovica-Kosovo, Alex Szoros-Romania, Vivek Kumar-India



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

701.

Let $x_1, x_2, \dots, x_{2021} > 0$ such that : $\sum_{k=1}^{2021} x_k = 1$. Prove that :

$$\sum_{k=1}^{2021} \frac{x_k + 2021}{x_k^2 + 1} \leq 2021^2$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Using Tangent Line Method to prove (*).

Since there is equality when $x_k = \frac{1}{2021}, k$

= $\overline{1, 2021}$, we consider the equation of tangent line to the function

$$f(x) = \frac{x + 2021}{x^2 + 1} \text{ at } x = \frac{1}{2021}, \text{ which is } y = -\frac{2021^2}{2021^2 + 1} \cdot x + 2021 \cdot \frac{2021^2 + 2}{2021^2 + 1}$$

$$\rightarrow \text{Let's prove that : } f(x) = \frac{x + 2021}{x^2 + 1} \leq -\frac{2021^2}{2021^2 + 1} \cdot x + 2021 \cdot \frac{2021^2 + 2}{2021^2 + 1}, \forall x \in (0, 1)$$

$$\leftrightarrow -2021^2 x^3 + 2021(2021^2 + 2)x^2 - (2 \cdot 2021^2 + 1)x + 2021 \geq 0$$

$$\leftrightarrow (2021x - 1)^2(2021 - x) \geq 0$$

Which is true for all $x \in (0, 1)$ $\rightarrow \frac{x_k + 2021}{x_k^2 + 1}$

$$\leq -\frac{2021^2}{2021^2 + 1} \cdot x_k + 2021 \cdot \frac{2021^2 + 2}{2021^2 + 1}, \forall k = \overline{1, 2021}.$$

$$\rightarrow \sum_{k=1}^{2021} \frac{x_k + 2021}{x_k^2 + 1} \leq 2021 \sum_{k=1}^{2021} \left(-\frac{2021}{2021^2 + 1} \cdot x_k + 2021 \cdot \frac{2021^2 + 2}{2021^2 + 1} \right)$$

$$= 2021 \left(-\frac{2021}{2021^2 + 1} + 2021 \cdot \frac{2021^2 + 2}{2021^2 + 1} \right) = 2021^2.$$

Therefore, $\sum_{k=1}^{2021} \frac{x_k + 2021}{x_k^2 + 1} \leq 2021^2.$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

702. If $a, b \geq 0$ then:

$$8(a^5 + b^5) \geq 5(a^2 - b^2)(a^3 - b^3) + \sqrt{8} \sqrt{a^2 + b^2}^5$$

Proposed by Asmat Qatea-Afghanistan

Solution 1 by Ravi Prakash-New Delhi-India

$$\text{Put } a = r \cos \theta, b = r \sin \theta, r \geq 0, 0 \leq \theta \leq \frac{\pi}{2}$$

The inequality becomes

$$8(\cos^5 \theta + \sin^5 \theta) \geq 5(\cos^2 \theta - \sin^2 \theta)(\cos^3 \theta - \sin^3 \theta) + \sqrt{8}$$

$$3(\cos \theta + \sin \theta) + 5 \sin^2 \theta \cos^2 \theta (\sin \theta + \cos \theta) \geq \sqrt{8}$$

$$(\cos \theta + \sin \theta)(3 + 2 \cos^2 \theta \sin^2 \theta - 3 \sin \theta \cos \theta) \geq \sqrt{8}$$

$$\text{Let } f(\theta) = (\cos \theta + \sin \theta) \left(3 - \frac{3}{2} \sin 2\theta + \frac{1}{2} \sin^2 2\theta \right), 0 \leq \theta \leq \frac{\pi}{2}$$

$$f'(\theta) = (\cos \theta - \sin \theta) \left(3 - \frac{3}{2} \sin 2\theta + \frac{1}{2} \sin^2 2\theta \right) + (\cos \theta + \sin \theta)(-3 \cos 2\theta + 2 \sin 2\theta + \cos 2\theta) =$$

$$= (\cos \theta - \sin \theta) \left(3 - \frac{3}{2} \sin 2\theta + \frac{1}{2} \sin^2 2\theta - 3(\cos \theta + \sin \theta)^2 + 2 \sin 2\theta (\cos \theta + \sin \theta)^2 \right) =$$

$$= (\cos \theta - \sin \theta) \left(3 - \frac{3}{2} \sin 2\theta + \frac{1}{2} \sin^2 2\theta - 3(1 + \sin 2\theta) + 2 \sin 2\theta + 2 \sin^2 2\theta \right) =$$

$$= (\cos \theta - \sin \theta) \left(\frac{5}{2} \sin^2 2\theta - \frac{5}{2} \sin 2\theta \right) =$$

$$= \frac{5}{2} \sin 2\theta (\sin \theta - \cos \theta)(1 - \sin 2\theta)$$

$$\text{For } 0 < \theta < \pi, \sin 2\theta > 0, 1 - \sin 2\theta \geq 0 \Rightarrow f'(\theta) = 0 \Rightarrow \theta = \frac{\pi}{4}$$

$$f'(\theta) = \begin{cases} > 0; 0 < \theta < \frac{\pi}{4} \\ < 0; \frac{\pi}{4} < \theta < \frac{\pi}{2} \end{cases}$$

Thus, $f'(\theta)$ is minimum when $\theta = \frac{\pi}{4} \Rightarrow f(\theta) \geq f\left(\frac{\pi}{4}\right)$

$$f(\theta) \geq \sqrt{2} \left(3 - \frac{3}{2} + \frac{1}{2} \right) = \sqrt{8}$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$8(a^5 + b^5) \stackrel{(*)}{\geq} 5(a^2 - b^2)(a^3 - b^3) + \sqrt{8}\sqrt{a^2 + b^2}^5$$

If $b = 0 \rightarrow (*) \leftrightarrow 3a^5 \geq \sqrt{8}a^5$ which is true ($\therefore 3 \geq \sqrt{8}$)

$$\text{If } b > 0, \text{ let } x = \frac{a}{b} \rightarrow (*) \leftrightarrow 8(x^5 + 1) - 5(x^2 - 1)(x^3 - 1) \geq \sqrt{8}\sqrt{x^2 + 1}^5$$

$$\leftrightarrow 3x^5 + 5x^3 + 5x^2 + 3 \geq \sqrt{8}\sqrt{x^2 + 1}^5 \leftrightarrow (3x^5 + 5x^3 + 5x^2 + 3)^2 \geq 8(x^2 + 1)^5$$

$$\leftrightarrow x^{10} - 10x^8 + 30x^7 - 55x^6 + 68x^5 - 55x^4 + 30x^3 - 10x^2 + 1 \geq 0$$

$$\leftrightarrow (x - 1)^4(x^6 + 4x^5 + 10x^3 + 4x + 1) \geq 0 \text{ which is true.}$$

$$\text{Therefore, } 8(a^5 + b^5) \geq 5(a^2 - b^2)(a^3 - b^3) + \sqrt{8}\sqrt{a^2 + b^2}^5$$

703. If $a, b, c \in (0, \frac{\pi}{2})$. Prove that:

$$\left(\sum \sqrt[3]{\sin a} \right)^6 + 27 \left(\sum \sqrt[3]{\cos^2 a} \right)^3 \leq 729$$

Proposed by Pavlos Trifon-Greece

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\left(\sum \sqrt[3]{\sin a} \right)^6 + 27 \left(\sum \sqrt[3]{\cos^2 a} \right)^3 \stackrel{\text{Hölder}}{\leq} \left(\sum 1 \right)^5 \left(\sum \sqrt[3]{\sin a}^6 \right)$$

$$+ 27 \left(\sum 1 \right)^2 \left(\sum \sqrt[3]{\cos^2 a}^3 \right) =$$

$$= 243 \left(\sum \sin^2 a + \sum \cos^2 a \right) = 243 \left[\sum (\sin^2 a + \cos^2 a) \right] = 243 \sum 1 = 729.$$

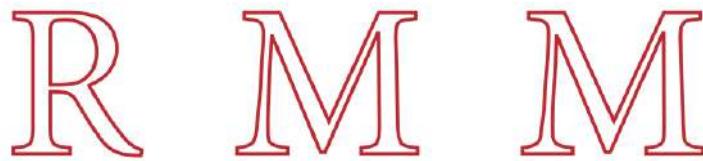
$$\text{Therefore, } \left(\sum \sqrt[3]{\sin a} \right)^6 + 27 \left(\sum \sqrt[3]{\cos^2 a} \right)^3 \leq 729.$$

Solution 2 by proposer

By Power mean inequality:

$$\sqrt[3]{\frac{\sqrt[3]{\sin^2 a} + \sqrt[3]{\sin^2 b} + \sqrt[3]{\sin^2 c}}{3}} \geq \frac{\sqrt[3]{\sin a} + \sqrt[3]{\sin b} + \sqrt[3]{\sin c}}{3}$$

$$\left(\sqrt[3]{\sin^2 a} + \sqrt[3]{\sin^2 b} + \sqrt[3]{\sin^2 c} \right)^3 \geq \frac{\left(\sqrt[3]{\sin a} + \sqrt[3]{\sin b} + \sqrt[3]{\sin c} \right)^6}{27}; \quad (1)$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Equality holds for $\sqrt[3]{\sin a} = \sqrt[3]{\sin b} = \sqrt[3]{\sin c}; \left(a, b, c \in \left(0, \frac{\pi}{2}\right) \right) \Rightarrow a = b = c.$

By Minkowski:

$$\begin{aligned} & \left(\sqrt[3]{\sin^2 a} + \sqrt[3]{\sin^2 b} + \sqrt[3]{\sin^2 c} \right)^3 + \left(\sqrt[3]{\cos^2 a} + \sqrt[3]{\cos^2 b} + \sqrt[3]{\cos^2 c} \right)^3 \leq \\ & \leq \left(\sum_{cyc} \sqrt[3]{\sin^2 a + \cos^2 a} \right)^3 = 27; (2) \end{aligned}$$

Equality holds for $\left(\frac{\sin a}{\sin b} = \frac{\cos a}{\cos b} \right) \& \left(\frac{\sin a}{\sin c} = \frac{\cos a}{\cos c} \right) \Leftrightarrow \sin(a - b) = \sin(a - c);$

$$(\because a - b, a - c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \Leftrightarrow a = b = c.$$

$$(1) \& (2) \Rightarrow \frac{\left(\sqrt[3]{\sin a} + \sqrt[3]{\sin b} + \sqrt[3]{\sin c} \right)^6}{27} + \left(\sqrt[3]{\cos^2 a} + \sqrt[3]{\cos^2 b} + \sqrt[3]{\cos^2 c} \right)^3 \leq 27$$

Therefore,

$$\left(\sum \sqrt[3]{\sin a} \right)^6 + 27 \left(\sum \sqrt[3]{\cos^2 a} \right)^3 \leq 729$$

Equality holds for $a = c.$

704. Let $a, b, c > 0$. Prove that:

$$: \min \left\{ 2 \left(\sum \frac{a}{b} \right) \left(\sum a \right), 3 \left(\sum \frac{a^2}{b} + \sum a \right) \right\} \geq 6 \sqrt{3 \sum a^2}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum \frac{a}{b} = \sum \frac{a^2}{ab} \stackrel{Bergstrom}{\geq} \frac{(\sum a)^2}{\sum ab} \stackrel{?}{\geq} \frac{3\sqrt{3\sum a^2}}{\sum a} \leftrightarrow (\sum a)^6 \geq 27 (\sum ab)^2 (\sum a^2)$$

$$\begin{aligned} \text{We have : } & (\sum a)^2 = \sum a^2 + 2 \sum ab \stackrel{AM-GM}{\geq} 3 \sqrt[3]{(\sum ab)^2 (\sum a^2)} \rightarrow (\sum a)^6 \\ & \geq 27 (\sum ab)^2 (\sum a^2) \\ & \rightarrow \sum \frac{a}{b} \geq \frac{3\sqrt{3\sum a^2}}{\sum a} \rightarrow 2 \left(\sum \frac{a}{b} \right) \left(\sum a \right) \geq 6 \sqrt{3 \sum a^2} \quad (1) \end{aligned}$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 \text{Now, let's prove : } \sum \frac{a^2}{b} &\geq \frac{3 \sum a^2}{\sum a} \leftrightarrow \sum \left(\frac{a^2}{b} - 2a + b \right) \geq \frac{3 \sum a^2}{\sum a} - \sum a \\
 &\leftrightarrow \sum \frac{(a-b)^2}{b} \geq \sum \frac{(a-b)^2}{a+b+c} \\
 &\leftrightarrow \sum \frac{(a+c)(a-b)^2}{b(a+b+c)} \geq 0 \text{ which is true} \rightarrow \sum \frac{a^2}{b} + \sum a \\
 &\geq \frac{3 \sum a^2}{\sum a} + \sum a \stackrel{\text{AM-GM}}{\geq} 2 \sqrt{\frac{3 \sum a^2}{\sum a} \cdot \sum a} = 2 \sqrt{3 \sum a^2} \\
 &\rightarrow 3 \left(\sum \frac{a^2}{b} + \sum a \right) \geq 6 \sqrt{3 \sum a^2} \quad (2) \\
 (1), (2) \rightarrow \min \left\{ 2 \left(\sum \frac{a}{b} \right) \left(\sum a \right), 3 \left(\sum \frac{a^2}{b} + \sum a \right) \right\} &\geq 6 \sqrt{3 \sum a^2}.
 \end{aligned}$$

705.

$$\text{Let } a, b > 0, A = \frac{\sum \frac{a^2}{b}}{\sqrt[4]{\frac{3 \sum a^3}{abc}} \cdot \sqrt{\sum a^2}} \text{ and } B = \frac{3abc}{\sum a^3}. \text{ Prove that } A \geq B$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\begin{aligned}
 \text{Since : } \sum \frac{a^2}{b} &\stackrel{\text{Bergstrom}}{\geq} \frac{(a+b+c)^2}{b+c+a} = \sum a \rightarrow \text{It's suffices to prove} \\
 &\cdot (\sum a^3) (\sum a) \geq 3abc \sqrt[4]{\frac{3 \sum a^3}{abc}} \cdot \sqrt{\sum a^2} \\
 &\leftrightarrow (\sum a^3)^3 (\sum a)^4 \geq 3^5 (abc)^3 (\sum a^2)^2
 \end{aligned}$$

We have

$$\begin{aligned}
 &\cdot (\sum a^3) (\sum a) \stackrel{\text{CBS}}{\geq} (\sum a^2)^2 \text{ and } (\sum a^3)^2 (\sum a)^3 \stackrel{\text{Am-GM}}{\geq} (3abc)^2 (3 \sqrt[3]{abc})^3 \\
 &= 3^5 (abc)^3
 \end{aligned}$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\rightarrow \left(\sum a^3 \right)^3 \left(\sum a \right)^4 \geq 3^5 (abc)^3 \left(\sum a^2 \right)^2 \rightarrow A \geq B.$$

706. If $a, b, c \in \left[0, \frac{\sqrt{3}}{3}\right]$. Prove that:

$$\frac{a^4}{3^{0,5625(b+c)^4} - 1} + \frac{b^4}{3^{(c^2+ca+a^2)^2} - 1} \stackrel{(*)}{\leq} \frac{1}{6} + \frac{c^4}{1 - 19683^{a^2b^2}}$$

Proposed by Pavlos Trifon-Greece

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$(*) \leftrightarrow \frac{a^4}{3^{\frac{9(b+c)^4}{16}} - 1} + \frac{b^4}{3^{(c^2+ca+a^2)^2} - 1} + \frac{c^4}{3^{9a^2b^2} - 1} \geq \frac{1}{6}$$

$$\text{Since } a, b, c \in \left[0, \frac{\sqrt{3}}{3}\right] \rightarrow 0 < \frac{9(b+c)^4}{16} \leq \frac{9}{16} \left(2 \cdot \frac{\sqrt{3}}{3}\right)^4 = 1, 0 < (c^2+ca+a^2)^2$$

$$\leq \left(3 \cdot \frac{1}{3}\right)^2 = 1, 0 < 9a^2b^2 \leq 1$$

$$\rightarrow 3^{\frac{9(b+c)^4}{16}} - 1$$

$$= (1+2)^{\frac{9(b+c)^4}{16}} - 1 \stackrel{\text{Bernoulli}}{\leq} 1 + 2 \cdot \frac{9(b+c)^4}{16}$$

$$- 1 \stackrel{\text{Power Mean}}{\leq} 9(b^4 + c^4) \quad (1)$$

$$3^{(c^2+ca+a^2)^2} - 1 = (1+2)^{(c^2+ca+a^2)^2} - 1 \stackrel{\text{Bernoulli}}{\leq} 1 + 2(c^2+ca+a^2)^2 - 1$$

$$\stackrel{\text{AM-GM}}{\leq} 2 \left(c^2 + \frac{c^2+a^2}{2} + a^2 \right)^2 =$$

$$= \frac{9}{2}(c^2+a^2)^2 \stackrel{\text{CBS}}{\leq} 9(c^4+a^4) \rightarrow 3^{(c^2+ca+a^2)^2} - 1 \leq 9(c^4+a^4) \quad (2)$$

$$3^{9a^2b^2} - 1 = (1+2)^{9a^2b^2} - 1 \stackrel{\text{Bernoulli}}{\leq} 1 + 2 \cdot 9a^2b^2 - 1 \stackrel{\text{AM-GM}}{\leq} 9(a^4+b^4) \quad (3)$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 (1), (2), (3) &\rightarrow \frac{a^4}{3^{\frac{9(b+c)^4}{16}} - 1} + \frac{b^4}{3^{(c^2+ca+a^2)^2} - 1} + \frac{c^4}{3^{9a^2b^2} - 1} \\
 &\geq \frac{1}{9} \left(\frac{a^4}{b^4 + c^4} + \frac{b^4}{c^4 + a^4} + \frac{c^4}{a^4 + b^4} \right) \stackrel{\text{Nesbitt}}{\geq} \frac{1}{9} \cdot \frac{3}{2} = \frac{1}{6}
 \end{aligned}$$

Therefore,

$$\frac{a^4}{3^{0.5625(b+c)^4} - 1} + \frac{b^4}{3^{(c^2+ca+a^2)^2} - 1} \geq \frac{1}{6} + \frac{c^4}{1 - 19683^{a^2b^2}}$$

Solution 2 by proposer

Lemma. For all $x, y, z > 0$, $2(x^2 + xy + y^2)^2 \leq 9(x^4 + y^4)$

Proof. $2(x^2 + xy + y^2)^2 \leq 9(x^4 + y^4) \Leftrightarrow$

$$2(x^4 + x^2y^2 + y^4 + 2x^3y + 2xy^3 + 2x^2y^2) \leq 9x^4 + 9y^4 \Leftrightarrow$$

$$4x^3y + 4xy^3 \leq 7x^4 + 7y^4 \Leftrightarrow 3(x^2 - y^2)^2 + 4(x^4 + y^4 - xy(x^2 + y^2)) \geq 0$$

Equality holds for $x = y$.

So, $a, b \in \left(0, \frac{\sqrt{3}}{3}\right] \Rightarrow a^2, ab, b^2 \leq \frac{1}{3} \Rightarrow a^2 + ab + b^2 \leq 1 \Rightarrow 0 < (a^2 + ab + b^2)^2 \leq 1$

From Bernoulli, we have:

$$\begin{aligned}
 3^{(a^2+ab+b^2)^2} &= (1+2)^{(a^2+ab+b^2)^2} \leq 1 + 2(a^2 + ab + b^2)^2 \stackrel{\text{Lemma}}{\leq} 1 + 9(a^4 + b^4) \Rightarrow \\
 \frac{9c^4}{3^{(a^2+ab+b^2)^2} - 1} &\geq \frac{c^4}{a^4 + b^4}
 \end{aligned}$$

Equality holds for $a^2 + ab + b^2 = 1$; $\left(a, b \in \left(0, \frac{\sqrt{3}}{3}\right]\right) \Leftrightarrow a = b = c = \frac{\sqrt{3}}{3}$

Now,

$$9 \sum_{cyc} \frac{c^4}{3^{(a^2+ab+b^2)^2} - 1} \geq \sum_{cyc} \frac{c^4}{a^4 + b^4} \stackrel{\text{Nesbitt}}{\geq} \frac{3}{2} \Leftrightarrow \sum_{cyc} \frac{c^4}{3^{(a^2+ab+b^2)^2} - 1} \geq \frac{1}{6}; (1)$$

$$a^2 + ab + b^2 \stackrel{\text{AGM}}{\geq} 3 \cdot \sqrt[3]{a^3b^3} = 3ab \Rightarrow \frac{c^4}{3^{(a^2+ab+b^2)^2} - 1} \leq \frac{c^4}{(19683)^{a^2b^2} - 1}; (2)$$

Equality holds for $a^2 = ab = b^2 \Leftrightarrow a = b$.

And



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$b^2 + bc + c^2 \geq \frac{3(b+c)^2}{4} \Rightarrow \frac{a^4}{3^{(b^2+bc+c^2)^2} - 1} \leq \frac{a^4}{3^{0.5625(b+c)^4} - 1}; (3)$$

Equality holds for $b = c$.

From (1),(2) and (3) it follows that:

$$\frac{a^4}{3^{0.5625(b+c)^4} - 1} + \frac{b^4}{3^{(c^2+ca+a^2)^2} - 1} \geq \frac{1}{6} + \frac{c^4}{1 - 19683^{a^2b^2}}$$

Equality holds for $a = b = c = \frac{\sqrt{3}}{3}$, $a = b, b = c \Leftrightarrow a = b = c = \frac{\sqrt{3}}{3}$

707. If $a, b > 0$ and $a \neq b$ then:

$$\frac{a^2 + 10ab + b^2}{6ab} \geq \frac{a+b}{a-b} \log\left(\frac{a}{b}\right)$$

Proposed by Asmat Qatea-Afghanistan

Solution 1 by Ravi Prakash-New Delhi-India

$$\frac{a}{b} + \frac{b}{a} + 10 \geq 6 \left(\frac{\frac{a}{b} + 1}{\frac{a}{b} - 1} \right) \log\left(\frac{a}{b}\right); (1)$$

Assume that $a > b$, otherwise we work to $\frac{a}{b}$.

$$\text{Let } f(t) = (t-1)(t^2 + 10t + 1) - 6t(t+1) \log t; t \geq 1$$

$$f(t) = t^3 + 9t^2 - 9t - 1 - 6(t^2 + t) \log t$$

$$f'(t) = 3t^2 + 12t - 15 - 6(2t + 1) \log t$$

$$f''(t) = 6\left(t - \frac{1}{t}\right) - 12 \log t$$

$$f'''(t) = 6\left(1 - \frac{1}{t}\right)^2 > 0, \forall t > 1 \Rightarrow f'' \nearrow \text{ in } [1, \infty) \Rightarrow f''(t) \geq f''(1) = 0, \forall t > 1$$

$$\Rightarrow f' \nearrow \text{ on } [1, \infty) \Rightarrow f'(t) > f'(1), \forall t > 1$$

$$\Rightarrow f \nearrow \text{ on } [1, \infty) \Rightarrow f(t) > f(1), \forall t > 1.$$

Thus,

$$t + \frac{1}{t} + 10 > \frac{t+1}{t-1} \log t; \forall t > 1$$

$$\text{Put } t = \frac{a}{b} \text{ to obtain (1)}$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Solution 2 by Kamel Gandouli Rezgui-Tunisia

$$\frac{a^2 + 10ab + b^2}{6ab} \geq \frac{a+b}{a-b} \log\left(\frac{a}{b}\right); (*)$$

Put $x = \frac{a}{b}$ then $(*) \Rightarrow \frac{x^2 + 10x + 1}{6x} \geq \frac{x+1}{x-1} \log x, x > 0$

$$\frac{x^2 + 10x + 1}{6x(x+1)} \stackrel{?}{\geq} \frac{\log x}{x-1}; x > 0$$

(I) For $x > 1 \Rightarrow x + \frac{1}{x} \geq 2 + \frac{1}{x^2} \geq \frac{2}{x} \Rightarrow$

$u(x) = x - 2 \log x; u(x) \nearrow, u(x) \geq 0$ and $a > 0 \Rightarrow$

$$v(x) = 3x^2 + 12x - 15 - (12x + 6) \log x, v(x) \nearrow, x \geq 1, v(x) \geq 0$$

$$3x^2 + 18x - 9 \geq (12x + 6) \log x + 6x + 6$$

$$w(x) = x^3 + 9x^2 - 9x - 1 - (6x^2 + 6x) \log x; w(x) \nearrow, w(x) \geq 0, x \geq 1$$

$$\frac{x^2 + 10x + 1}{6x(x+1)} \geq \frac{\log x}{x-1} \text{ true.}$$

(II) For $0 < x < 1; x + \frac{1}{x} \geq 2 \Rightarrow$

$u(x) = x - 2 \log x; u(x) \nearrow, u(x) \leq 0$ and $a > 0 \Rightarrow$

$$v(x) = 3x^2 + 12x - 15 - (12x + 6) \log x, v(x) \nearrow, 0 < x < 1, v(x) \leq 0$$

$$3x^2 + 18x - 9 \geq (12x + 6) \log x + 6x + 6$$

$$w(x) = x^3 + 9x^2 - 9x - 1 - (6x^2 + 6x) \log x; w(x) \nearrow, w(x) \geq 0, 0 < x < 1$$

$$\frac{x^2 + 10x + 1}{6x(x+1)} \geq \frac{\log x}{x-1}, x > 0 \text{ true.}$$

Solution 3 by Marian Dincă-Romania

$$\frac{a^2 + 10ab + b^2}{6ab} \geq \frac{a+b}{a-b} \log\left(\frac{a}{b}\right); (*)$$

$$F(a, b) = \frac{a^2 + 10ab + b^2}{6ab} - \frac{a+b}{a-b} \log\left(\frac{a}{b}\right)$$

$F(a, b) = F(b, a)$. Let $\frac{a}{b} = x \geq 1$ because is symmetric.

$$f(x) = \frac{x^2 + 10x + 1}{6x} - \frac{x+1}{x-1} \log x$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$g(x) = \frac{x^2 + 10x + 1}{6x} \cdot \frac{x-1}{x+1} - \log x = \frac{1}{6} \left(x + 10 + \frac{1}{x} \right) \left(1 - \frac{2}{x+1} \right) - \log x$$

$$g'(x) = \frac{1}{6} \left[\left(1 - \frac{1}{x^2} \right) \left(1 - \frac{2}{x+1} \right) + \left(x + 10 + \frac{1}{x} \right) \cdot \frac{2}{(x+1)^2} \right] - \frac{1}{x} =$$

$$= \frac{1}{6} \left[\frac{(x^2 - 1)(x - 1)}{x^2(x + 1)} + \frac{2(x^2 + 10x + 1)}{x(x + 1)^2} \right] - \frac{1}{x} =$$

$$= \frac{(x^2 - 1)^2 + (x^2 + 10x + 1)2x}{6x^2(x + 1)^2} - \frac{1}{x} =$$

$$= \frac{x^4 - 4x^3 + 6x^2 - 4x + 1}{6x^2(x + 1)^2} = \frac{(x - 1)^4}{6x^2(x + 1)^2} \geq 0$$

$$g(x) \geq g(1) = 0$$

$$g(x) \geq 0 \Leftrightarrow f(x) \geq 0$$

$$f(x) = \frac{x^2 + 10x + 1}{6x} - \frac{x+1}{x-1} \log x = \frac{x^2 + 10x + 1}{6x} - (x+1) \cdot \frac{\log x - \log 1}{x-1}$$

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} \frac{x^2 + 10x + 1}{6x} - \lim_{x \rightarrow 1} (x+1) \cdot \lim_{x \rightarrow 1} \frac{\log x - \log 1}{x-1} =$$

$$= 2 - 2 \lim_{x \rightarrow 1} \left(\frac{1}{x} \right) = 0$$

Equality holds for $x = 1 \Leftrightarrow a = b$.

708. Let $a, b, c, d, x, y, z, t > 0$. Prove that:

$$\left(\sum_{a,b,c,d} a \right) \left(\sum_{x,y,z,t} x \right) \leq 2 \sqrt{\left(\sum_{a,b,c,d} a^2 \right) \left(\sum_{x,y,z,t} x^2 \right)} + 2(at + bx + cy + dz)$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\left(\sum_{a,b,c,d} a \right) \left(\sum_{x,y,z,t} x \right) \leq 2 \sqrt{\left(\sum_{a,b,c,d} a^2 \right) \left(\sum_{x,y,z,t} x^2 \right)} + 2(at + bx + cy + dz); (*)$$

$$\text{Let } u = \sqrt{\frac{x^2 + y^2 + z^2 + t^2}{a^2 + b^2 + c^2 + d^2}}, p = \frac{x}{u}, q = \frac{y}{u}, r = \frac{z}{u} \text{ and } s = \frac{t}{u}$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\text{We have : } \sum_{p,q,r,s} p^2 = \frac{1}{u^2} \sum_{x,y,z,t} x^2 = \sum_{a,b,c,d} a^2$$

$$\begin{aligned} \rightarrow (*) \leftrightarrow & \left(\sum_{a,b,c,d} a \right) \left(\sum_{p,q,r,s} p \right) \leq 2 \left(\sum_{a,b,c,d} a^2 \right) + 2(as + bp + cq + dr) \\ & = (a+s)^2 + (b+p)^2 + (c+q)^2 + (d+r)^2 \end{aligned}$$

We have :

$$\begin{aligned} & (a+s)^2 + (b+p)^2 + (c+q)^2 \\ & + (d+r)^2 \stackrel{CBS}{\geq} \frac{1}{4} [(a+s) + (b+p) + (c+q) + (d+r)]^2 = \\ & = \frac{1}{4} \left[\left(\sum_{a,b,c,d} a \right) + \left(\sum_{p,q,r,s} p \right) \right]^2 \stackrel{AM-GM}{\geq} \left(\sum_{a,b,c,d} a \right) \left(\sum_{p,q,r,s} p \right) \rightarrow (*) \text{ is true.} \end{aligned}$$

Therefore,

$$\left(\sum_{a,b,c,d} a \right) \left(\sum_{x,y,z,t} x \right) \leq 2 \sqrt{\left(\sum_{a,b,c,d} a^2 \right) \left(\sum_{x,y,z,t} x^2 \right)} + 2(at + bx + cy + dz).$$

709. If $x, y \in R^*$ with $x^2(1 - xy) + y^2(1 + xy) = 0$, then

find $\min(\sqrt{x^2 + y^2})$

Proposed by Neculai Stanciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$x^2(1 - xy) + y^2(1 + xy) \stackrel{(*)}{=} 0$$

$$\text{Let } x = ay \rightarrow (*) \leftrightarrow a^2(1 - ay^2) + 1 + ay^2 = 0 \leftrightarrow y^2 = \frac{a^2 + 1}{a^3 - a} > 0 \rightarrow a^3 - a > 0$$

$$\leftrightarrow a \in (-1, 0) \cup (1, \infty)$$

$$\rightarrow \sqrt{x^2 + y^2} = \sqrt{y^2(a^2 + 1)} = \sqrt{\frac{(a^2 + 1)^2}{a^3 - a}} = \frac{a^2 + 1}{\sqrt{a^3 - a}} = f(a), \quad a \in (-1, 0) \cup (1, \infty)$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\text{We have : } f'(a) = \frac{2a\sqrt{a^3 - a} - \frac{(a^2 + 1)(3a^2 - 1)}{2\sqrt{a^3 - a}}}{a^3 - a} = \frac{a^4 - 6a^2 + 1}{2\sqrt{a^3 - a}^3}$$

$$= \frac{(a^2 - (\sqrt{2} - 1)^2)(a^2 - (\sqrt{2} + 1)^2)}{2\sqrt{a^3 - a}^3} =$$

$$= \frac{(a - (\sqrt{2} - 1))(a + (\sqrt{2} - 1))(a - (\sqrt{2} + 1))(a + (\sqrt{2} + 1))}{2\sqrt{a^3 - a}^3}$$

$\rightarrow f$ is decreasing on $[-1, -(\sqrt{2} - 1)]$ and $[1, \sqrt{2} + 1]$, increasing on $[-(\sqrt{2} - 1), 0]$ and $[\sqrt{2} + 1, \infty]$

$$\rightarrow \min_{a \in (-1, 0) \cup (1, \infty)} f(a) = \min\{f(-\sqrt{2} + 1), f(\sqrt{2} + 1)\} = \min\{2, 2\} = 2.$$

$$\begin{aligned} \text{Equality if } a = -\sqrt{2} + 1 \rightarrow y^2 = 2 + \sqrt{2} \leftrightarrow y = \pm\sqrt{2 + \sqrt{2}} \text{ and } \\ x = \pm(\sqrt{2} - 1)\sqrt{2 + \sqrt{2}} = \pm\sqrt{2 - \sqrt{2}} \end{aligned}$$

$$\text{Or if } a = \sqrt{2} + 1 \rightarrow y^2 = 2 - \sqrt{2} \leftrightarrow y = \pm\sqrt{2 - \sqrt{2}} \text{ and }$$

$$x = \pm(\sqrt{2} + 1)\sqrt{2 - \sqrt{2}} = \pm\sqrt{2 + \sqrt{2}}$$

$$\text{Therefore, } \min(\sqrt{x^2 + y^2}) = 2.$$

710. If $x, y > 0, x^2 + 2y^2 = 3$ then:

$$\left(\frac{x+y}{\sqrt{y^2+3}} + \frac{y}{\sqrt{x^2+3}} \right)^2 \leq \frac{3(x+2y)}{4y\sqrt{x}}$$

Proposed by Daniel Sitaru-Romania

Solution by Hikmat Mammadov-Azerbaijan

$$x^2 + 2y^2 = 3, 0 < x < \sqrt{3}, 0 < y < \frac{\sqrt{3}}{2}$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{cases} x^2 + 3 \geq 2x + 2 \\ y^2 + 3 \geq 2y + 2 \end{cases} \Rightarrow \begin{cases} \frac{1}{x^2 + 3} \leq \frac{1}{2x + 2} \\ \frac{1}{y^2 + 3} \leq \frac{1}{2y + 2} \end{cases} \Rightarrow \begin{cases} \frac{y^2}{x^2 + 3} \leq \frac{y^2}{2x + 2} \\ \frac{(x+y)^2}{y^2 + 3} \leq \frac{(x+y)^2}{2y + 2} \end{cases}$$

$$\begin{cases} \frac{y}{\sqrt{x^2 + 3}} \leq \frac{y}{\sqrt{2x + 2}} \\ \frac{x+y}{\sqrt{y^2 + 3}} \leq \frac{y}{\sqrt{2x + 2}} \end{cases} \Rightarrow \frac{x+y}{\sqrt{y^2 + 3}} + \frac{y}{\sqrt{x^2 + 3}} \leq \frac{x+y}{\sqrt{2y + 2}} + \frac{y}{\sqrt{2x + 2}}$$

$$\left(\frac{x+y}{\sqrt{y^2 + 3}} + \frac{y}{\sqrt{x^2 + 3}} \right)^2 \leq \left(\frac{x+y}{\sqrt{2y + 2}} + \frac{y}{\sqrt{2x + 2}} \right)^2$$

$$\left(\frac{x+y}{\sqrt{y^2 + 3}} + \frac{y}{\sqrt{x^2 + 3}} \right)^2 \leq \frac{3}{4} \left[-\sqrt{\frac{\sqrt{x}}{y} + \frac{2}{\sqrt{x}}} + \left(\sqrt{\frac{\sqrt{x}}{y} + \frac{2}{\sqrt{x}}} - \frac{2x+2y}{\sqrt{6y+6}} - \frac{2y}{\sqrt{6x+6}} \right) \right]^2$$

$$T = \sqrt{\frac{\sqrt{x}}{y} + \frac{2}{\sqrt{x}}} - \frac{2x+2y}{\sqrt{6y+6}} - \frac{2y}{\sqrt{6x+6}}; 0 < x < \sqrt{3} \Rightarrow \frac{1}{\sqrt{x}} \in (0, \infty) \Rightarrow T \in [0, \infty)$$

$$\Rightarrow \left(\frac{x+y}{\sqrt{y^2 + 3}} + \frac{y}{\sqrt{x^2 + 3}} \right)^2 \leq \frac{3}{4} \left[-\sqrt{\frac{\sqrt{x}}{y} + \frac{2}{\sqrt{x}}} \right]^2$$

Therefore,

$$\left(\frac{x+y}{\sqrt{y^2 + 3}} + \frac{y}{\sqrt{x^2 + 3}} \right)^2 \leq \frac{3(x+2y)}{4y\sqrt{x}}$$

Equality holds for $x = y = 1$.

711. If $0 \leq a \leq b$ then:

$$ab(4 + (a+b)^2) \tan^{-1}(\sqrt{ab}) \leq (1+ab)(a+b)^2 \tan^{-1}\left(\frac{a+b}{2}\right)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Kamel Gandouli Rezgui-Tunisia

$$\begin{aligned} & (a+b)^2(1+ab) - ab(4 + (a+b)^2) = \\ & = (a+b)^2(1+ab-ab) - 4ab = (a+b)^2 - 4ab = (a-b)^2 \geq 0 \\ & \Rightarrow (a+b)^2(1+ab) \geq ab(4 + (a+b)^2) \geq 0 \end{aligned}$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\left. \begin{array}{l} \frac{a+b}{2} \geq \sqrt{ab} \\ \tan^{-1} x = u(x) \nearrow \text{in } (0, \infty) \end{array} \right\} \Rightarrow \tan^{-1} \left(\frac{a+b}{2} \right) \geq \tan^{-1} (\sqrt{ab}) \geq 0$$

Hence,

$$ab(4 + (a+b)^2) \tan^{-1}(\sqrt{ab}) \leq (1+ab)(a+b)^2 \tan^{-1} \left(\frac{a+b}{2} \right), \forall a, b \geq 0.$$

Solution 2 by Surjeet Singhania-India

$$\begin{aligned} \text{Let: } f(x) &= \frac{x^2}{1+x^2} \tan^{-1} x, x \geq 0 \\ f'(x) &= \frac{x^2}{(1+x^2)^2} + \frac{2x}{(1+x^2)^2} \tan^{-1} x > 0, \forall x > 0 \Rightarrow f \nearrow \text{ and since } \frac{a+b}{2} \geq \sqrt{ab} \Rightarrow \\ f\left(\frac{a+b}{2}\right) &\geq f(\sqrt{ab}) \Rightarrow \frac{(a+b)^2}{4\left(1+\frac{(a+b)^2}{4}\right)} \tan^{-1}\left(\frac{a+b}{2}\right) \geq \frac{ab}{1+ab} \tan^{-1}(\sqrt{ab}) \end{aligned}$$

Hence,

$$ab(4 + (a+b)^2) \tan^{-1}(\sqrt{ab}) \leq (1+ab)(a+b)^2 \tan^{-1} \left(\frac{a+b}{2} \right), \forall a, b \geq 0.$$

Solution 3 by Soumitra Mandal-Chandar Nagore-India

$$\begin{aligned} \text{Let: } f(x) &= \frac{x^2}{1+x^2} \tan^{-1} x, x \geq 0 \\ f'(x) &= \frac{1}{1+x^2} - \frac{1}{(1+x^2)^2} + \frac{2x}{(1+x^2)^2} \tan^{-1} x = \frac{x^2 + 2x \cdot \tan^{-1} x}{(1+x^2)^2} > 0, \forall x > 0 \\ &\Rightarrow f \nearrow \text{ and since } \frac{a+b}{2} \geq \sqrt{ab} \Rightarrow \\ f\left(\frac{a+b}{2}\right) &\geq f(\sqrt{ab}) \Rightarrow \frac{\left(\frac{a+b}{2}\right)^2}{1+\left(\frac{a+b}{2}\right)^2} \tan^{-1}\left(\frac{a+b}{2}\right) \geq \frac{ab}{1+ab} \tan^{-1}(\sqrt{ab}) \end{aligned}$$

Hence,

$$ab(4 + (a+b)^2) \tan^{-1}(\sqrt{ab}) \leq (1+ab)(a+b)^2 \tan^{-1} \left(\frac{a+b}{2} \right), \forall a, b \geq 0.$$

712. If $a, b > 0$ and $a \neq b$ then:

$$\left(\frac{b}{a} \right)^{\frac{b+a}{b-a}} \geq e^{\frac{8ab-a^2-b^2}{3ab}}$$

Proposed by Asmat Qatea-Afghanistan



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Solution 1 by Ravi Prakash-New Delhi-India

Assume that $b > a$ and put $\frac{b}{a} = t$ so that

$$\frac{t+1}{t-1} \geq e^{\frac{8}{3} - \frac{1}{3t} - \frac{t}{3}} \Leftrightarrow \frac{t+1}{t-1} \log t \geq \frac{8}{3} - \frac{1}{3t} - \frac{t}{3} \Leftrightarrow 3(t+1) \log t \geq \left(8 - \frac{1}{t} - t\right)(t-1)$$

$$\text{Let: } f(t) = 3(t+1) \log t - 8t + 8 + 1 + t^2 - \frac{1}{t} - t =$$

$$= 3(t+1) \log t - 9t + 9 + t^2 - \frac{1}{t}; t \geq 1$$

$$f'(t) = -\frac{3}{t^2} + \frac{3}{t} + 2 - \frac{2}{t^3} = \frac{3}{t^2}(t-1) + \frac{2}{t^3}(t^3-1) > 0, \forall t > 1$$

$$\Rightarrow f''(t) > 0, \forall t > 1 \Rightarrow f'(t) > f'(1), \forall t > 1 \Rightarrow f'(t) > 0, \forall t > 1$$

$\Rightarrow f$ increases on $[1, \infty)$

$$\text{Hence, } \frac{t+1}{t-1} \log t \geq \frac{8}{3} - \frac{1}{3t} - \frac{t}{3} \Leftrightarrow \frac{t+1}{t-1} \geq e^{\frac{8}{3} - \frac{1}{3t} - \frac{t}{3}} \text{ and therefore,}$$

$$\left(\frac{b}{a}\right)^{\frac{b+a}{b-a}} \geq e^{\frac{8ab-a^2-b^2}{3ab}}$$

Solution 2 by Adrian Popa-Romania

Assume that $a > b \Rightarrow \frac{a}{b} \in (0, 1)$ and then

$$\left(\frac{b}{a}\right)^{\frac{b+a}{b-a}} \geq e^{\frac{8ab-a^2-b^2}{3ab}} \Leftrightarrow \left(\frac{b}{a}\right)^{\frac{b+1}{a-1}} \geq e^{\frac{8}{3} - \frac{1}{3}(\frac{b+a}{a+b})} \Leftrightarrow \frac{8}{3} - \frac{1}{3}\left(\frac{b}{a} + \frac{a}{b}\right) \stackrel{AGM}{\leq} \frac{8}{3} - \frac{2}{3} = 2$$

Let: $\frac{b}{a} = x$ then we must to prove: $x^{\frac{x+1}{x-1}} \geq e^2, \forall x > 0 \Leftrightarrow \frac{x+1}{x-1} \log x \geq 2; (1)$

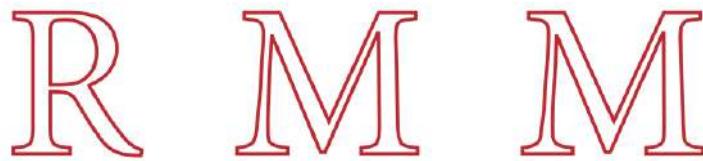
We known that: $\frac{2x}{2+x} \leq \log(1+x), \forall x > 0; (2).$

Case 1) If $x > 1 \Rightarrow \frac{b}{a} > 1 \Rightarrow b > a$ and from (2) pit $x \rightarrow x-1$ we get

$$\frac{2(x-1)}{x+1} \leq \log x \Rightarrow \frac{x+1}{x-1} \log x \geq \frac{x+1}{x-1} \cdot \frac{2(x-1)}{x+1} = 2$$

Case 2) If $x < 1 \Rightarrow \frac{b}{a} < 1 \Rightarrow b < a$ and $x-1 > 0$ then we must to prove:

$$x^{\frac{x+1}{x-1}} \geq e^2. \text{ Let } y = \frac{1}{x} \Rightarrow y > 1 \Rightarrow \left(\frac{1}{y}\right)^{\frac{1}{y-1}} \geq e^2 \Rightarrow$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$y^{\frac{y+1}{y-1}} \geq e^2 \text{ true by case 1)$$

Therefore,

$$\left(\frac{b}{a}\right)^{\frac{b+a}{b-a}} \geq e^{\frac{8ab-a^2-b^2}{3ab}}$$

Solution 3 by Soumitra Mandal-Chandar Nagore-India

WLOG, let us assume $b \geq a$ and let $x = \frac{b}{a}$, $f(x) = \frac{x+1}{x-1} \log x - \frac{8}{3} + \frac{x}{3} + \frac{1}{3x}; \forall x \geq 1$

$$f'(x) = \frac{x+1}{x(x+1)} - \frac{2 \log x}{3x^2} = \frac{x^2 - 1 - 2x \log x}{x(x-1)^2} + \frac{x^2 - 1}{3x^2}; \forall x \geq 1$$

$$\begin{aligned} \text{Let: } \varphi(x) &= x^2 - 1 - 2x \log x; \forall x \geq 1, \varphi'(x) = 2x - 2 - 2 \log x = \\ &= 2(x - 1 - \log x); \forall x \geq 1 \end{aligned}$$

$$\begin{aligned} \varphi''(x) &= \frac{2(x-1)}{x} \geq 0; \forall x \geq 1 \Rightarrow \varphi'(x) \text{ increasing } \varphi'(x) \geq \varphi'(1) = 0 \\ &\Rightarrow \varphi(x) \text{ increasing } \Rightarrow \varphi(x) \geq \varphi(1) = 0 \end{aligned}$$

$$\frac{x^2 - 1 - 2x \log x}{x(x-1)^2} \geq 0 \text{ and } \frac{x^2 - 1}{3x^2} \geq 0; \forall x \geq 1$$

$$\begin{aligned} f'(x) \geq 0 &\Rightarrow f \text{ - increasing } \Rightarrow f(x) \geq f(1) = \lim_{x \rightarrow 1^+} \frac{x+1}{x-1} \log x - \frac{8}{3} + \frac{1}{3} + \frac{1}{3} = \\ &= \lim_{x \rightarrow 1^+} \frac{x \log x + \log x}{x-1} - 2 \stackrel{(0)}{=} \lim_{x \rightarrow 1^+} \left(1 + \frac{1}{x} + \log x\right) - 1 = 0 \end{aligned}$$

$$\text{Thus, } \frac{x+1}{x-1} \log x \geq \frac{8}{3} - \frac{1}{3} \left(x + \frac{1}{x}\right) \text{ and therefore,}$$

$$\left(\frac{b}{a}\right)^{\frac{b+a}{b-a}} \geq e^{\frac{8}{3} - \frac{1}{3} \left(\frac{b+a}{a+b}\right)} \Rightarrow \left(\frac{b}{a}\right)^{\frac{b+a}{b-a}} \geq e^{\frac{8ab-a^2-b^2}{3ab}}$$

713. Prove that: $\forall n \in N^*$,

$$\frac{1}{n} \cdot \zeta_n \left(-\frac{1}{2}\right) \leq \sqrt{\frac{n+1}{2}}, \text{ where } \zeta_n(s) = \sum_{k=1}^n \frac{1}{k^s}$$

Proposed by Amrit Awasthi-India



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{1}{n} \cdot \zeta_n \left(-\frac{1}{2} \right) \stackrel{(*)}{\leq} \sqrt{\frac{n+1}{2}}$$

$$\text{We have : } (*) \leftrightarrow \sum_{k=1}^n \sqrt{k} \leq n \sqrt{\frac{n+1}{2}}$$

We know that : $\forall x, y \geq 0, \sqrt{x} + \sqrt{y} \leq \sqrt{2(x+y)}$ (CBS)

$x = k \in [1, n]$ and $y = n - k + 1 \rightarrow \sqrt{k} + \sqrt{n-k+1} \leq \sqrt{2(n+1)}, \forall k \in [1, n]$

$$\rightarrow \sum_{k=1}^n \sqrt{k} = \frac{1}{2} \sum_{k=1}^n (\sqrt{k} + \sqrt{n-k+1}) \leq \frac{1}{2} \sum_{k=1}^n \sqrt{2(n+1)} = n \sqrt{\frac{n+1}{2}}$$

$$\text{Therefore, } \frac{1}{n} \cdot \zeta_n \left(-\frac{1}{2} \right) \leq \sqrt{\frac{n+1}{2}}.$$

714. Prove that: $\forall n \in N^*$,

$$H_n + \ln(H_n) \geq \ln(2) + \ln(n), \quad H_n \geq \frac{2n}{n+1}$$

Proposed by Amrit Awasthi-India

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We know that : $\forall x, y \geq 0, \frac{1}{x} + \frac{1}{y} \geq \frac{4}{x+y}$ (CBS)

$x = k \in [1, n]$ and $y = n - k + 1 \rightarrow \frac{1}{k} + \frac{1}{n-k+1} \geq \frac{4}{n+1}, \forall k \in [1, n]$

$$\rightarrow H_n = \frac{1}{2} \sum_{k=1}^n \left(\frac{1}{k} + \frac{1}{n-k+1} \right) \geq \frac{1}{2} \sum_{k=1}^n \frac{4}{n+1} = \frac{2n}{n+1}$$

$$\rightarrow \ln(H_n) \geq \ln\left(\frac{2n}{n+1}\right) = \ln(2) + \ln(n) - \ln(n+1) \quad (1)$$

Also, we have : $\forall k \geq 1, \frac{1}{k} \geq \int_k^{k+1} \frac{dt}{t} \rightarrow H_n = \sum_{k=1}^n \frac{1}{k} \geq \sum_{k=1}^n \int_k^{k+1} \frac{dt}{t} = \int_1^{n+1} \frac{dt}{t}$

$$= \ln(n+1) \quad (2)$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\rightarrow (1) + (2) \rightarrow H_n + \ln(H_n) \geq \ln(2) + \ln(n)$$

$$\text{Therefore, } \forall n \in N^*, H_n + \ln(H_n) \geq \ln(2) + \ln(n), H_n \geq \frac{2n}{n+1}.$$

Solution 2 by Alex Szoros-Romania

$$(1 + 2 + 3 + \dots + n) \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \right) \stackrel{CBS}{\geq} n^2, \forall n \geq 1$$

$$\frac{n(n+1)}{2} \cdot H_n \geq n^2 \Rightarrow H_n \geq \frac{2n}{n+1}; \forall n \geq 1$$

$$H_n \geq \frac{2n}{n+1} \Rightarrow \log H_n \geq \log 2 + \log n - \log(n+1)$$

$$\log(n+1) + \log H_n \geq \log 2 + \log n; (1)$$

$\forall k \in N^*; f: [k, k+1] \rightarrow \mathbb{R}, f(x) = \log x \stackrel{MVT}{\implies} \exists c \in (k, k+1) \text{ such that:}$

$$f(k+1) - f(k) = (k+1-k)f'(c)$$

$$\log(k+1) - \log k = \frac{1}{c} \in \left(\frac{1}{k+1}, \frac{1}{k} \right) \Rightarrow$$

$$\log(k+1) - \log k < \frac{1}{k}; \forall k \geq 1 \Rightarrow \sum_{k=1}^n (\log(k+1) - \log k) \leq \sum_{k=1}^n \frac{1}{k}; (2)$$

From (1),(2) it follows that $H_n + \log(H_n) \geq \log 2 + \log n$

715. For all $x \in [-1, 1]$, $|ax|\sqrt{1-x^2} \leq 1$ prove that:

$$|a| \leq \frac{x^2 + 1}{4} + \frac{4}{x^2 + 1}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution 1 by Kamel Gandouli Rezgui-Tunisia

$$\text{Let: } f(x) = \frac{1}{|x|\sqrt{1-x^2}}, x \in (-1, 1), f'(x) = \frac{2x^2 - 1}{x^2(1-x^2)^{\frac{3}{2}}}$$

$$\min_{x \in [-1, 1]} f(x) = \frac{1}{\frac{1}{\sqrt{2}}\sqrt{1-\frac{1}{2}}} = 2 \text{ for } x = \pm \frac{1}{\sqrt{2}}, x \in (-1, 1)$$

$$|ax|\sqrt{1-x^2} \leq 1 \Rightarrow x = \frac{1}{\sqrt{2}} \Rightarrow \left| \frac{a}{\sqrt{2}} \right| \cdot \frac{1}{\sqrt{2}} \leq 1 \Rightarrow |a| \leq 2$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\frac{x^2 + 1}{4} + \frac{4}{x^2 + 1} \stackrel{AGM}{\geq} 2, \forall x \in \mathbb{R} \Rightarrow |a| \leq \frac{x^2 + 1}{4} + \frac{4}{x^2 + 1}$$

Solution 2 by Ravi Prakash-New Delhi-India

$$|ax|\sqrt{1-x^2} \leq 1 \Rightarrow |a||x|\sqrt{1-x^2} \leq 1. \text{ Put: } x = \sin \theta, \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Rightarrow$$

$$|a||\sin 2\theta| \leq 2; (1)$$

It is clearly true for $\theta \in \left\{-\frac{\pi}{2}, 0, \frac{\pi}{2}\right\}$. For $0 < |\theta| < \frac{\pi}{2}$, (1) gives

$$|a| \leq \frac{2}{|\sin 2\theta|} = |2 \csc(2\theta)|$$

|a| must be less than the value of |2 csc(2θ)|, which is 2 for $\theta = \frac{\pi}{4}$.

$$\therefore |a| \leq 2 \stackrel{AGM}{\leq} \frac{1}{4}(x^2 + 1) + \frac{4}{x^2 + 1}, \forall x > 0$$

716. Let $x \in [-1, 1]$ such that $|ax + b|\sqrt{1-x^2} \leq 1$. Prove that:

$$|a| \leq x^2 - 2x + 3$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Ravi Prakash-New Delhi-India

For $x \in [-1, 1]$ such that $|ax + b|\sqrt{1-x^2} \leq 1$; (1) put $x = 0$ so that $|b| \leq 1$.

If $b = 0$, then $|ax|\sqrt{1-x^2} \leq 1$. Suppose $0 < |b| \leq 1$.

If $a \geq 2, b > 0$ then for $x = \frac{1}{\sqrt{2}}$: $Lhs_{(1)} = \left(\frac{a}{\sqrt{2}} + b\right) \cdot \frac{1}{\sqrt{2}} = \frac{a}{2} + \frac{b}{\sqrt{2}} > 1$.

If $a \geq 2, b < 0$ then for $x = -\frac{1}{\sqrt{2}}$: $Lhs_{(1)} = \left(-\frac{a}{\sqrt{2}} + b\right) \cdot \frac{1}{\sqrt{2}} = \frac{a}{2} - \frac{b}{\sqrt{2}} > 1$.

If $a \leq -2, b > 0$ we use $x = -\frac{1}{\sqrt{2}}$ and if $a \leq -2, b < 0$ we use $x = \frac{1}{\sqrt{2}}$ to show that:

$$Lhs > 1$$

$a \not\leq 2$ and $a \not\geq -2 \Rightarrow -2 < a < 2 \Leftrightarrow |a| < 2 \leq (x-1)^2 + 2$ or $|a| \leq x^2 - 2x + 3$

717. Let $a, b, c \in \mathbb{R}, n \geq 1, n \in \mathbb{N}$ such that:

$$|ax^{2n} + bx + c|\sqrt{1-x^2} \leq 1, \forall |x| \leq 1.$$

Prove that: $|b + c| \leq 3^n$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Ravi Prakash-New Delhi-India

$$|ax^{2n} + bx + c|\sqrt{1-x^2} \leq 1, \forall |x| \leq 1, \text{ put } x = 0 \Rightarrow |c| \leq 1$$

and put $x = \pm \frac{1}{\sqrt{2}}$ to obtain $\frac{1}{\sqrt{2}} \left| \frac{a}{2^n} + \frac{b}{\sqrt{2}} + c \right| \leq 1$ and



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\frac{1}{\sqrt{2}} \left| \frac{a}{2^n} - \frac{b}{\sqrt{2}} + c \right| \leq 1$$

$$|b| = \frac{1}{\sqrt{2}} \left| \left(\frac{a}{2^n} + \frac{b}{\sqrt{2}} + c \right) - \left(\frac{a}{2^n} - \frac{b}{\sqrt{2}} + c \right) \right| \leq \frac{1}{\sqrt{2}} \left| \frac{a}{2^n} + \frac{b}{\sqrt{2}} + c \right| + \frac{1}{\sqrt{2}} \left| \frac{a}{2^n} - \frac{b}{\sqrt{2}} + c \right| \leq 2$$

$$\therefore |b + c| \leq |b| + |c| \leq 3 \Rightarrow |b + c| \leq 3^n$$

718. Prove that:

$$\log\left(\frac{\pi}{7}\right) > \cos e > \log\left(\frac{1}{\pi}\right)$$

Proposed by Olimjon Jalilov-Uzbekistan

Solution 1 by Adrian Popa-Romania

$$\cos e > \log\left(\frac{1}{\pi}\right) \Leftrightarrow \cos e > \log 1 - \log \pi \Leftrightarrow \cos e > -\log \pi$$

$$\left. \begin{array}{l} e < \pi \\ \frac{\pi}{2} < e < \pi \end{array} \right\} \Rightarrow -1 < \cos e < 0$$

$$\left. \begin{array}{l} \cos e > \cos \pi = -1 \\ \log \pi > \log e = 1 \Rightarrow -\log \pi < -1 \end{array} \right\} \Rightarrow \cos e > -\log \pi \Rightarrow \cos e > \log\left(\frac{1}{\pi}\right)$$

$$\cos e > -\log \pi > ? \log\left(\frac{\pi}{7}\right) \Leftrightarrow \cos(\pi - e) < \log\left(\frac{7}{\pi}\right) \Leftrightarrow \log\left(\frac{7}{\pi}\right) < \log e = 1 \Leftrightarrow \cos(\pi - e) < 1, \text{ which is true.}$$

Solution 2 by Hikmat Mammadov-Azerbaijan

$$\pi > e > \frac{5\pi}{6} \Rightarrow -1 < \cos e < -\frac{\sqrt{3}}{2}$$

$$\pi > e \Rightarrow \pi^{-1} < \Rightarrow \log\left(\frac{1}{\pi}\right) < -1$$

$$3 < \pi \Rightarrow \frac{1}{7} < \frac{\pi}{7}$$

$$1 > \frac{\sqrt{3}}{2} \Rightarrow e < e^{\frac{\sqrt{3}}{2}} \Rightarrow \frac{1}{e} > e^{-\frac{\sqrt{3}}{2}}$$

$$\frac{\pi}{7} > \frac{3}{7} > \frac{2}{5} > e^{-\frac{\sqrt{3}}{2}} \Rightarrow \frac{\pi}{7} > e^{-\frac{\sqrt{3}}{2}} \Rightarrow \log\left(\frac{\pi}{7}\right) > -\frac{\sqrt{3}}{2}$$

So, we get:



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\log\left(\frac{1}{\pi}\right) < -1 < \cos e < -\frac{\sqrt{3}}{2} < \log\left(\frac{\pi}{7}\right)$$

$$\Rightarrow \log\left(\frac{1}{\pi}\right) < \cos e < \log\left(\frac{\pi}{7}\right) \Leftrightarrow \log\left(\frac{\pi}{7}\right) > \cos e > \log\left(\frac{1}{\pi}\right)$$

719. If $x, y, z, t > 0$ such that $\frac{x^n}{1+x^n} + \frac{y^n}{1+y^n} + \frac{z^n}{1+z^n} + \frac{t^n}{1+t^n} = 3$ and $n > 0$ then

$$\sqrt{x^n y^n} + \sqrt{x^n z^n} + \sqrt{x^n t^n} + \sqrt{y^n z^n} + \sqrt{y^n t^n} + \sqrt{z^n t^n} \leq 2\sqrt{(xyzt)^n}$$

Proposed by Marin Chirciu-Romania

Solution by Amrit Awasthi-India

Substitute: $\frac{1}{x^n} = a^2, \frac{1}{y^n} = b^2, \frac{1}{z^n} = c^2, \frac{1}{t^n} = d^2$. Therefore,

$$\sum_{cyc} \frac{x^n}{1+x^n} = \sum_{cyc} \frac{1}{1+a^2} = \sum_{cyc} \left(1 - \frac{a^2}{1+a^2}\right) = 3 \Rightarrow \sum_{cyc} \frac{a^2}{1+a^2} = 1$$

Now, applying Bergstrom's we get

$$1 = \sum_{cyc} \frac{a^2}{1+a^2} \geq \frac{(a+b+c+d)^2}{4+a^2+b^2+c^2+d^2} \Rightarrow$$

$$4 + a^2 + b^2 + c^2 + d^2 \geq (a+b+c+d)^2 \Rightarrow$$

$$4 \geq 2(ab+ac+ad+bc+bd+cd) \Rightarrow$$

$$2 \geq \frac{1}{\sqrt{x^n y^n}} + \frac{1}{\sqrt{x^n z^n}} + \frac{1}{\sqrt{x^n t^n}} + \frac{1}{\sqrt{y^n z^n}} + \frac{1}{\sqrt{y^n t^n}} + \frac{1}{\sqrt{z^n t^n}} \Rightarrow$$

$$2\sqrt{(xyzt)^n} \geq \sqrt{x^n y^n} + \sqrt{x^n z^n} + \sqrt{x^n t^n} + \sqrt{y^n z^n} + \sqrt{y^n t^n} + \sqrt{z^n t^n}$$

720. If $0 < x \leq y \leq e \leq z \leq t$ then:

$$e^x + e^y + e^z + e^t \geq e^{x+y-e} + 3 \cdot \sqrt[3]{e^{z+t+e}}$$

Proposed by Daniel Sitaru-Romania

Solution by Kamel Gandouli Rezgui-Tunisia

$$e^x + e^y - e^e \stackrel{?}{\geq} e^{x+y-e}$$

Let $f(x) = e^x + e^y - e^e - e^{x+y-e}$ then $f'(x) = e^x - e^{x+y-e}$

$x + y - e \leq x$ because $e \geq y \Rightarrow f'(x) \geq 0 \Rightarrow f \nearrow$.



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\lim_{x \rightarrow 0^+} f(x) = e^y - e^e - e^{y-e}$$

Let $g(y) = e^y - e^e - y^{y-e}$ then $g'(y) = e^y - e^{y-e} \leq 0 \Rightarrow g \searrow \Rightarrow 1 - \frac{1}{e^e} \leq g(y) \leq e^e$

$\Rightarrow g(y) \geq 0$ because $1 - \frac{1}{e^e} > 0 \Rightarrow f(x) \geq g(y) \geq 1 - \frac{1}{e^e} > 0$

$$\Rightarrow e^x + e^y - e^e \geq e^{x+y-e}$$

$$\begin{aligned} e^x + e^y + e^z + e^t &= e^x + e^y - e^e + e^z + e^t + e^e \geq e^{x+y-e} + 3 \cdot \sqrt[3]{e^z e^t e^e} \geq \\ &\geq e^{x+y-e} + 3 \cdot \sqrt[3]{e^{z+t+e}} \end{aligned}$$

721. Given $a, x, y, z \in \mathbb{R}$ such that

$$xy + yz + zx - z - 2a^2 = x + y + z - 2a = 0.$$

Prove that: $|axyz| < \sqrt{3}$.

Proposed by Dang Le Gia Khanh-An Giang-Vietnam

Solution 1 by Kamel Gandouli Rezgui-Tunisia

$$x + y + z = 2a \Rightarrow x^2 + y^2 + z^2 + 2(xy + yz + zx) = 4a^2$$

$$\Rightarrow x^2 + y^2 + z^2 + 2(z + 2a^2) = 4a^2$$

$$\Rightarrow x^2 + y^2 + z^2 = -2|z|, z \leq 0$$

$$\Rightarrow |x|^2 + |y|^2 + |z|^2 = 2|z| \Rightarrow |x|^2 + |y|^2 = 2|z| - |z|^2 = |z|(2 - |z|) \text{ and } |z| \leq 2$$

$$\Rightarrow 2|x||y| \leq |z|(2 - |z|) \leq 1 \text{ because } \forall x \geq 0, x(2 - x) \leq 1 \Rightarrow |x||y| \leq \frac{1}{2}$$

$$|2a| = |x + y + z| \leq |x| + |y| + |z| \leq \sqrt{3}\sqrt{(|x|^2 + |y|^2 + |z|^2)} \leq$$

$$\leq \sqrt{3}\sqrt{2|z|} \leq \sqrt{6}\sqrt{|z|}$$

$$\Rightarrow 2|a| \leq \sqrt{6}\sqrt{|z|} \Rightarrow |a| \leq \frac{\sqrt{6}}{2}\sqrt{2} = \sqrt{3}$$

$$\Rightarrow |axyz| = |x||y||z||a| \leq \frac{1}{2} \cdot 2 \cdot \sqrt{3} = \sqrt{3}$$

Solution 2 by Jamal Issah-Ghana

$$xy + yz + zx - z - 2a^2 = 0 \Rightarrow xy + yz + zx = z + 2a^2; (1)$$

$$x + y + z - 2a = 0 \Rightarrow x + y = 2a - z; (2)$$

$$\text{From (1)} xy + (x + y)z = z + 2a^2$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$xy + (2a - z)z = z + 2a^2$$

$$xy + 2az - z^2 = z + 2a^2 \Rightarrow z^2 + 2a^2 + z - 2az - xy = 0$$

$$z^2 + (1 - 2a)z + 2a^2 - xy \Rightarrow (1 - 2a)^2 - 4(2a^2 - xy) = 0$$

$$\Rightarrow 4a^2 + 4a - 1 = 4xy$$

Hence, back to the initial question,

$$z^2 + (1 - 2a)z + 2a^2 - xy = 0 \Rightarrow 4z^2 + 4(1 - 2a)z + 8a^2 - 4xy = 0$$

$$\Rightarrow 4z^2 + 4(1 - 2a)z + 8a^2 - 4a^2 - 4a + 1 = 0$$

$$\Rightarrow 4z^2 + 4(1 - 2a)z + 4a^2 - 4a + 1 = 0$$

$$\Rightarrow 4z^2 + 4(1 - 2a)z + (2a - 1)^2 = 0$$

Let $(1 - 2a), (2a - 1) = 1$. So that the equation conforms to $4z^2 \pm 4z \pm 1 = 0$

When $2a - 1 = 1 \Rightarrow a = 1$

$$2a - 1 = -(1 - 2a) \Rightarrow a = 1 \Rightarrow 4z^2 - 4z + 1 = 0 \Rightarrow z = \frac{1}{2}$$

But: $4a^2 + 4a - 1 = 4xy$, since $a = 1 \Rightarrow xy = \frac{7}{4}$.

$$\Rightarrow axyz = 1 \cdot \frac{7}{4} \cdot \frac{1}{2} \Rightarrow |axyz| = |0.875|. \text{ Hence, } |axyz| < \sqrt{3}.$$

722. Let $\alpha, \beta \in \mathbb{R}$ and $\alpha\beta \neq 0$. Exists a function $f(x) = |\alpha x + \beta| \sqrt{1 - x^2}$

such that

$$|f(x)| \leq 1, \forall x \in [-1, 1]?$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Ravi Prakash-New Delhi-India

$$\text{Put } x = \sin \theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \Rightarrow |ax + b| \sqrt{1 - x^2} \leq 1, \forall x \in [-1, 1]$$

$$|a \sin \theta + b| \cos \theta \leq 1 \Rightarrow |a \sin \theta \cos \theta + b \cos \theta| \leq 1; (1)$$

$$\because \cos \theta \geq 0, \forall \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \text{ and } |a| \leq 2.$$

$$\text{Let: } f(\theta) = \frac{1}{2} \sin 2\theta + \cos \theta, \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$f'(\theta) = \cos 2\theta - \sin \theta = (1 + \sin \theta)(2 \sin \theta - 1)$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$f'(\theta) = 0 \Rightarrow \sin \theta = \frac{1}{2}, \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$f\left(-\frac{\pi}{2}\right) = 0 = f\left(\frac{\pi}{2}\right)$$

$$f\left(\frac{\pi}{6}\right) = \frac{1}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{4}$$

$$\text{Thus, } \left| \frac{4}{3\sqrt{3}}(x+1)\sqrt{1-x^2} \right| \leq 1$$

So, we may take $a = b = \frac{4\sqrt{3}}{9}$ other values are possible.

723. Let $a, b, c \in \mathbb{R}$ such that $|ax^2 + bx + c| \leq 1, \forall x \in [0, 1]$. Prove that:

$$\max\{|a|, |b|, |2a+b|, 8|c|\} \leq 8$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Ravi Prakash-New Delhi-India

Put $x \in \{-1, 0, \frac{1}{2}, 1\}$ to obtain $|c| \leq 1, |a+b+c| \leq 1, |a-b+c| \leq 1$

$$\left| \frac{1}{4}a + \frac{1}{2}b + c \right| \leq 1 \Rightarrow |c| \leq 1$$

$$2|b| = |(a+b+c) - (a-b+c)| \leq |a+b+c| + |a-b+c| \leq 1 + 1 = 2$$

$$\Rightarrow |b| \leq 1. \text{ Also, } |a+2b+4c| \leq 4 \Rightarrow -4 \leq a+2b+4c \leq 4$$

$$-4 \leq -4c \leq 4 \Rightarrow -8 \leq a+2b \leq 8 \Rightarrow |a+2b| \leq 8$$

Now, we show that $|a| \leq 8$

$$-1 \leq a+b+c \leq 1 \Rightarrow -4 \leq 4a+4b+4c \leq 4 \Rightarrow -4 \leq -a-2b-4c \leq 4$$

$\Rightarrow -8 \leq 3a+2b \leq 8$. Adding with $-8 \leq -a-2b \leq 8$, we get

$$-16 \leq 2a \leq 16 \Rightarrow |a| \leq 8. \text{ Thus,}$$

$$\max\{|a|, |b|, |2a+b|, 8|c|\} \leq 8$$

724. If $e < a \leq b$ then:

$$4^{a+3b} \cdot (a+3b)^{3a+b} \leq 4^{3a+b} \cdot (3a+b)^{a+3b}$$

Proposed by Daniel Sitaru-Romania



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Solution 1 by Florentin Vișescu-Romania

Let $f: (0, \infty) \rightarrow \mathbb{R}$, $f(x) = \frac{\log x}{x}$, then $f'(x) = \frac{1-\log x}{x^2}$, $\forall x \in (0, \infty)$

$$f'(x) = 0 \Rightarrow x = e.$$

x	0	e	∞
$f'(x)$	+	0	-
$f(x)$	\nearrow	$\frac{1}{e}$	\searrow

On $(0, e)$, f –increasing and (e, ∞) , f –decreasing

$x = e$ maxim point and $f(x) \leq \frac{1}{e}$, $\forall x \in (0, \infty)$

$$\frac{\log x}{x} \leq \frac{1}{e}, \forall x \in (0, \infty)$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{\log x}{x} = 0$$

$$e < a \leq b \Rightarrow \frac{a+3b}{4} > \frac{3a+b}{4} > e \Rightarrow f\left(\frac{a+3b}{4}\right) \leq f\left(\frac{3a+b}{4}\right)$$

$$\frac{\log\left(\frac{a+3b}{4}\right)}{\frac{a+3b}{4}} \geq \frac{\log\left(\frac{3a+b}{4}\right)}{\frac{3a+b}{4}} \Leftrightarrow (3a+b) \log\left(\frac{a+3b}{4}\right) \leq (a+3b) \log\left(\frac{3a+b}{4}\right)$$

$$\left(\frac{a+3b}{4}\right)^{3a+b} \leq \left(\frac{3a+b}{4}\right)^{a+3b} \Leftrightarrow 4^{a+3b} \cdot (a+3b)^{3a+b} \leq 4^{3a+b} \cdot (3a+b)^{a+3b}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $f(x) = \frac{\log(x)}{x}$, $x \geq e$. We have : $f'(x) = \frac{1 - \log(x)}{x^2} \leq 0$, $\forall x \geq e$

$\rightarrow f$ –decreasing on $[e, \infty[$

$$\text{Since : } \frac{a+3b}{4} \geq \frac{3a+b}{4} \geq e \rightarrow \frac{\log\left(\frac{a+3b}{4}\right)}{\frac{a+3b}{4}} \leq \frac{\log\left(\frac{3a+b}{4}\right)}{\frac{3a+b}{4}}$$

$$\Leftrightarrow (3a+b) \log\left(\frac{a+3b}{4}\right) \leq (a+3b) \log\left(\frac{3a+b}{4}\right)$$

$$\Leftrightarrow \left(\frac{a+3b}{4}\right)^{3a+b} \leq \left(\frac{3a+b}{4}\right)^{a+3b} \Leftrightarrow 4^{a+3b} \cdot (a+3b)^{3a+b} \leq 4^{3a+b} \cdot (3a+b)^{a+3b}$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\text{Therefore, } 4^{a+3b} \cdot (a+3b)^{3a+b} \leq 4^{3a+b} \cdot (3a+b)^{a+3b}.$$

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

For $e < a \leq b$ we give $a+3b = x, b+3a = y, x \geq y \geq 4e$, hence

$$4^{a+3b} \cdot (a+3b)^{3a+b} \leq 4^{3a+b} \cdot (3a+b)^{a+3b}$$

$$\Leftrightarrow 4^x \cdot x^y \leq 4^y \cdot y^x \text{ and since } x \geq y \geq 4e \text{ we give } x = ky, k \geq 1.$$

$$\text{Hence, } 4^x \cdot x^y \leq 4^y \cdot y^x \Leftrightarrow 4^{ky}(ky)^y \leq 4^y y^{ky} \Leftrightarrow$$

$$4^k \cdot ky \leq 4 \cdot y^k \Leftrightarrow 4^{k-1} \cdot k \leq y^{k-1}$$

$$4^{k-1} \cdot k \leq (4e)^{k-1} \Leftrightarrow k \leq e^{k-1}, 4^{k-1} \geq 1, k \geq 1.$$

725. Let $a, b, c \in R$ such that $(*) : |ax^2 + bx + c| \leq 1, \forall |x| \leq 1$.

Prove that : $|cx^2 + bx + a| \leq 2, \forall |x| \leq 1$.

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\begin{aligned} x = 0 \text{ in } (*) \rightarrow |c| \leq 1, x = 1 \text{ in } (*) \rightarrow |a + b + c| \leq 1 \text{ and } x = -1 \text{ in } (*) \\ \rightarrow |a - b + c| \leq 1. \end{aligned}$$

$$\begin{aligned} \text{we have : } |cx^2 + bx + a| &= \left| c(x^2 - 1) + \frac{1}{2}(a + b + c)(x + 1) + \frac{1}{2}(a - b + c)(1 - x) \right| \\ &\stackrel{\Delta}{\leq} |c| \cdot |x^2 - 1| + \frac{1}{2}|a + b + c| \cdot |x + 1| + \frac{1}{2}|a - b + c| \cdot |1 - x| \stackrel{|x| \leq 1}{\leq} 1 \cdot (1 - x^2) \\ &\quad + \frac{1}{2} \cdot 1 \cdot (x + 1) + \frac{1}{2} \cdot 1 \cdot (1 - x) = \\ &= 2 - x^2 \leq 2, \forall |x| \leq 1. \end{aligned}$$

$$\text{Therefore, } |cx^2 + bx + a| \leq 2, \forall |x| \leq 1.$$

726. Let $a, b, c \in R$ such that : $|ax^2 + bx + c| \sqrt{1 - x^2} \leq 1, \forall |x| \leq 1$.

Prove that : $|a + b + c| \leq 3$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{Let } f(x) = ax^2 + bx + c, \forall |x| \leq 1 \rightarrow |f(x)| \leq \frac{1}{\sqrt{1 - x^2}}, \forall |x| < 1.$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\text{We have : } f(0) = c, f\left(\frac{\sqrt{3}}{2}\right) = \frac{3}{4}a + \frac{\sqrt{3}}{2}b + c, f\left(-\frac{\sqrt{3}}{2}\right) = \frac{3}{4}a - \frac{\sqrt{3}}{2}b + c$$

$$\rightarrow a = \frac{2}{3} \left[f\left(\frac{\sqrt{3}}{2}\right) + f\left(-\frac{\sqrt{3}}{2}\right) - 2f(0) \right] \text{ and } b = \frac{\sqrt{3}}{3} \left[f\left(\frac{\sqrt{3}}{2}\right) - f\left(-\frac{\sqrt{3}}{2}\right) \right]$$

$$\text{Also, we have : } |f(0)| \leq 1, \left| f\left(\frac{\sqrt{3}}{2}\right) \right| \leq 2 \text{ and } \left| f\left(-\frac{\sqrt{3}}{2}\right) \right| \leq 2$$

$$\rightarrow |a + b + c| =$$

$$\begin{aligned} & \left| \frac{2}{3} \left(f\left(\frac{\sqrt{3}}{2}\right) + f\left(-\frac{\sqrt{3}}{2}\right) - 2f(0) \right) + \frac{\sqrt{3}}{3} \left(f\left(\frac{\sqrt{3}}{2}\right) - f\left(-\frac{\sqrt{3}}{2}\right) \right) + f(0) \right| = \\ &= \left| \frac{2 + \sqrt{3}}{3} f\left(\frac{\sqrt{3}}{2}\right) + \frac{2 - \sqrt{3}}{3} f\left(-\frac{\sqrt{3}}{2}\right) - \frac{1}{3} f(0) \right| \stackrel{\Delta}{\leq} \frac{2 + \sqrt{3}}{3} \cdot \left| f\left(\frac{\sqrt{3}}{2}\right) \right| \\ & \quad + \frac{2 - \sqrt{3}}{3} \cdot \left| f\left(-\frac{\sqrt{3}}{2}\right) \right| + \frac{1}{3} |f(0)| \\ & \leq \frac{2 + \sqrt{3}}{3} \cdot 2 + \frac{2 - \sqrt{3}}{3} \cdot 2 + \frac{1}{3} \cdot 1 = 3. \end{aligned}$$

$$\text{Therefore, } |a + b + c| \leq 3.$$

727.

$$a, b, c > 0, a + b + c = 3, \Omega(a) = \prod_{k=1}^n \left(1 + \frac{k}{an^2} \right), n \in \mathbb{N}, n \geq 1$$

Prove that:

$$a\Omega(a) + b\Omega(b) + c\Omega(c) \leq e^{\frac{1}{2a}} + e^{\frac{1}{2b}} + e^{\frac{1}{2c}}$$

Proposed by Daniel Sitaru-Romania

Solution by Kamel Gandouli Rezgui-Tunisia

$$\begin{aligned} \Omega(a) &= \prod_{k=1}^n \left(1 + \frac{k}{an^2} \right) \stackrel{AGM}{\leq} \left(\frac{1}{n} \sum_{k=1}^n \left(1 + \frac{k}{an^2} \right) \right)^n = \left(\frac{n + \frac{n^2+n}{2an^2}}{n} \right)^n = \left(1 + \frac{n^2+n}{2an^3} \right)^n = \\ &= e^{n \log \left(1 + \frac{n^2+n}{2an^3} \right)}. \text{ Let } a_n = e^{n \log \left(1 + \frac{n^2+n}{2an^3} \right)} \nearrow \text{ and } a_n \geq 0 \Rightarrow \end{aligned}$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$e^{n \log\left(1 + \frac{n^2+n}{2an^3}\right)} \leq \lim_{n \rightarrow \infty} e^{n \log\left(1 + \frac{n^2+n}{2an^3}\right)} \leq e^{\frac{1}{2a}}$$

$$\Omega(a) \leq e^{\frac{1}{2a}}. \text{ Analogous, } \Omega(b) \leq e^{\frac{1}{2b}} \text{ and } \Omega(c) \leq e^{\frac{1}{2c}}.$$

$$\text{If } a \leq b \leq c \Rightarrow \Omega(a) \geq \Omega(b) \geq \Omega(c)$$

$$a\Omega(a) + b\Omega(b) + c\Omega(c) \stackrel{\text{Chebyshev}}{\leq} \frac{a+b+c}{3}(\Omega(a) + \Omega(b) + \Omega(c))$$

Therefore,

$$a\Omega(a) + b\Omega(b) + c\Omega(c) \leq e^{\frac{1}{2a}} + e^{\frac{1}{2b}} + e^{\frac{1}{2c}}$$

728.

Let $a, b, c \in \mathbb{R}$ such that $\left| x^3 \cdot \int_0^1 (ax^2 + bx + c) dx \right| \sqrt{1 - x^2} \leq 1, \forall x \leq 1$.

Prove that: $|2a + 3b + 6c| \leq 48$.

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution 1 by Ravi Prakash-New Delhi-India

$$\begin{aligned} & \left| x^3 \cdot \int_0^1 (ax^2 + bx + c) dx \right| \sqrt{1 - x^2} \leq 1, \forall x \leq 1 \\ & \Rightarrow \left| x^3 \left(\frac{a}{3} + \frac{b}{2} + c \right) \right| \sqrt{1 - x^2} < -1, \forall x \in [-1, 1]; (1) \end{aligned}$$

It holds for $x \in \{-1, 0, 1\}$

For $0 < |x| < 1$, put $x = \sin \theta, 0 < |\theta| < \frac{\pi}{2}$, (1) becomes

$$|2a + 3b + 6c| |\sin^3 \theta| \cos \theta \leq 6 \text{ for } 0 < |\theta| < \frac{\pi}{2}.$$

$$|2a + 3b + 6c| \sin^3 \theta \cos \theta \leq 6 \text{ for } 0 < \theta < \frac{\pi}{2}$$

Let $f(\theta) = \sin^3 \theta \cos \theta, 0 < \theta \leq \frac{\pi}{2}$ then

$$\begin{aligned} f'(\theta) &= 3 \sin^2 \theta \cos^2 \theta - \sin^4 \theta = \sin^2 \theta (3 \cos^2 \theta - 1 + \cos^2 \theta) = \\ &= \sin^2 \theta (2 \cos \theta + 1)(2 \cos \theta - 1) \end{aligned}$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$f'(\theta) \text{ is } \begin{cases} > 0, \text{ if } \theta \in \left(0, \frac{\pi}{3}\right) \\ 0, \text{ if } \theta = \frac{\pi}{3} \\ < 0, \text{ if } \theta \in \left(\frac{\pi}{3}, \frac{\pi}{2}\right) \end{cases}$$

As $\max(|2a + 3b + 6c| \sin^3 \theta \cos \theta) \leq 6$

$$|2a + 3b + 6c| \cdot \frac{3\sqrt{3}}{16} \leq 6 \Rightarrow |2a + 3b + 6c| \leq \frac{32}{\sqrt{3}} \leq 48.$$

Solution 2 by Kamel Gandouli Rezgui-Tunisia

$$\begin{aligned} & \left| x^3 \cdot \int_0^1 (ax^2 + bx + c) dx \right| \sqrt{1 - x^2} \leq 1, \forall x \leq 1 \\ & \Rightarrow \left| x^3 \left(\frac{a}{3} + \frac{b}{2} + c \right) \right| \sqrt{1 - x^2} < -1, \forall x \in [-1, 1] \\ & \Rightarrow \left| x^3 \cdot \frac{2a + 3b + 6c}{6} \right| \sqrt{1 - x^2} \leq 1, \forall x \in [-1, 1] \\ & \Rightarrow |2a + 3b + 6c| \leq \frac{6}{|x^3| \sqrt{1 - x^2}}, \forall x \in [-1, 1]; (1) \end{aligned}$$

$$\begin{aligned} \text{Let } f(x) &= x^6(1 - x^2) - \frac{1}{64} = x^6 - x^8 - \frac{1}{64}; |x| \leq 1 \\ f'(x) &= 6x^5 - 8x^7 = x^5(6 - 8x^2) \end{aligned}$$

$$f'(x) = 0 \Rightarrow x \in \left\{ -\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2} \right\}$$

$\ln \left[0, \frac{\sqrt{3}}{2} \right]$ is continue, decreasing and $f(0) < 0, f\left(\frac{\sqrt{3}}{2}\right) > 0 \Rightarrow \exists \alpha \in \left[0, \frac{\sqrt{3}}{2} \right]$ such that

$$f(\alpha) = 0 \Rightarrow \alpha^6(1 - \alpha^2) = \frac{1}{64} \Rightarrow \alpha^3 \sqrt{1 - \alpha^2} = \frac{1}{8}; (2)$$

From (1),(2) it follows that $|2a + 3b + 6c| \leq 48$.

729.

Let $a, b \in \mathbb{R}$ such that $\left| \int_{-1}^x (at + b) dt \right| \sqrt{1 - x^2} \leq 1, \forall |x| \leq 1$.

Prove that: $|b| \leq \frac{3}{2}$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Kamel Gandouli Rezgui-Tunisia

$$\left| \int_{-1}^x (at + b) dt \right| \sqrt{1 - x^2} \leq 1, \forall |x| \leq 1$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\Rightarrow \left| \left[a \frac{t^2}{2} + bt \right]_{-1}^x \right| \sqrt{1 - x^2} \leq 1, \forall |x| \leq 1$$

$$\left| \frac{a}{2} x^2 + bx - \frac{a}{2} + b \right| \sqrt{1 - x^2} \leq 1, \forall |x| \leq 1$$

$$x = 0 \Rightarrow \left| b - \frac{a}{2} \right| \leq 1 \Rightarrow \left| \frac{a}{2} - b \right| \leq 1; (1)$$

$$x = \frac{4}{5} \Rightarrow \left| \frac{16}{50}a + \frac{4}{5}b - \frac{a}{2} + b \right| \cdot \frac{3}{5} \leq 1$$

$$\Rightarrow \left| -\frac{9}{50}a + \frac{9}{5}b \right| \leq \frac{5}{3} \Rightarrow \left| -\frac{a}{2} + 5b \right| \leq \frac{5}{3} \cdot \frac{25}{9} = \frac{125}{27}; (2)$$

$$\text{From (1),(2) we get: } |4b| \leq \frac{125}{27} + 1 = \frac{152}{27}$$

$$\text{So, } |b| \leq \frac{152}{108} = \frac{38}{27} \leq \frac{3}{2}$$

730. If $a, b, x, y > 0$ then:

$$32ab(ax + by)^4 \leq (a + b)^4(8a^2x^4 + ab(x + y)^4 + 8b^2y^4)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\begin{aligned} (a + b)^4(8a^2x^4 + ab(x + y)^4 + 8b^2y^4) &\stackrel{AGM}{\geq} \\ &\geq (a + b)^4 \left(8a^2x^4 + ab(2\sqrt{xy})^4 + 8b^2y^4 \right) = \\ &= 8(a + b)^2(a^2 + 2ab + b^2)(a^2x^4 + 2abx^2y^2 + b^2y^4) \stackrel{CBS}{\geq} \\ &\geq 8(a + b)^2(a^2x^2 + 2abxy + b^2y^2)^2 \stackrel{AGM}{\geq} \\ &\geq 8 \cdot 4ab(ax + by)^4 = 32ab(ax + by)^4 \end{aligned}$$

Therefore,

$$32ab(ax + by)^4 \leq (a + b)^4(8a^2x^4 + ab(x + y)^4 + 8b^2y^4)$$

Equality holds for $a = b$.

Solution 2 by Kunihiko Chikaya-Tokyo-Japan

$$\text{Let: } f(t) = \frac{8a^2 + ab(1+t)^4 + 8b^2t^4}{(a+bt)^4}; \left(t = \frac{x}{y} > 0 \right)$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 f'(t) &= \frac{(4ab(1+t)^3 + 32b^2t^3)(a+bt)^4 - (8a^2 + ab(1+t)^4 + 8b^2t^4) \cdot 4(a+bt)^3}{(a+bt)^8} \\
 &= 4b \cdot \frac{(a+bt)(a(1+t)^3 + 8bt^3) - (8a^2 + ab(1+t)^4 + 8b^2t^4)}{(a+bt)^5} = \\
 &= \frac{4ab(t-1) \left((a+7b)t^3 + 3(a-b)t^2 + 3(a-b)t - (7a+b) \right)}{(a+bt)^5} = \\
 &= \frac{4ab((a+7b)t^2 + 4(a+b)t + 7a+b)}{(a+bt)^5} \cdot (t-1)
 \end{aligned}$$

Since $a, b > 0$, for $t > 0$, the sign of $f'(t)$ coincides with $t-1$, thus $f(t)$ has a local minimum at $t = 1$, which is also the minimum value. Therefore,

$$(a+b)^2 \geq 4ab, \text{ here by obtain}$$

$$f(t) \geq f(1) = \frac{8(a+b)^2}{(a+b)^4} \geq \frac{32ab}{(a+b)^4}.$$

Equality holds if and only if $t = 1 \Leftrightarrow a = b$.

731. Let $\alpha, \beta, \gamma, \delta \in R$ such that : $|\alpha x^3 + \beta x^2 + \gamma x + \delta| \leq 1, \forall |x| \leq 1$.

Prove that : a) $|3\alpha x^2 + 2\beta x + \gamma| \leq 9, \forall |x| \leq 1$

b) $|\delta x^3 + \gamma x^2 + \beta x + \alpha| \leq 4, \forall |x| \leq 1$.

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

a) Firstly, let's prove the lemma : $|a+b| + |a-b| = 2 \cdot \max\{|a|, |b|\}, \forall a, b \in R$
WLOG we may assume that : $a \geq b \geq 0 \rightarrow |a+b| + |a-b| = (a+b) + (a-b) = 2a = 2|a| = 2 \cdot \max\{|a|, |b|\}$.
 $\rightarrow \forall a, b \in R, |a+b| + |a-b| = 2 \cdot \max\{|a|, |b|\}$.

Now, let $f(x) = \alpha x^3 + \beta x^2 + \gamma x + \delta, x \in [-1, 1] \rightarrow |f(x)| \leq 1, \forall x \in [-1, 1]$

We have : $f(1) = \alpha + \beta + \gamma + \delta, f(-1) = -\alpha + \beta - \gamma + \delta, f\left(\frac{1}{2}\right) = \frac{\alpha}{8} + \frac{\beta}{4} + \frac{\gamma}{2} + \delta, f\left(-\frac{1}{2}\right) = -\frac{\alpha}{8} + \frac{\beta}{4} - \frac{\gamma}{2} + \delta$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 \rightarrow \alpha &= \frac{2}{3} \left[f(1) - f(-1) - 2f\left(\frac{1}{2}\right) + 2f\left(-\frac{1}{2}\right) \right], \beta = \frac{2}{3} \left[f(1) + f(-1) - f\left(\frac{1}{2}\right) - f\left(-\frac{1}{2}\right) \right] \\
 \gamma &= \frac{1}{6} \left[8f\left(\frac{1}{2}\right) - 8f\left(-\frac{1}{2}\right) - f(1) + f(-1) \right], \delta = \frac{1}{6} \left[4f\left(\frac{1}{2}\right) + 4f\left(-\frac{1}{2}\right) - f(1) - f(-1) \right] \\
 &\rightarrow |3ax^2 + 2\beta x + \gamma| = \\
 &= \left| 3 \cdot \frac{2}{3} \left[f(1) - f(-1) - 2f\left(\frac{1}{2}\right) + 2f\left(-\frac{1}{2}\right) \right] x^2 \right. \\
 &\quad + 2 \cdot \frac{2}{3} \left[f(1) + f(-1) - f\left(\frac{1}{2}\right) - f\left(-\frac{1}{2}\right) \right] x \\
 &\quad \left. + \frac{1}{6} \left[8f\left(\frac{1}{2}\right) - 8f\left(-\frac{1}{2}\right) - f(1) + f(-1) \right] \right| = \\
 &= \left| \left(2x^2 - \frac{1}{6} + \frac{4}{3}x \right) f(1) - \left(2x^2 - \frac{1}{6} - \frac{4}{3}x \right) f(-1) - \left(4x^2 - \frac{4}{3} + \frac{4}{3}x \right) f\left(\frac{1}{2}\right) \right. \\
 &\quad \left. + \left(4x^2 - \frac{4}{3} - \frac{4}{3}x \right) f\left(-\frac{1}{2}\right) \right| \leq \\
 &\stackrel{\Delta}{\leq} \left| 2x^2 - \frac{1}{6} + \frac{4}{3}x \right| \cdot |f(1)| + \left| 2x^2 - \frac{1}{6} - \frac{4}{3}x \right| \cdot |f(-1)| + \left| 4x^2 - \frac{4}{3} + \frac{4}{3}x \right| \cdot \left| f\left(\frac{1}{2}\right) \right| \\
 &\quad + \left| 4x^2 - \frac{4}{3} - \frac{4}{3}x \right| \cdot \left| f\left(-\frac{1}{2}\right) \right| \leq \\
 &\stackrel{|f(x)| \leq 1}{\leq} \left| \left(2x^2 - \frac{1}{6} \right) + \frac{4}{3}x \right| + \left| \left(2x^2 - \frac{1}{6} \right) - \frac{4}{3}x \right| + \left| \left(4x^2 - \frac{4}{3} \right) + \frac{4}{3}x \right| + \left| \left(4x^2 - \frac{4}{3} \right) - \frac{4}{3}x \right| \\
 &\stackrel{\text{Lemma}}{\leq} 2 \cdot \max \left\{ \left| 2x^2 - \frac{1}{6} \right|, \frac{4}{3}|x| \right\} + 2 \cdot \max \left\{ \left| 4x^2 - \frac{4}{3} \right|, \frac{4}{3}|x| \right\} \stackrel{|x| \leq 1}{\leq} 2 \left(2 - \frac{1}{6} \right) + 2 \left(4 - \frac{4}{3} \right) \\
 &= 9
 \end{aligned}$$

Therefore, $|3ax^2 + 2\beta x + \gamma| \leq 9, \forall |x| \leq 1$.

$$\begin{aligned}
 b) \quad &\text{We have : } |\delta x^3 + \gamma x^2 + \beta x + \alpha| = \\
 &= \left| \frac{1}{6} \left[4f\left(\frac{1}{2}\right) + 4f\left(-\frac{1}{2}\right) - f(1) - f(-1) \right] x^3 \right. \\
 &\quad + \frac{1}{6} \left[8f\left(\frac{1}{2}\right) - 8f\left(-\frac{1}{2}\right) - f(1) + f(-1) \right] x^2 \\
 &\quad + \frac{2}{3} \left[f(1) + f(-1) - f\left(\frac{1}{2}\right) - f\left(-\frac{1}{2}\right) \right] x \\
 &\quad \left. + \frac{2}{3} \left[f(1) - f(-1) - 2f\left(\frac{1}{2}\right) + 2f\left(-\frac{1}{2}\right) \right] \right| = \\
 &= \left| \frac{1}{6} (-x^3 - x^2 + 4x + 4) f(1) - \frac{1}{6} (x^3 - x^2 - 4x + 4) f(-1) \right. \\
 &\quad - \frac{2}{3} (-x^3 - 2x^2 + x + 2) f\left(\frac{1}{2}\right) + \frac{2}{3} (x^3 - 2x^2 - x + 2) f\left(-\frac{1}{2}\right) \left. \right| \stackrel{\Delta}{\leq} \\
 &\quad \frac{1}{6} |-x^3 - x^2 + 4x + 4| \cdot |f(1)| + \frac{1}{6} |x^3 - x^2 - 4x + 4| \cdot |f(-1)| +
 \end{aligned}$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 & + \frac{2}{3} |-x^3 - 2x^2 + x + 2| \cdot \left| f\left(\frac{1}{2}\right) \right| \\
 & + \frac{2}{3} |x^3 - 2x^2 - x + 2| \cdot \left| f\left(-\frac{1}{2}\right) \right| \stackrel{|f(x)| \leq 1}{\lesssim} \frac{1}{6} |-x^3 - x^2 + 4x + 4| + \\
 & + \frac{1}{6} |x^3 - x^2 - 4x + 4| + \frac{2}{3} |-x^3 - 2x^2 + x + 2| + \frac{2}{3} |x^3 - 2x^2 - x + 2| = \\
 & = \frac{1}{6} |(x+1)(4-x^2)| + \frac{1}{6} |(1-x)(4-x^2)| + \frac{2}{3} |(x+2)(1-x^2)| + \frac{2}{3} |(2-x)(1-x^2)| \\
 & \stackrel{|x| \leq 1}{=} \frac{1}{6}(x+1)(4-x^2) + \frac{1}{6}(1-x)(4-x^2) + \frac{2}{3}(x+2)(1-x^2) + \frac{2}{3}(2-x)(1-x^2) \\
 & = \frac{1}{6} \cdot 2 \cdot (4-x^2) + \frac{2}{3} \cdot 4 \cdot (1-x^2)
 \end{aligned}$$

Therefore, $|\delta x^3 + \gamma x^2 + \beta x + \alpha| \leq 4 - 3x^2 \leq 4, \forall |x| \leq 1$.

732. Let $\alpha, \beta, \gamma, \delta \in R$ and $\alpha\beta\gamma\delta \neq 0$.

Exists a function $f(x) = \alpha x^3 + \beta x^2 + \gamma x + \delta$ such that

$$|f(x)| \leq 1, \forall |x| \leq 1?$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let's prove that the function $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{2}{3}x - \frac{1}{2}$,

$x \in [-1, 1]$ satisfies the given problem.

We have : $f'(x) = x^2 + x + \frac{2}{3} = \left(x + \frac{1}{2}\right)^2 + \frac{5}{12} > 0, \forall x \in [-1, 1]$

$\rightarrow f$ – increasing on $[-1, 1]$.

$\rightarrow \min_{x \in [-1, 1]} \{f(x)\} = f(-1) = -1$ and $\max_{x \in [-1, 1]} \{f(x)\} = f(1) = 1 \rightarrow$

$$|f(x)| \leq 1, \forall |x| \leq 1.$$

$\text{Therefore, there exists a function } f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{2}{3}x - \frac{1}{2},$

$x \in [-1, 1]$ such that $|f(x)| \leq 1, \forall |x| \leq 1$.



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

733. If $A_1A_2 \dots A_n (n \geq 3)$ is convex polygon with $a_k, k = \overline{1, n}$ the sides' lengths and s the semi-primeter, $m \in [1, \infty)$, $S_n(m) = \sum_{k=1}^n a_k^m$, then prove that:

$$\sum_{k=1}^n \frac{S_n(m) - a_k^m}{(s - a_k)^m} \geq \frac{2^m n(n-1)}{(n-2)^m}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by Soumitra Mandal-Chandar Nagore-India

$$s = \frac{1}{2} \sum_{k=1}^n a_k \Rightarrow \frac{a_1^m + a_2^m + \dots + a_{n-1}^m}{n-1} \geq \left(\frac{a_1 + a_2 + \dots + a_{n-1}}{n-1} \right)^m$$

$x \rightarrow x^m$ is convex function for all $m \geq 2$, then

$$\frac{a_1^m + a_2^m + \dots + a_{n-1}^m}{n-1} \geq \left(\frac{2s - a_n}{n-1} \right)^m \Rightarrow$$

$$\frac{S_n(m) - a_n^m}{(s - a_n)^m} \geq \frac{1}{(n-1)^{m-1}} \left(\frac{2s - a_n}{s - a_n} \right)^m$$

$$\begin{aligned} \sum_{k=1}^n \frac{S_n(m) - a_k^m}{(s - a_k)^m} &\geq \frac{1}{(n-1)^{m-1}} \sum_{k=1}^n \left(\frac{2s - a_k}{s - a_k} \right)^m \geq \frac{1}{(n-1)^{m-1}} \cdot \frac{1}{n^{m-1}} \cdot \left(\sum_{k=1}^n \frac{2s - a_k}{s - a_k} \right)^m = \\ &= \frac{1}{(n-1)^{m-1}} \cdot \frac{1}{n^{m-1}} \cdot \left(s \cdot \sum_{k=1}^n \frac{1}{s - a_k} + n \right)^m \stackrel{\text{Bergstrom}}{\geq} \\ &\geq \frac{1}{(n-1)^{m-1}} \cdot \frac{1}{n^{m-1}} \left(\frac{n^2 s}{ns - 2s} + n \right)^m = \\ &= \frac{1}{(n-1)^{m-1}} \cdot \frac{1}{n^{m-1}} \left(\frac{n^2}{n-2} + n \right)^m = \frac{2^m n(n-1)}{(n-2)^m} \end{aligned}$$

734. Let $a, b, c \in R$ such that : $|ax^2 + bx + c| \leq 1, \forall x \in [0, 1]$.

Prove that : $\max\{|a|, |b|, |2a+b|, 8|c|\} \leq 8$.

Proposed by Nguyen Van Canh-Ben Tre-Vietnam



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $f(x) = ax^2 + bx + c \rightarrow |f(x)| \leq 1, \forall x \in [0, 1]$.

We have : $f(0) = c, f\left(\frac{1}{2}\right) = \frac{a}{4} + \frac{b}{2} + c$ and $f(1) = a + b + c$

$$\rightarrow a = 2\left(f(1) - 2f\left(\frac{1}{2}\right) + f(0)\right), b = -f(1) + 4f\left(\frac{1}{2}\right) - 3f(0) \text{ and } c = f(0).$$

$$|a| = 2 \left| f(1) - 2f\left(\frac{1}{2}\right) + f(0) \right| \stackrel{\Delta}{\geq} 2 \left(|f(1)| + 2 \left| f\left(\frac{1}{2}\right) \right| + |f(0)| \right) \stackrel{|f(x)| \leq 1, \forall x \in [0, 1]}{\geq} 2(1 + 2 \cdot 1 + 1) = 8.$$

$$|b| = \left| -f(1) + 4f\left(\frac{1}{2}\right) - 3f(0) \right| \stackrel{\Delta}{\geq} |f(1)| + 4 \left| f\left(\frac{1}{2}\right) \right| + 3|f(0)| \stackrel{|f(x)| \leq 1, \forall x \in [0, 1]}{\leq} 1 + 4 \cdot 1 + 3 \cdot 1 = 8.$$

$$|2a + b| = \left| 3f(1) - 4f\left(\frac{1}{2}\right) + f(0) \right| \stackrel{\Delta}{\geq} 3|f(1)| + 4 \left| f\left(\frac{1}{2}\right) \right| + |f(0)| \stackrel{|f(x)| \leq 1, \forall x \in [0, 1]}{\geq} 3 \cdot 1 + 4 \cdot 1 + 1 = 8.$$

$$\text{Also, } 8|c| = 8|f(0)| \stackrel{|f(0)| \leq 1}{\leq} 8 \cdot 1 = 8.$$

Therefore, $\max\{|a|, |b|, |2a + b|, 8|c|\} \leq 8$.

735. Let $a, b \in \mathbb{R}$ such that $|ax^3 + bx| \leq 1, \forall x \in [-1, 1]$

Prove that: $|a| \leq 4, |b| \leq 3$.

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution 1 by Kamel Gandouli Rezgui-Tunisia

$$|ax^3 + bx| \leq 1, \forall x \in [-1, 1] \Rightarrow |ax^2 + b| \leq \frac{1}{|x|}, \forall x \in [-1, 1]$$

For $x = 1 \Rightarrow |a + b| \leq 1 \Rightarrow |-a - b| \leq 1$; (1)

For $x = \frac{1}{2} \Rightarrow \left| \frac{a}{4} + b \right| \leq 2 \Rightarrow |a + 4b| \leq 8$; (2)

From (1),(2) we get: $|3b| \leq 9 \Rightarrow |b| \leq 3$.

$$|a| = |a + b - b| \leq |a + b| + |b| \leq 4.$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Solution 2 by Ravi Prakash-New Delhi-India

Let $f(x) = ax^3 + bx$, then $|f(x)| \leq 1, \forall x \in [-1, 1]$

$$f\left(\frac{1}{2}\right) = \frac{a}{8} + \frac{b}{2}; \quad (1)$$

$$f(1) = a + b; \quad (2)$$

From (1),(2), we get:

$$\frac{3}{4}a = f(1) - 2f\left(\frac{1}{2}\right)$$

$$3b = 8f\left(\frac{1}{2}\right) - f(1) \Rightarrow a = \frac{4}{3}\left(f(1) - 2f\left(\frac{1}{2}\right)\right)$$

$$\Rightarrow |a| \leq \frac{4}{3}\left[|f(1)| + 2\left|f\left(\frac{1}{2}\right)\right|\right] = 4 \text{ and } 3|b| \leq 8\left[\left|f\left(\frac{1}{2}\right)\right| + |f(1)|\right] = 9 \\ \Rightarrow |b| \leq 3.$$

736. Let $a, b, c \in R$ such that:

$$|ax^2 + bx + c|\sqrt{1-x^2} \leq 1, \forall x \in [-1, 1].$$

Prove that : $|a| \leq 4$.

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $f(x) = ax^2 + bx + c, x \in [-1, 1] \rightarrow |f(x)| \leq \frac{1}{\sqrt{1-x^2}}, \forall x \in (-1, 1)$

$$\rightarrow |f(0)| \leq 1, \left|f\left(\frac{\sqrt{3}}{2}\right)\right| \leq 2 \text{ and } \left|f\left(-\frac{\sqrt{3}}{2}\right)\right| \leq 2$$

$$\text{With : } f(0) = c, f\left(\frac{\sqrt{3}}{2}\right) = \frac{3}{4}a + \frac{\sqrt{3}}{2}b + c \text{ and } f\left(-\frac{\sqrt{3}}{2}\right) = \frac{3}{4}a - \frac{\sqrt{3}}{2}b + c \\ \rightarrow a = \frac{2}{3}\left[f\left(\frac{\sqrt{3}}{2}\right) + f\left(-\frac{\sqrt{3}}{2}\right) - 2f(0)\right] \\ \rightarrow |a| \stackrel{\Delta}{\leq} \frac{2}{3}\left(\left|f\left(\frac{\sqrt{3}}{2}\right)\right| + \left|f\left(-\frac{\sqrt{3}}{2}\right)\right| + 2|f(0)|\right) \leq \frac{2}{3}(2 + 2 + 2 \cdot 1) = 4.$$

Therefore, $|a| \leq 4$.



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Solution 2 by Adrian Popa-Romania

$$x \in [-1, 1], \text{ let } x = \sin t, t \in [0, 2\pi]$$

$$|a \sin^2 t + b \sin t + c| \cdot |\cos t| \leq 1, \forall t \in [0, 2\pi]$$

$$\text{If } t = 0 \Rightarrow |c| \leq 1 \Rightarrow -1 \leq c \leq 1$$

$$\text{If } t = \frac{\pi}{3} \Rightarrow \left| \frac{3a}{4} + \frac{\sqrt{3}b}{2} + c \right| \cdot \frac{1}{2} \leq 1 \Rightarrow |3a + 2\sqrt{3}b + 4c| \leq 8; (1)$$

$$\text{If } t = \frac{4\pi}{3} \Rightarrow \left| \frac{3a}{4} - \frac{\sqrt{3}}{2}b + c \right| \cdot \frac{1}{2} \leq 1 \Rightarrow |3a - 2\sqrt{3}b + 4c| \leq 8; (2)$$

$$\Rightarrow \begin{cases} -8 \leq 3a + 2\sqrt{3}b + 4c \leq 8 \\ -8 \leq 3a - 2\sqrt{3}b + 4c \leq 8 \end{cases} \Rightarrow -16 \leq 6a + 8c \leq 16$$

$$\text{If } c \leq 1 \Rightarrow -16 < 6a + 8 \Rightarrow -4 < a \text{ ad if } c = -1 \Rightarrow a \leq 4. \text{ Thus, } |a| = 4.$$

737. Let $a, b \in R$ such that $|ax + b|\sqrt{1-x^2} \leq 1, \forall |x| \leq 1$. Prove that:

$$|a + b| \leq 2.$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{Let } f(x) = ax + b, x \in [-1, 1] \rightarrow |f(x)| \leq \frac{1}{\sqrt{1-x^2}}, \forall x \in (-1, 1)$$

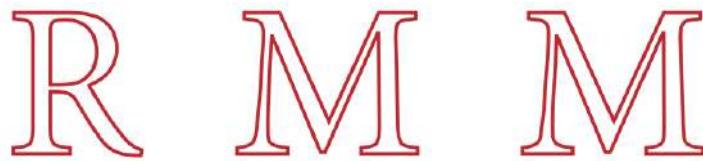
$$\rightarrow \left| f\left(\pm \frac{\sqrt{2}}{2}\right) \right| \leq \sqrt{2} \quad (1)$$

$$\text{With : } f\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}a + b \text{ and } f\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}}{2}a + b$$

$$\rightarrow a = \frac{\sqrt{2}}{2} \left(f\left(\frac{\sqrt{2}}{2}\right) - f\left(-\frac{\sqrt{2}}{2}\right) \right) \text{ and } b = \frac{1}{2} \left(f\left(\frac{\sqrt{2}}{2}\right) + f\left(-\frac{\sqrt{2}}{2}\right) \right)$$

$$\begin{aligned} \rightarrow |a + b| &= \left| \frac{\sqrt{2}+1}{2} f\left(\frac{\sqrt{2}}{2}\right) - \frac{\sqrt{2}-1}{2} f\left(-\frac{\sqrt{2}}{2}\right) \right| \stackrel{\Delta}{\leq} \frac{\sqrt{2}+1}{2} \left| f\left(\frac{\sqrt{2}}{2}\right) \right| \\ &\quad + \frac{\sqrt{2}-1}{2} \left| f\left(-\frac{\sqrt{2}}{2}\right) \right| \leq \end{aligned}$$

$$\stackrel{(1)}{\leq} \frac{\sqrt{2}+1}{2} \cdot \sqrt{2} + \frac{\sqrt{2}-1}{2} \cdot \sqrt{2} = 2. \quad \text{Therefore, } |a + b| \leq 2.$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

738.

Let $f(x) = ax^{2n} + bx + c$, ($a, b, c \in \mathbb{R}$, $n \in \mathbb{N}^*$), $f(0), f(1), f(-1) \in [-1, 1]$.

Prove that : $|f(x)| \leq \frac{2n-1}{\sqrt[2n-1]{4^n \cdot n^{2n}}} + 1, \forall |x| \leq 1$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have : $f(0) = c$, $f(1) = a + b + c$ and $f(-1) = a - b + c$

$$\rightarrow a = \frac{1}{2}(f(1) + f(-1) - 2 \cdot f(0)), b = \frac{1}{2}(f(1) - f(-1)) \text{ and } c = f(0).$$

$$\rightarrow |f(x)| = \left| \frac{1}{2}(f(1) + f(-1) - 2 \cdot f(0))x^{2n} + \frac{1}{2}(f(1) - f(-1))x + f(0) \right| =$$

$$= \left| \frac{1}{2}x(x^{2n-1} + 1)f(1) - \frac{1}{2}x(1 - x^{2n-1})f(-1) + (1 - x^{2n})f(0) \right| \leq$$

$$\stackrel{\Delta}{\leq} \frac{1}{2}|x| \cdot |x^{2n-1} + 1| \cdot |f(1)| + \frac{1}{2}|x| \cdot |1 - x^{2n-1}| \cdot |f(-1)| + |1 - x^{2n}| \cdot |f(0)| \leq$$

$$\stackrel{|x| \leq 1}{\leq} \frac{1}{2}|x| \cdot (x^{2n-1} + 1) \cdot 1 + \frac{1}{2}|x| \cdot (1 - x^{2n-1}) \cdot 1 + (1 - x^{2n}) \cdot 1$$

$$= |x| + 1 - x^{2n} \stackrel{?}{\leq} \frac{2n-1}{\sqrt[2n-1]{4^n \cdot n^{2n}}} + 1$$

$$\leftrightarrow \frac{2n-1}{\sqrt[2n-1]{4^n \cdot n^{2n}}} + x^{2n} \geq |x| \text{ which is true from AM - GM :}$$

$$\frac{2n-1}{\sqrt[2n-1]{4^n \cdot n^{2n}}} + x^{2n} = \underbrace{\frac{1}{\sqrt[2n-1]{4^n \cdot n^{2n}}} + \dots + \frac{1}{\sqrt[2n-1]{4^n \cdot n^{2n}}}}_{2n-1 \text{ times}} + |x|^{2n} \stackrel{AM-GM}{\leq}$$

$$\geq 2n \cdot \sqrt[2n]{\left(\frac{1}{\sqrt[2n-1]{4^n \cdot n^{2n}}}\right)^{2n-1} \cdot |x|^{2n}} = |x|$$

$$\text{Therefore, } |f(x)| \leq \frac{2n-1}{\sqrt[2n-1]{4^n \cdot n^{2n}}} + 1, \forall |x| \leq 1.$$

739. If $n \in \mathbb{N}^* - \{1\}$, $a \in \mathbb{R}_+$, $b, c, d, m, p \in \mathbb{R}_+^*$, $x_k \in \mathbb{R}_+^*$, $k = \overline{1, n}$,

$X_{n,m} = \sum_{k=1}^n x_k^m$, $X_{n,p} = \sum_{k=1}^n x_k^p$ such that $c \cdot X_{n,p} > d \cdot \max_{1 \leq k \leq n} x_k^p$ then prove:



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\sum_{k=1}^n \frac{a \cdot X_{n,m} + b \cdot x_k^m}{c \cdot X_{n,p} - d \cdot x_k^p} \geq \frac{n(an+b)}{(cn-d)} \cdot \frac{X_{n,m}}{X_{n,p}}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\begin{aligned} WLOG, we may assume that : x_1 &\geq x_2 \geq \dots \geq x_n \rightarrow a \cdot X_{n,m} + b \cdot x_1^m \\ &\geq a \cdot X_{n,m} + b \cdot x_2^m \geq \dots \geq a \cdot X_{n,m} + b \cdot x_n^m \end{aligned}$$

$$And : \frac{1}{c \cdot X_{n,p} - d \cdot x_1^p} \geq \frac{1}{c \cdot X_{n,p} - d \cdot x_2^p} \geq \dots \geq \frac{1}{c \cdot X_{n,p} - d \cdot x_n^p}$$

→ From Chebyshev's inequality, we have :

$$\begin{aligned} \sum_{k=1}^n \frac{a \cdot X_{n,m} + b \cdot x_k^m}{c \cdot X_{n,p} - d \cdot x_k^p} &\geq \frac{1}{n} \left[\sum_{k=1}^n (a \cdot X_{n,m} + b \cdot x_k^m) \right] \left(\sum_{k=1}^n \frac{1}{c \cdot X_{n,p} - d \cdot x_k^p} \right) \stackrel{CBS}{\geq} \\ &\geq \frac{1}{n} \cdot (an+b) X_{n,m} \cdot \frac{n^2}{\sum (c \cdot X_{n,p} - d \cdot x_k^p)} = \frac{n(an+b) X_{n,m}}{(cn-d) X_{n,p}}. \end{aligned}$$

$$Therefore, \quad \sum_{k=1}^n \frac{a \cdot X_{n,m} + b \cdot x_k^m}{c \cdot X_{n,p} - d \cdot x_k^p} \geq \frac{n(an+b)}{(cn-d)} \cdot \frac{X_{n,m}}{X_{n,p}}$$

740. Let $x, y \in [1, \frac{\pi}{2}]$ such that :

$$\sqrt[5]{\sin^{10} x + \sin^{10} y} + \sqrt[5]{\cos^{10} x + \cos^{10} y} = \sqrt[5]{2} \text{ and}$$

$$x^2 + (2 + \sqrt[3]{y})^{\sqrt[3]{x}} \leq 3y + 1. \text{ Find } x \text{ and } y.$$

Proposed by Pavlos Trifon-Greece

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sqrt[5]{\sin^{10} x + \sin^{10} y} + \sqrt[5]{\cos^{10} x + \cos^{10} y} \stackrel{(1)}{\cong} \sqrt[5]{2} \text{ and } x^2 + (2 + \sqrt[3]{y})^{\sqrt[3]{x}} \stackrel{(2)}{\cong} 3y + 1.$$

By Power Mean inequality, we know that : $\sqrt[5]{\frac{a^5 + b^5}{2}} \stackrel{(*)}{\geq} \frac{a+b}{2}, \forall a, b \geq 0,$

with equality iff $a = b$.



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 & \rightarrow \sqrt[5]{2} \stackrel{(1)}{\cong} \sqrt[5]{\sin^{10} x + \sin^{10} y} + \sqrt[5]{\cos^{10} x + \cos^{10} y} \\
 & = \sqrt[5]{2} \left(\sqrt[5]{\frac{(\sin^2 x)^5 + (\sin^2 y)^5}{2}} + \sqrt[5]{\frac{(\cos^2 x)^5 + (\cos^2 y)^5}{2}} \right) \geq \\
 & \stackrel{(*)}{\geq} \sqrt[5]{2} \left(\frac{\sin^2 x + \sin^2 y}{2} + \frac{\cos^2 x + \cos^2 y}{2} \right) = \\
 & \sqrt[5]{2} \cdot \frac{(\sin^2 x + \cos^2 x) + (\sin^2 y + \cos^2 y)}{2} = \sqrt[5]{2}.
 \end{aligned}$$

With equality iff $\sin^2 x = \sin^2 y$ and $\cos^2 x = \cos^2 y \leftrightarrow x = y$.

$$\begin{aligned}
 & \rightarrow (2) \leftrightarrow 3x + 1 \geq x^2 + (2 + \sqrt[3]{x})^{\sqrt[3]{x}} \\
 & = x^2 + [1 + (1 + \sqrt[3]{x})]^{\sqrt[3]{x}} \stackrel{\text{Bernoulli}, x \geq 1}{\geq} x^2 + 1 + \sqrt[3]{x}(1 + \sqrt[3]{x}) = \\
 & = (x^2 + \sqrt[3]{x^2} + \sqrt[3]{x}) + 1 \stackrel{\text{AM-GM}}{\geq} 3\sqrt[3]{x^2 \cdot \sqrt[3]{x^2} \cdot \sqrt[3]{x}} + 1 = 3x + 1.
 \end{aligned}$$

With equality iff $x^2 = \sqrt[3]{x^2} = \sqrt[3]{x} \leftrightarrow x = 1$.

Therefore, $x = y = 1$.

Solution 2 by proposer

$$\begin{aligned}
 & \sqrt[5]{\sin^{10} x + \sin^{10} y} + \sqrt[5]{\cos^{10} x + \cos^{10} y} = \\
 & = \sqrt[5]{(\sin^2 x)^5 + (\sin^2 y)^5} + \sqrt[5]{(\cos^2 x)^5 + (\cos^2 y)^5} \stackrel{\text{Minkovski}}{\geq} \\
 & \geq \sqrt[5]{(\sin^2 x + \cos^2 x)^5 + (\sin^2 y + \cos^2 y)^5} = \sqrt[5]{2}
 \end{aligned}$$

Equality holds for

$$\frac{\sin^2 x}{\sin^2 y} = \frac{\cos^2 x}{\cos^2 y}; x, y \in (0, \frac{\pi}{2}) \Leftrightarrow \sin(x - y) = 0, \forall x, y \in (0, \frac{\pi}{2}) \Leftrightarrow x = y.$$

So, $\sqrt[5]{\sin^{10} x + \sin^{10} y} + \sqrt[5]{\cos^{10} x + \cos^{10} y} = \sqrt[5]{2} \Leftrightarrow x = y$.

For $x = y$ and $x^2 + (2 + \sqrt[3]{y})^{\sqrt[3]{x}} \leq 3y + 1 \Rightarrow x^2 + (2 + \sqrt[3]{x})^{\sqrt[3]{x}} \leq 3x + 1$; (1)

But: $x^2 + \sqrt[3]{x} + (\sqrt[3]{x})^2 \stackrel{\text{AGM}}{\geq} 3 \cdot \sqrt[3]{x^2 \cdot \sqrt[3]{x} \cdot (\sqrt[3]{x})^2} = 3\sqrt[3]{x^3} = 3x \Rightarrow$

$x^2 + \sqrt[3]{x} + (\sqrt[3]{x})^2 \geq 3x \Rightarrow (1 + \sqrt[3]{x})^{\sqrt[3]{x}} \geq 3x - x^2$; (2)



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Equality holds for $x^2 = (\sqrt[3]{x})^2 \Leftrightarrow x = 1$.

$$(2 + \sqrt[3]{x})^{\frac{3}{\sqrt{x}}} = (1 + (1 + \sqrt[3]{x}))^{\frac{3}{\sqrt{x}}} \stackrel{\text{Bernoulli}}{\geq} 1 + (1 + \sqrt[3]{x})^3 \sqrt{x} \stackrel{(2)}{\geq} 1 + 3x - x^2 \Rightarrow x^2 + (2 + \sqrt[3]{x})^{\frac{3}{\sqrt{x}}} \geq 1 + 3x; (3)$$

Equality holds for $\sqrt[3]{x} = 1 \Leftrightarrow x = 1$.

From (1) and (3) it follows that

$$x^2 + (2 + \sqrt[3]{x})^{\frac{3}{\sqrt{x}}} = 3x + 1 \Rightarrow x = 1 \Rightarrow x = y = 1.$$

741. Let $a, b \in \mathbb{R}$ such that $\frac{|ax+b|}{1+x^2} \leq 1, \forall x \in \mathbb{R}$.

Prove that: $|a| \leq 2, |b| \leq 1$.

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution 1 by Ravi Prakash-New Delhi-India

$$\frac{|ax+b|}{1+x^2} \leq 1, \forall x \in [-1, 1]$$

For $x = 0$, so that $|b| \leq 1$.

$$\text{For } x \neq 0, \text{ we have: } \frac{\left| \frac{a+b}{x} \right|}{|x| + \frac{1}{|x|}} \leq 1 \Leftrightarrow$$

$$-\left(|x| + \frac{1}{|x|} \right) \leq a + \frac{b}{x} \leq |x| + \frac{1}{|x|}$$

Put $x = \{-1, 1\} \Rightarrow \begin{cases} -2 \leq a+b \leq 2 \\ -2 \leq a-b \leq 2 \end{cases}$ and adding, we get:

$$-4 \leq 2a \leq 4 \Rightarrow |a| \leq 2.$$

Therefore, $|a| \leq 2$ and $|b| \leq 1$.

Solution 2 by Hikmat Mammadov-Azerbaijan

For $x = 0 \Rightarrow |b| \leq 1$; (1)

For $x = 1 \Rightarrow |a+b| \leq 2$; (2)

For $x = -1 \Rightarrow |a-b| \leq 2$; (3)

$$(1) \Rightarrow |b| = \frac{|a \cdot 0 + b|}{a^2 + 1} \leq 1 \Rightarrow \frac{|b|}{1} \leq 1 \Rightarrow |b| \leq 1$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

From (2) and (3), we get:

$$|a| = \left| \frac{2a}{a} \right| = \frac{|a+b+a-b|}{2} \leq \frac{|a+b| + |a-b|}{2} \leq \frac{2+2}{2} = 2 \Rightarrow |a| \leq 2.$$

Therefore, $|a| \leq 2$ and $|b| \leq 1$.

742. Let $a, b, c \in \mathbb{R}$ such that $|ax^4 + bx^2 + c| \leq 1, \forall |x| \leq 1$. Prove that:

$$\max\{|4a + 2b|, |2b|, |16c|\} \leq 16.$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Ravi Prakash-New Delhi-India

$$\text{Let } f(x) = ax^4 + bx^2 + c, f(0) = c, f(1) = a + b + c$$

$$f\left(\frac{1}{\sqrt{2}}\right) = \frac{a}{4} + \frac{b}{2} + c$$

$$\begin{cases} a + b = f(1) - f(0) \\ a + 2b = 4f\left(\frac{1}{\sqrt{2}}\right) - 4f(0) \end{cases} \Rightarrow |a + 2b| \leq 4f(1) + 4f(0) = 8$$

$$\Rightarrow |2a + 4b| \leq 16. \text{ Also, } b = 4f\left(\frac{1}{2}\right) - f(1) - 3f(0)$$

$$\Rightarrow |b| = 4 + 1 + 3 = 8 \Rightarrow |2b| \leq 16.$$

$$\text{Also, } a = 4f\left(\frac{1}{\sqrt{2}}\right) - 2f(0) - 2f(1) \Rightarrow |a| \leq 4 + 2 + 2 = 8 \Rightarrow |2a| \leq 16$$

Thus,

$$\max\{|4a + 2b|, |2b|, |16c|\} \leq 16.$$

743. If $x \in \mathbb{R}$ then:

$$e^{e^{-x}+x} + e^{e^{-x}-x} \geq 2 \cosh x \cdot e^{\operatorname{sech} x}$$

Proposed by Daniel Sitaru-Romania

Solution by Kamel Gandouli Rezgui-Tunisia

$$\text{Let } f(y) = e^y - ye^{\frac{1}{y}}, y \geq 1 \Rightarrow \frac{1}{y} \leq 1 \Rightarrow e^{\frac{1}{y}} \leq e \Rightarrow ye^{\frac{1}{y}} \leq ey$$

$$\Rightarrow e^y - ye^{\frac{1}{y}} \geq e^y - ey \geq 0, \text{ because } y \geq 1 \Rightarrow \forall y \geq 1: e^y \geq ye^{\frac{1}{y}}$$

$$\cosh x \geq 1, \forall x \in \mathbb{R} \Rightarrow e^{\cosh x} \geq \cosh x \cdot e^{\frac{1}{\cosh x}}, \forall x \in \mathbb{R}$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 2e^{\cosh x} &\geq 2 \cosh x \cdot e^{\cosh x} = 2 \cosh x e^{\operatorname{sech} x} \\
 e^{e^{-x}+x} + e^{e^{-x}-x} &\stackrel{AGM}{\geq} 2\sqrt{e^{e^{-x}+x} \cdot e^{e^{-x}-x}} = 2\sqrt{e^{e^{-x}+e^x}} = \\
 &= 2\sqrt{e^{2\cosh x}} = 2e^{\cosh x} \geq 2 \cosh x \cdot e^{\operatorname{sech} x}
 \end{aligned}$$

744. If $x \geq 0$ then:

$$\frac{e^x}{\sqrt[3]{1+e^{-x}}} + \frac{e^{-x}}{\sqrt[3]{1+e^x}} \geq \frac{2}{\sqrt[3]{1+\operatorname{sech} x}}$$

Proposed by Daniel Sitaru-Romania

Solution by Kamel Gandouli Rezgui-Tunisia

$$\begin{aligned}
 \frac{e^x}{\sqrt[3]{1+e^{-x}}} + \frac{e^{-x}}{\sqrt[3]{1+e^x}} &\stackrel{AGM}{\geq} 2 \sqrt{\frac{e^x}{\sqrt[3]{1+e^{-x}}} \cdot \frac{e^{-x}}{\sqrt[3]{1+e^x}}} = \frac{2}{\sqrt[3]{\frac{x}{e^2} + e^{-\frac{x}{2}}}} \\
 x \geq 0 \Rightarrow e^{\frac{x}{2}} + e^{-\frac{x}{2}} &\geq 2 \Rightarrow e^{\frac{x}{2}} + e^{-\frac{x}{2}} - 1 \geq 1 \\
 \left(e^{\frac{x}{2}} + e^{-\frac{x}{2}} - 1\right)(e^x + e^{-x}) &\geq (e^x + e^{-x}) \geq 2 \\
 \Rightarrow \left(e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right)(e^x + e^{-x}) &\geq e^x + e^{-x} + 2 \\
 1 + \frac{2}{e^x + e^{-x}} \leq e^{\frac{x}{2}} + e^{-\frac{x}{2}} &\Rightarrow 1 + \operatorname{sech} x \leq e^{\frac{x}{2}} + e^{-\frac{x}{2}} \Rightarrow \\
 \frac{2}{\sqrt[3]{\frac{x}{e^2} + e^{-\frac{x}{2}}}} &\geq \frac{2}{\sqrt[3]{1+\operatorname{sech} x}} \\
 \text{Therefore } \frac{e^x}{\sqrt[3]{1+e^{-x}}} + \frac{e^{-x}}{\sqrt[3]{1+e^x}} &\geq \frac{2}{\sqrt[3]{1+\operatorname{sech} x}}
 \end{aligned}$$

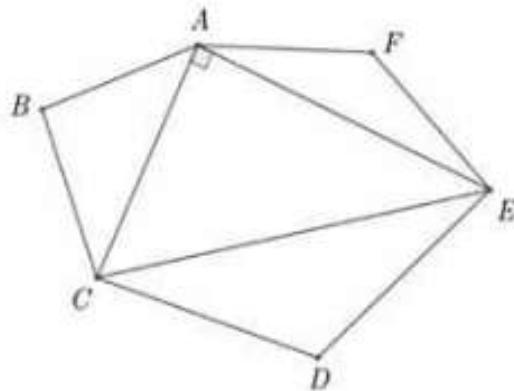
745. If in $ABCDEF$ – convex hexagon, $AB = BC$, $CD = DE$ and

$EF = FA$, ΔACE is a right triangle, then find the minimum value of

$$\Omega = \frac{BC}{BE} + \frac{DE}{DA} + \frac{FA}{FC}$$

Proposed by Dang Le Gia Khanh-Vietnam

Solution 1 by proposer



Apply the Ptolemy's inequality to the quadrilateral $ABCE$, we have:

$$AB \cdot CE + BC \cdot AE \geq AC \cdot BE$$

$$AB = BC \Rightarrow BC(CE + AE) \geq AC \cdot BE \Leftrightarrow \frac{BC}{BE} \geq \frac{AC}{CE + EA}$$

Similarly, we have:

$$\frac{DE}{DA} \geq \frac{CE}{EA + AC}, \frac{FA}{FC} \geq \frac{EA}{AC + CE}$$

Hence,

$$\frac{BC}{BE} + \frac{DE}{DA} + \frac{FA}{FC} \geq \frac{AC}{CE + EA} + \frac{CE}{EA + AC} + \frac{EA}{AC + CE}$$

Let $AC = c, CE = a, EA = b, \angle ACE = \alpha, \angle AEC = \beta$ ($\because \alpha + \beta = 90^\circ$)

Thus, we get:

$$\begin{aligned} \Omega &= \frac{BC}{BE} + \frac{DE}{DA} + \frac{FA}{FC} = \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}; (a, b, c > 0 \text{ and } a^2 = b^2 + c^2) \\ \Omega &= \frac{1}{b+\frac{c}{a}} + \frac{\frac{b}{a}}{\frac{c}{a}+1} + \frac{\frac{c}{a}}{1+\frac{b}{a}} = \frac{1}{\sin \alpha + \sin \beta} + \frac{\sin \alpha}{\sin \beta + 1} + \frac{\sin \beta}{1 + \sin \alpha} = \\ &= \frac{1}{\sin \alpha + \sin \beta} + (\sin \alpha + \sin \beta + 1) \left(\underbrace{\frac{1}{\sin \alpha + 1} + \frac{1}{\sin \beta + 1}}_{\text{By CBS}} \right) - 2 \geq \\ &\geq \frac{1}{\sin \alpha + \sin \beta} + \frac{4(\sin \alpha + \sin \beta + 1)}{\sin \alpha + \sin \beta + 2} - 2 \end{aligned}$$

Let $t = \sin \alpha + \sin \beta = \sin \alpha + \cos \alpha = \sqrt{2} \sin\left(\alpha + \frac{\pi}{4}\right)$

$$\Rightarrow 1 < t \leq \sqrt{2} \Rightarrow \Omega \geq \frac{1}{t} + \frac{4(t+1)}{t+2} - 2 = f(t)$$

$$f'(t) = -\frac{1}{t^2} + \frac{4}{(t+2)^2};$$

$f'(t) = 0 \Leftrightarrow t = 2$ (eliminated).

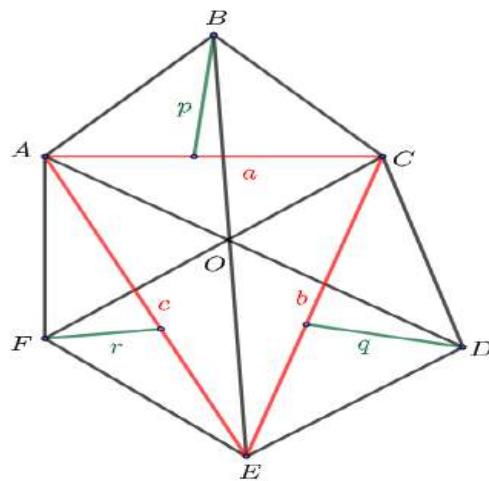
t	1	$\sqrt{2}$
$f'(t)$		-
$f(t)$		$f(\sqrt{2})$

Therefore,

$$\Omega \geq \min_{(1, \sqrt{2}]} f(t) = f(\sqrt{2}) = \frac{5\sqrt{2} - 4}{2}$$

$$\min\left(\frac{BC}{BE} + \frac{DE}{DA} + \frac{FA}{FC}\right) = \frac{5\sqrt{2} - 4}{2}$$

Solution 2 by Hikmat Mammadov-Azerbaijan



Let $AE = c, CE = b, AC = a, BC = \sqrt{a^2 + p^2}, BE = \sqrt{a^2 + (2b + p)^2}$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\Rightarrow \Omega = \frac{BE}{BC} = \frac{\sqrt{a^2 + (2b+p)^2}}{\sqrt{a^2 + p^2}} = \frac{\sqrt{1 + \left(\frac{2b}{a} + \frac{p}{a}\right)^2}}{\sqrt{1 + \left(\frac{p}{a}\right)^2}}$$

$$x = \frac{p}{a}, z = \frac{b}{a} \Rightarrow \Omega = \frac{\sqrt{1 + (2z+x)^2}}{\sqrt{1 + x^2}}$$

$$\frac{d\Omega}{dx} = \frac{2(2z+x)}{2\sqrt{1 + (2z+x)^2}\sqrt{1+x^2}} - \frac{2x\sqrt{1 + (2z+x)^2}}{2(1+x^2)\sqrt{1+x^2}} = 0$$

$$\Rightarrow (1+x^2)(2z+x) = x[1 + (2z+x)^2] \Rightarrow x^2 + 2z - 1 = 0$$

$$x = \sqrt{z^2 + 1} - 1 = 1, \text{ for } z = \frac{b}{a} = 1 \Rightarrow$$

$$\Omega_{max} = \frac{\sqrt{1 + (2+1)^2}}{\sqrt{1+1}} = \sqrt{5}$$

$$\left[\frac{BC}{BE}\right]_{min} = \frac{1}{\Omega_{max}} = \frac{1}{\sqrt{5}}, \text{ similarly, } \left[\frac{DE}{DA}\right]_{min} = \frac{1}{\sqrt{5}}, \text{ with } b = a, c = \sqrt{2}a$$

$$\Rightarrow \begin{cases} FA = \sqrt{c^2 + r^2} \\ FC = c + r \end{cases} \Rightarrow \omega = \frac{FA}{FC} = \frac{\sqrt{c^2 + r^2}}{c + r} = \frac{\sqrt{1 + \left(\frac{r}{c}\right)^2}}{1 + \frac{r}{c}}$$

$$\Rightarrow y = \frac{r}{c} \Rightarrow \omega = \frac{\sqrt{1+y^2}}{1+y} \Rightarrow \frac{d\omega}{dy} = \frac{\frac{y(1+y)}{\sqrt{1+y^2}} - \sqrt{1+y^2}}{(1+y)^2} = 0$$

$$\Rightarrow y^2 + y = 1 + y^2 \Rightarrow y = 1.$$

$$\omega_{min} = \frac{1}{\sqrt{2}}$$

$$\min \left(\frac{BC}{BE} + \frac{DE}{DA} + \frac{FA}{FC} \right) = \frac{5\sqrt{2} - 4}{2}$$

746. If $x, y, a, b > 0$ then:

$$\frac{2x^2}{\sqrt{ab}} + \frac{4y^2}{a+b} \geq \frac{4(x+y)^2}{(\sqrt{a} + \sqrt{b})^2} + \sqrt{ab} \left(\frac{x}{\sqrt{ab}} - \frac{2y}{a+b} \right)^2$$

Proposed by Daniel Sitaru-Romania



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Solution 1 by George Florin Șerban-Romania

$$\begin{aligned}
 \frac{x}{y} = t \Rightarrow \frac{2t^2}{\sqrt{ab}} + \frac{4}{a+b} &\geq \frac{4(t+1)^2}{(\sqrt{a} + \sqrt{b})^2} + \frac{t^2}{\sqrt{ab}} - \frac{4t}{a+b} + \frac{4\sqrt{ab}}{(a+b)^2} \\
 \frac{t^2}{\sqrt{ab}} + \frac{4t+4}{a+b} - \frac{4\sqrt{ab}}{(a+b)^2} &\geq \frac{4(t+1)^2}{(\sqrt{a} + \sqrt{b})^2} \\
 \frac{(a+b)^2 t^2 + (4t+4)(a+b)\sqrt{ab} - 4ab}{\sqrt{ab}(a+b)^2} &\geq \frac{4(t+1)^2}{(\sqrt{a} + \sqrt{b})^2} \\
 (\sqrt{a} + \sqrt{b})^2 [(a+b)^2 t^2 + (4t+4)(a+b)\sqrt{ab} - 4ab] &\geq 4(t+1)^2 \sqrt{ab}(a+b)^2 \\
 [(a+b)^3 - 2\sqrt{ab}(a^2 + b^2 + 2ab)]t^2 - [4\sqrt{ab}(a+b)^2 - 8ab(a+b)]t + \\
 + 4ab(a+b - 2\sqrt{ab}) &\geq 0 \\
 (a+b)^2(a+b - 2\sqrt{ab})t^2 - 4\sqrt{ab}(a+b)(a+b - 2\sqrt{ab})t + 4ab(\sqrt{a} - \sqrt{b})^2 &\geq 0 \\
 (\sqrt{a} - \sqrt{b})^2 [(a+b)^2 t^2 - 4\sqrt{ab}(a+b)t + 4ab] &\geq 0 \\
 (\sqrt{a} - \sqrt{b})^2 [(a+b)t - 2\sqrt{ab}]^2 &\geq 0 \\
 \text{If } (\sqrt{a} - \sqrt{b})^2 = 0 \Rightarrow a = b & \\
 \text{If } [(a+b)t - 2\sqrt{ab}]^2 = 0 \Rightarrow (a+b)t = 2\sqrt{ab} \Rightarrow \frac{x}{y} = \frac{2\sqrt{ab}}{a+b} & \\
 \text{Equality holds for } a = b \text{ or } \frac{x}{y} = \frac{2\sqrt{ab}}{a+b}. &
 \end{aligned}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\begin{aligned}
 \left(\frac{2x^2}{\sqrt{ab}} + \frac{4y^2}{a+b} \right) - \frac{4(x+y)^2}{(\sqrt{a} + \sqrt{b})^2} \\
 = \frac{2}{(\sqrt{a} + \sqrt{b})^2} \cdot \left[\left(\frac{(\sqrt{a} + \sqrt{b})^2}{\sqrt{ab}} - 2 \right) x^2 + 2 \left(\frac{(\sqrt{a} + \sqrt{b})^2}{a+b} - 1 \right) y^2 - 4xy \right] \\
 = \frac{2}{(\sqrt{a} + \sqrt{b})^2} \cdot \left(\frac{a+b}{\sqrt{ab}} \cdot x^2 - 4xy + \frac{4\sqrt{ab}}{a+b} \cdot y^2 \right) \\
 = \frac{2(a+b)\sqrt{ab}}{(\sqrt{a} + \sqrt{b})^2} \cdot \left(\frac{x^2}{ab} - \frac{4xy}{\sqrt{ab}(a+b)} + \frac{4y^2}{(a+b)^2} \right) =
 \end{aligned}$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 &= \frac{2(a+b)}{(\sqrt{a} + \sqrt{b})^2} \cdot \sqrt{ab} \left(\frac{x}{\sqrt{ab}} - \frac{2y}{a+b} \right)^2 \stackrel{CBS}{\geq} 1 \cdot \sqrt{ab} \left(\frac{x}{\sqrt{ab}} - \frac{2y}{a+b} \right)^2 \\
 &= \sqrt{ab} \left(\frac{x}{\sqrt{ab}} - \frac{2y}{a+b} \right)^2.
 \end{aligned}$$

Therefore,

$$\frac{2x^2}{\sqrt{ab}} + \frac{4y^2}{a+b} \geq \frac{4(x+y)^2}{(\sqrt{a} + \sqrt{b})^2} + \sqrt{ab} \left(\frac{x}{\sqrt{ab}} - \frac{2y}{a+b} \right)^2.$$

Solution 3 by Ravi Prakash-New Delhi-India

Let $A = \frac{a+b}{2}$, $G = \sqrt{ab}$, $x = Gt_1$, $y = At_2$. The inequality becomes:

$$\frac{2(Gt_1)^2}{G} + \frac{4(At_2)^2}{2A} \geq \frac{4(Gt_1 + At_2)^2}{2A + 2G} + G \left(\frac{Gt_1}{G} - \frac{2At_2}{2A} \right)^2$$

$$2(Gt_1^2 + At_2^2) - G(t_1 - t_2)^2 \geq \frac{2(Gt_1 + At_2)^2}{A + G}$$

$$2(Gt_1^2 + At_2^2) - G(t_1^2 + t_2^2 - 2t_1t_2) \geq \frac{2(G^2t_1^2 + A^2t_2^2 - 2t_1t_2)}{A + G}$$

$$(A + G)[Gt_1^2 + (2A - G)t_2^2] + 2(A + G)Gt_1t_2 - 2G^2t_1^2 - 2A^2t_2^2 - 4GAt_1t_2 \geq 0$$

$$G(A - G)t_1^2 + G(A - G)t_2^2 + 2G(G - A)t_1t_2 \geq 0$$

$$G(A - G)(t_1 - t_2)^2 \geq 0, \text{ which is true as } G > 0 \text{ and } A \geq G.$$

747. If $a, b, c > 1$ and $d < 0$, then:

$$(\log_{a^2bc} a)^d + (\log_{b^2ca} b)^d + (\log_{c^2ab} c)^d \geq \frac{3}{4^d}$$

Proposed by Pavlos Trifon-Greece

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$(\log_{a^2bc} a)^d + (\log_{b^2ca} b)^d + (\log_{c^2ab} c)^d \stackrel{(*)}{\geq} \frac{3}{4^d}$$

Let $x = \log a$, $y = \log b$, $z = \log c$, $t = -d \rightarrow x, y, z, t > 0$

$$(*) \leftrightarrow \left(\frac{x}{2x + y + z} \right)^{-t} + \left(\frac{y}{2y + z + x} \right)^{-t} + \left(\frac{z}{2z + x + y} \right)^{-t} \geq \frac{3}{4^{-t}} \leftrightarrow$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\sum_{cyc} \left(\frac{2x+y+z}{x} \right)^t \geq 3 \cdot 4^t$$

$$\begin{aligned} \sum_{cyc} \left(\frac{2x+y+z}{x} \right)^t &\stackrel{AM-GM}{\geq} 3 \sqrt[3]{\prod_{cyc} \left(\frac{2x+y+z}{x} \right)^t} \\ &= 3 \cdot \sqrt[3]{\frac{[(x+y)+(x+z)][(y+z)+(y+x)][(z+x)+(z+y)]}{xyz}}^t \geq \\ &\stackrel{Cesaro}{\geq} 3 \cdot \sqrt[3]{\frac{8(x+y)(y+z)(z+x)}{xyz}}^t \stackrel{Cesaro}{\geq} 3 \cdot \sqrt[3]{\frac{8 \cdot 8xyz}{xyz}}^t = 3 \cdot 4^t \rightarrow (*) \text{ is true.} \end{aligned}$$

Therefore, $(\log_{a^2bc} a)^d + (\log_{b^2ca} b)^d + (\log_{c^2ab} c)^d \geq \frac{3}{4^d}$.

Solution 2 by proposer

Lemma. If $x, y, z > 0$, then

$$\sum_{cyc} \frac{x}{2x+y+z} \leq \frac{3}{4}$$

$$\begin{aligned} \text{Proof. } \sum_{cyc} \frac{1}{2x+y+z} &\stackrel{\text{Bergstrom}}{\geq} \frac{(1+1+1)^2}{(2x+y+z) + (x+2y+z) + (x+y+2z)} \\ &= \frac{9}{4(x+y+z)}; (1) \end{aligned}$$

Equality holds for $x = y = z$.

$$\begin{aligned} \sum_{cyc} \frac{x}{2x+y+z} \leq \frac{3}{4} \Leftrightarrow \sum_{cyc} \left(1 - \frac{x}{2x+y+z} \right) &\geq 3 - \frac{3}{4} \Leftrightarrow \\ \sum_{cyc} \frac{1}{2x+y+z} &\geq \frac{9}{4(x+y+z)} \Leftrightarrow (1) \end{aligned}$$

$$\begin{aligned} \text{Now, } \log_{a^2bc} a + \log_{ab^2c} b + \log_{abc^2} c &= \frac{\log a}{\log(a^2bc)} + \frac{\log b}{\log(ab^2c)} + \frac{\log c}{\log(abc^2)} = \\ &= \frac{\log a}{2\log a + \log b + \log c} + \frac{\log b}{\log a + 2\log b + \log c} + \frac{\log c}{\log a + \log b + 2\log c} \end{aligned}$$

Let $x = \log a, y = \log b, z = \log c; x, y, z > 0$, then we have:



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\sum_{cyc} \frac{x}{2x+y+z} \stackrel{\text{Lemma}}{\leq} \frac{3}{4}; (2)$$

Equality holds for $x = y = z \Leftrightarrow a = b = c$. For $1 > 0 > d$, we get:

$$\begin{aligned} \left(\frac{(\log_{a^2bc} a)^d + (\log_{ab^2c} b)^d + (\log_{abc^2} c)^d}{3} \right)^{\frac{1}{d}} &\leq \frac{\log_{a^2bc} a + \log_{ab^2c} b + \log_{abc^2} c}{3} \\ \Leftrightarrow \left(\frac{(\log_{a^2bc} a)^d + (\log_{ab^2c} b)^d + (\log_{abc^2} c)^d}{3} \right)^{\frac{1}{d}} &\leq \frac{1}{4} \\ \Leftrightarrow (\log_{a^2bc} a)^d + (\log_{ab^2c} b)^d + (\log_{abc^2} c)^d &\geq \frac{3}{4^d} \end{aligned}$$

Equality holds for $a = b = c$.

748. Let $a \in \mathbb{R}, n \in \mathbb{N}, n \geq 2$. Prove that:

$$-(1+a^2)^n \leq (2a)^n + (1-a^2)^n \leq (1+a^2)^n$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Ravi Prakash-New Delhi-India

Put $a = \tan \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$, we get:

$$\begin{aligned} \left| \left(\frac{2a}{1+a^2} \right)^n + \left(\frac{1-a^2}{1+a^2} \right)^n \right| &= \left| \left(\frac{2 \tan \theta}{1+\tan^2 \theta} \right)^n + \left(\frac{1-\tan^2 \theta}{1+\tan^2 \theta} \right)^n \right| \leq \\ &\leq |(\sin 2\theta)^n + (\cos 2\theta)^n| \leq |\sin 2\theta|^n + |\cos 2\theta|^n \leq \sin^2 \theta + \cos^2 \theta = 1, n \geq 1 \end{aligned}$$

$$|\sin 2\theta|, |\cos 2\theta| \leq 1$$

$$\text{Hence, } |(2a)^n + (1-a^2)^n| \leq (1+a^2)^n.$$

Therefore,

$$-(1+a^2)^n \leq (2a)^n + (1-a^2)^n \leq (1+a^2)^n$$

749. Let $0 < a < b < c; x \geq 0$. Prove that:

$$a^2 \frac{(x-b)(x-c)}{(a-b)(a-c)} + b^2 \frac{(x-c)(x-a)}{(b-c)(b-a)} + c^2 \frac{(x-a)(x-b)}{(c-a)(c-b)} \geq 2x - 1$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Solution by Ravi Prakash-New Delhi-India

$$\text{Let: } f(x) = a^2 \frac{(x-b)(x-c)}{(a-b)(a-c)} + b^2 \frac{(x-c)(x-a)}{(b-c)(b-a)} + c^2 \frac{(x-a)(x-b)}{(c-a)(c-b)} - x^2$$

$f(x)$ – is a quadratic polynomial. Also,

$$f(a) = 0; f(b) = 0; f(c) = 0 \Rightarrow f(x) \equiv 0 \Rightarrow$$

$$a^2 \frac{(x-b)(x-c)}{(a-b)(a-c)} + b^2 \frac{(x-c)(x-a)}{(b-c)(b-a)} + c^2 \frac{(x-a)(x-b)}{(c-a)(c-b)} \equiv x^2$$

Also, $(x-1)^2 \geq 0 \Rightarrow x^2 \geq 2x-1$. Thus,

$$a^2 \frac{(x-b)(x-c)}{(a-b)(a-c)} + b^2 \frac{(x-c)(x-a)}{(b-c)(b-a)} + c^2 \frac{(x-a)(x-b)}{(c-a)(c-b)} \geq 2x-1$$

750. For $x > 0$, prove that:

$$\log_{2020} x \leq \frac{1}{2021} \sum_{k=0}^{2021} \left(\log_{2020} k + \frac{1}{k} (x-k) - \frac{1}{2! k^2} (x-k)^2 + \frac{2}{3! k^3} (x-k)^3 \right)$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Adrian Popa-Romania

$$\log_{2020} x \leq \frac{1}{2021} \sum_{k=0}^{2021} \left(\log_{2020} k + \frac{1}{k} (x-k) - \frac{1}{2! k^2} (x-k)^2 + \frac{2}{3! k^3} (x-k)^3 \right)$$

$$\frac{\log x}{\log 2021} \leq \frac{1}{2021} \sum_{k=0}^{2021} \left(\frac{\log k}{\log 2021} + \frac{1}{k} (x-k) - \frac{1}{2! k^2} (x-k)^2 + \frac{2}{3! k^3} (x-k)^3 \right)$$

$$\frac{1}{\log 2021} \sum_{k=1}^{2021} \log \left(\frac{x}{k} \right) \leq \sum_{k=1}^{2021} \left(\frac{\frac{x}{k} - 1}{1!} - \frac{\left(\frac{x}{k} - 1 \right)^2}{2!} + \frac{2 \left(\frac{x}{k} - 1 \right)^3}{3!} \right)$$

Let $f(y) = \log y$ and using Taylor expansion, we get:

$$f(y) = \frac{f'(1)}{1!} (y-1) + \frac{f''(y)}{2!} (y-1)^2 + \frac{f'''(y)}{3!} (y-1)^3 + \dots$$

$$f'(y) = \frac{1}{y}, f'(1) = 1$$

$$f''(y) = -\frac{1}{y^2}, f''(1) = -1$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$f'''(y) = \frac{2}{y^3}, f'''(1) = 1$$

Hence,

$$\log y < \frac{(y-1)!}{1!} - \frac{(y-1)^2}{2!} + \frac{2(y-1)^3}{3!}$$

Let $y = \frac{x}{k}$, then we get:

$$\sum_{k=1}^{2021} \log\left(\frac{x}{k}\right) < \sum_{k=1}^{2021} \left(\frac{\frac{x}{k}-1}{1!} - \frac{\left(\frac{x}{k}-1\right)^2}{2!} + \frac{2\left(\frac{x}{k}-1\right)^3}{3!} \right); \quad (1)$$

But $\log 2020 > \log e = 1$, thus,

$$\frac{1}{\log 2020} \sum_{k=1}^{2021} \log\left(\frac{x}{k}\right) < \sum_{k=1}^{2021} \log\left(\frac{x}{k}\right); \quad (2)$$

From (1) and (2) it follows that:

$$\log_{2020} x \leq \frac{1}{2021} \sum_{k=0}^{2021} \left(\log_{2020} k + \frac{1}{k} (x-k) - \frac{1}{2! k^2} (x-k)^2 + \frac{2}{3! k^3} (x-k)^3 \right)$$

751. Let $a, b, c, d \in \mathbb{R}$ and $f(x) = x^4 + ax^3 + bx^2 + cx + d \geq 0$,

$f'(x), f''(x), f'''(x) \geq 0, \forall x \in \mathbb{R}$. Prove that:

$$\begin{aligned} f(x) &\geq f(x_0) + (4x_0^3 + 3ax_0^2 + 2bx_0 + c)(x - x_0) + \\ &+ (6x_0^2 + 3ax_0 + b)(x - x_0)^2 + (4x + a)x_0(x - x_0)^3; \quad \forall x, x_0 \in \mathbb{R}, x \geq x_0 \end{aligned}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Soumitra Mandal-Chandar Nagore-India

Taylor's Expansion:

$$\begin{aligned} f(x) &= f(x - x_0 + x_0) = f(x_0) = \\ &= f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2!}f''(x_0) + \frac{(x - x_0)^3}{3!}f'''(x_0) + O((x - x_0)^4) \\ f(x) &\geq f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2!}f''(x_0) + \frac{(x - x_0)^3}{3!}f'''(x_0) = \\ &= f(x_0) + (x - x_0)(4x_0^3 + 3ax_0^2 + 2bx_0 + c) + \frac{12x_0^2 + 6ax_0 + 2b}{2!}(x - x_0)^2 + \end{aligned}$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$+\frac{24x_0 + 6a}{3!}(x - x_0)^3 = f(x_0) + (4x_0^3 + 3ax_0^2 + 2bx_0 + c)(x - x_0) + \\ +(6x_0^2 + 3ax_0 + b)(x - x_0)^2 + (4x + a)x_0(x - x_0)^3$$

752. Find all $m \in \mathbb{R}$ such that:

$$\max\{e^{-mx^2} + e^{mx^2}\} \leq 2; \forall x \in [-1, 1]$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution 1 by Amrit Awasthi-India

$$x \in [-1, 1] \Rightarrow y = x^2 \in [0, 1]$$

Using AM-GM Inequality, we get:

$$S = e^{-my} + e^{my} \geq 2 \cdot \sqrt{e^{-my} \cdot e^{my}} = 2$$

But it's given that $\max\{S\} \leq 2$, hence we must have

$$e^{-my} = e^{my} \Leftrightarrow -my = my \Leftrightarrow m = 0.$$

Solution 2 by Hikmat Mammadov-Azerbaijan

$$\text{Let } e^{mx^2} = S; e^{-mx^2} = \frac{1}{S}; \forall S > 0$$

By AM-GM, we have: $S + \frac{1}{S} \geq 2$. Equality holds for $S = 1$.

$$\forall x \in [-1, 1], \forall y \in \mathbb{R} \Rightarrow e^{mx^2} > 0, \text{ so, } e^{mx^2} + e^{-mx^2} = S + \frac{1}{S} \geq 2$$

$$\Rightarrow 2 \leq S + \frac{1}{S} \leq 2 \Rightarrow S = \frac{1}{S} = 1 \Rightarrow e^{mx^2} = 1 \Rightarrow mx^2 = 0 \Rightarrow m = 0.$$

753. If the minimum value of

$$f(x, y) = \sqrt{x^2 + y^2 - 10x - 10y + 50} + \sqrt{y^2 - 4y + 20} \\ + \sqrt{x^2 - 14x + 74}$$

Is m and occurs at $x = \alpha, y = \beta$, then find $m + 4\alpha + 3\beta$.

Proposed by Rajeev Rastogi-India

Solution by Hikmat Mammadov-Azerbaijan

$$f(x, y) = \sqrt{x^2 + y^2 - 10x - 10y + 50} + \sqrt{y^2 - 4y + 20} + \sqrt{x^2 - 14x + 74} \\ \sqrt{x^2 + y^2 - 10x - 10y + 50} = \sqrt{(x - 5)^2 + (5 - y)^2}$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\sqrt{y^2 - 4y + 20} = \sqrt{4^2 + (y - 2)^2}$$

$$\sqrt{x^2 - 14x + 74} = \sqrt{(7 - x)^2 + 5^2}$$

Now, using Minkowski's inequality,

$$\sqrt{a^2 + b^2} + \sqrt{c^2 + d^2} + \sqrt{e^2 + f^2} \geq \sqrt{(a + c + e)^2 + (b + d + f)^2}$$

$$\begin{aligned} f(x, y) &= \sqrt{x^2 + y^2 - 10x - 10y + 50} + \sqrt{y^2 - 4y + 20} + \sqrt{x^2 - 14x + 74} = \\ &= \sqrt{(x - 5)^2 + (5 - y)^2} + \sqrt{4^2 + (y - 2)^2} + \sqrt{(7 - x)^2 + 5^2} \geq \\ &\geq \sqrt{(x - 5 + 4 + 7 - x)^2 + (5 - y + y - 2 + 5)^2} = \sqrt{6^2 + 8^2} = \sqrt{100} = 10. \end{aligned}$$

Hence, $m = 10$ occurs when

$$\frac{x - 5}{5 - y} = \frac{4}{y - 2} = \frac{7 - x}{5} = \frac{6}{8} \Rightarrow \begin{cases} x = \frac{13}{4} = \alpha \\ y = \frac{22}{3} = \beta \end{cases} \Rightarrow \begin{cases} 4\alpha = 13 \\ 3\beta = 22 \end{cases}$$

$$\text{So, } m + 4\alpha + 3\beta = 10 + 13 + 22 = 45$$

754. Let $\alpha < \beta < \gamma$. Prove that:

$$1) \frac{\alpha^2}{|\alpha - \beta||\alpha - \gamma|} + \frac{\beta^2}{|\beta - \alpha||\beta - \gamma|} + \frac{\gamma^2}{|\gamma - \alpha||\gamma - \beta|} \geq \cos(\alpha^2 + \beta^2 + \gamma^2)$$

$$2) |\alpha\beta(\alpha - \beta)| + |\beta\gamma(\gamma - \beta)| + |\gamma\alpha(\gamma - \alpha)| \geq |(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)|$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \alpha - \beta < 0 &\Rightarrow |\alpha - \beta| = \beta - \alpha \text{ and } |\beta - \alpha| = \beta - \alpha, \text{ and } \alpha - \gamma < 0 \Rightarrow |\alpha - \gamma| \\ &= \gamma - \alpha \text{ and } |\gamma - \alpha| = \gamma - \alpha, \text{ and } \beta - \gamma < 0 \Rightarrow |\beta - \gamma| \\ &= \gamma - \beta \text{ and } |\gamma - \beta| = \gamma - \beta \\ &\therefore \frac{\alpha^2}{|\alpha - \beta||\alpha - \gamma|} + \frac{\beta^2}{|\beta - \alpha||\beta - \gamma|} + \frac{\gamma^2}{|\gamma - \alpha||\gamma - \beta|} \\ &= \frac{\alpha^2}{(\beta - \alpha)(\gamma - \alpha)} + \frac{\beta^2}{(\beta - \alpha)(\gamma - \beta)} + \frac{\gamma^2}{(\gamma - \alpha)(\gamma - \beta)} \\ &= \frac{\alpha^2(\gamma - \beta) + \beta^2(\gamma - \alpha) + \gamma^2(\beta - \alpha)}{(\beta - \alpha)(\gamma - \beta)(\gamma - \alpha)} - 1 + 1 \end{aligned}$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 &= \frac{\alpha^2(\gamma - \beta) + \beta^2(\gamma - \alpha) + \gamma^2(\beta - \alpha) - (\beta - \alpha)(\gamma - \beta)(\gamma - \alpha)}{(\beta - \alpha)(\gamma - \beta)(\gamma - \alpha)} + 1 \\
 &= \frac{2\beta^2(\gamma - \alpha)}{(\beta - \alpha)(\gamma - \beta)(\gamma - \alpha)} + 1 = \frac{2\beta^2}{(\beta - \alpha)(\gamma - \beta)} + 1 \\
 &\geq 1 [\because \beta - \alpha, \gamma - \beta > 0 \Rightarrow (\beta - \alpha)(\gamma - \beta) > 0] \\
 &\quad \geq \cos(\alpha^2 + \beta^2 + \gamma^2)
 \end{aligned}$$

$$\Rightarrow \boxed{\frac{\alpha^2}{|\alpha - \beta||\alpha - \gamma|} + \frac{\beta^2}{|\beta - \alpha||\beta - \gamma|} + \frac{\gamma^2}{|\gamma - \alpha||\gamma - \beta|} \geq \cos(\alpha^2 + \beta^2 + \gamma^2) \forall \alpha < \beta < \gamma}$$

Now, $\alpha < \beta < \gamma \Rightarrow$ **4 cases are possible :** (i) $0 \leq \alpha < \beta < \gamma$, (ii) $\alpha < 0 \leq \beta < \gamma$
 (iii) $\alpha < \beta < 0 \leq \gamma$, (iv) $\alpha < \beta < \gamma < 0$

$$\begin{aligned}
 &\text{Now, } |\alpha\beta(\alpha - \beta)| + |\beta\gamma(\gamma - \beta)| + |\gamma\alpha(\gamma - \alpha)| \\
 &\geq |\alpha\beta(\alpha - \beta) + \beta\gamma(\gamma - \beta) + \gamma\alpha(\gamma - \alpha)| \\
 &\geq |\alpha\beta(\alpha - \beta) + \beta\gamma(\gamma - \beta) + \gamma\alpha(\gamma - \alpha)| (\because |x + y| \geq |x + y| \forall x, y \in \mathbb{C}) \\
 &\stackrel{?}{\geq} |(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)| \\
 &\Leftrightarrow |\alpha\beta(\alpha - \beta) + \beta\gamma(\gamma - \beta) + \gamma\alpha(\gamma - \alpha)|^2 \stackrel{?}{\geq} |(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)|^2 \\
 &\Leftrightarrow (\alpha\beta(\alpha - \beta) + \beta\gamma(\gamma - \beta) + \gamma\alpha(\gamma - \alpha))^2 \stackrel{?}{\geq} (\alpha - \beta)^2(\beta - \gamma)^2(\gamma - \alpha)^2 \\
 &\Leftrightarrow 4(-\alpha^2\beta^3\gamma + 2\alpha^2\beta^2\gamma^2 - \alpha^2\beta\gamma^3 + \alpha\beta^4\gamma - \alpha\beta^3\gamma^2 - \alpha\beta^2\gamma^3 + \alpha\beta\gamma^4) \stackrel{?}{\geq} 0 \\
 &\Leftrightarrow -\alpha^2\beta\gamma(\beta^2 + \gamma^2 - 2\beta\gamma) + \alpha\beta\gamma(\beta^3 + \gamma^3 - \beta\gamma(\beta + \gamma)) \stackrel{?}{\geq} 0 \\
 &\Leftrightarrow -\alpha^2\beta\gamma(\beta - \gamma)^2 + \alpha\beta\gamma(\beta + \gamma)(\beta - \gamma)^2 \stackrel{?}{\geq} 0 \Leftrightarrow \alpha\beta\gamma(\beta - \gamma)^2(\beta + \gamma - \alpha) \stackrel{?}{\geq} 0 \\
 &\Leftrightarrow \alpha\beta\gamma(\beta + \gamma - \alpha) \stackrel{?}{\geq} 0 \quad (*) \\
 &\quad \beta - \alpha > 0
 \end{aligned}$$

Now, for case (i), $\alpha\beta\gamma \geq 0$ and $\beta + \gamma - \alpha \stackrel{\beta - \alpha > 0}{\geq} \gamma > 0 \Rightarrow \alpha\beta\gamma(\beta + \gamma - \alpha) \geq 0$

$\Rightarrow (*)$ is true and for case (iii), $\alpha\beta > 0$ and $\gamma \geq 0 \Rightarrow \alpha\beta\gamma$

$$\begin{aligned}
 &\stackrel{\beta - \alpha > 0}{\geq} 0 \text{ and } \beta + \gamma - \alpha \stackrel{\beta - \alpha > 0}{\geq} \gamma \geq 0
 \end{aligned}$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\Rightarrow \alpha\beta\gamma(\beta + \gamma - \alpha) \geq 0 \Rightarrow (*) \text{ is true}$$

\therefore for cases (i) and (iii), $|\alpha\beta(\alpha - \beta)| + |\beta\gamma(\gamma - \beta)| + |\gamma\alpha(\gamma - \alpha)| \geq |(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)|$

and we shall now focus on

cases (ii) and (iv) and treat them separately

$$\begin{aligned}
 \boxed{\text{Case (ii)}} \quad & \alpha < 0 \leq \beta < \gamma \text{ and then, } |\alpha\beta(\alpha - \beta)| = \alpha\beta(\alpha - \beta), |\beta\gamma(\gamma - \beta)| \\
 &= \beta\gamma(\gamma - \beta), |\gamma\alpha(\gamma - \alpha)| = -\gamma\alpha(\gamma - \alpha) \text{ and } |(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)| \\
 &= (\alpha - \beta)(\beta - \gamma)(\gamma - \alpha) \\
 \therefore & |\alpha\beta(\alpha - \beta)| + |\beta\gamma(\gamma - \beta)| + |\gamma\alpha(\gamma - \alpha)| \geq |(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)| \\
 \Leftrightarrow & \alpha\beta(\alpha - \beta) + \beta\gamma(\gamma - \beta) - \gamma\alpha(\gamma - \alpha) \geq (\alpha - \beta)(\beta - \gamma)(\gamma - \alpha) \\
 \Leftrightarrow & 2\alpha\beta(\alpha - \beta) \geq 0 \rightarrow \text{true}
 \end{aligned}$$

$$\begin{aligned}
 \because \alpha\beta \leq 0 \text{ and } \alpha - \beta < 0 \Rightarrow \alpha\beta(\alpha - \beta) \geq 0 \Rightarrow & |\alpha\beta(\alpha - \beta)| + |\beta\gamma(\gamma - \beta)| + |\gamma\alpha(\gamma - \alpha)| \\
 &\geq |(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)|
 \end{aligned}$$

$$\begin{aligned}
 \boxed{\text{Case (iv)}} \quad & \alpha < \beta < \gamma < 0 \text{ and then, } |\alpha\beta(\alpha - \beta)| = -\alpha\beta(\alpha - \beta), |\beta\gamma(\gamma - \beta)| \\
 &= \beta\gamma(\gamma - \beta), |\gamma\alpha(\gamma - \alpha)| = \gamma\alpha(\gamma - \alpha) \text{ and } |(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)| \\
 &= (\alpha - \beta)(\beta - \gamma)(\gamma - \alpha) \\
 \therefore & |\alpha\beta(\alpha - \beta)| + |\beta\gamma(\gamma - \beta)| + |\gamma\alpha(\gamma - \alpha)| \geq |(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)| \\
 \Leftrightarrow & -\alpha\beta(\alpha - \beta) + \beta\gamma(\gamma - \beta) + \gamma\alpha(\gamma - \alpha) \geq (\alpha - \beta)(\beta - \gamma)(\gamma - \alpha) \\
 \Leftrightarrow & 2\alpha\gamma(\gamma - \alpha) \geq 0 \rightarrow \text{true}
 \end{aligned}$$

$$\begin{aligned}
 \because \alpha\gamma > 0 \text{ and } \gamma - \alpha > 0 \Rightarrow \alpha\gamma(\gamma - \alpha) > 0 \Rightarrow & |\alpha\beta(\alpha - \beta)| + |\beta\gamma(\gamma - \beta)| + |\gamma\alpha(\gamma - \alpha)| \\
 &> |(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)|
 \end{aligned}$$

\therefore combining, $\boxed{\text{for cases (ii) and (iv), } |\alpha\beta(\alpha - \beta)| + |\beta\gamma(\gamma - \beta)| + |\gamma\alpha(\gamma - \alpha)| \geq |(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)|}$

\therefore combining all 4 cases, $\boxed{|\alpha\beta(\alpha - \beta)| + |\beta\gamma(\gamma - \beta)| + |\gamma\alpha(\gamma - \alpha)| \geq |(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)| \forall \alpha < \beta < \gamma}$ (QED)

Solution 2 by Ravi Prakash-New Delhi-India

$$\begin{aligned}
 1) \quad & \frac{\alpha^2}{|\alpha - \beta||\alpha - \gamma|} + \frac{\beta^2}{|\beta - \alpha||\beta - \gamma|} + \frac{\gamma^2}{|\gamma - \alpha||\gamma - \beta|} \geq \\
 & \geq \left| \frac{\alpha^2}{(\alpha - \beta)(\gamma - \alpha)} + \frac{\beta^2}{(\alpha - \beta)(\beta - \gamma)} + \frac{\gamma^2}{(\beta - \gamma)(\gamma - \alpha)} \right| =
 \end{aligned}$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 &= \left| \frac{\alpha^2(\beta - \gamma) + \beta^2(\gamma - \alpha) + \gamma^2(\alpha - \beta)}{(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)} \right| = \\
 &= \left| \frac{\alpha^2(\beta - \gamma) - \alpha(\beta^2 - \gamma^2) + \beta\gamma(\beta - \gamma)}{(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)} \right| = \\
 &= \left| \frac{(\beta - \gamma)(\alpha - \beta)(\alpha - \gamma)}{(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)} \right| = |-1| = 1 \geq \cos(\alpha^2 + \beta^2 + \gamma^2) \\
 2) \quad &\frac{|\alpha\beta(\alpha - \beta)| + |\beta\gamma(\gamma - \beta)| + |\gamma\alpha(\gamma - \alpha)|}{|(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)|} \geq \\
 &\geq \frac{|\alpha\beta(\alpha - \beta) - \beta\gamma(\gamma - \beta) + \gamma\alpha(\gamma - \alpha)|}{|(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)|} = \\
 &= \frac{|\alpha\beta(\alpha - \beta) + (\alpha - \beta)\gamma^2 - \gamma(\alpha^2 - \beta^2)|}{|(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)|} = \\
 &= \frac{|(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)|}{|(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)|} = 1
 \end{aligned}$$

Therefore,

$$|\alpha\beta(\alpha - \beta)| + |\beta\gamma(\gamma - \beta)| + |\gamma\alpha(\gamma - \alpha)| \geq |(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)|$$

755. Let $a, b, c > 0$ and $ab + bc + ca = 3c^2$ then prove:

$$\frac{1}{3} \left(\frac{a^4 + b^4 - 2c^4}{a^4 + b^4 + c^4} \right) \leq \frac{a^3 + b^3 - 2c^3}{a^3 + b^3 + c^3} \leq 3 \left(\frac{a^2 + b^2 - 2c^2}{a^2 + b^2 + c^2} \right)$$

Proposed by Choy Fai Lam-Hong Kong

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\begin{aligned}
 \frac{1}{3} \left(\frac{a^4 + b^4 - 2c^4}{a^4 + b^4 + c^4} \right) &\stackrel{(**)}{\leq} \frac{a^3 + b^3 - 2c^3}{a^3 + b^3 + c^3} \stackrel{(*)}{\leq} 3 \left(\frac{a^2 + b^2 - 2c^2}{a^2 + b^2 + c^2} \right) \\
 3c^2 = ab = bc + ca \Leftrightarrow 4c^2 &= (c + a)(c + b) \stackrel{AGM}{\leq} \left(\frac{(c + a) + (c + b)}{2} \right)^2 \Leftrightarrow \\
 2c \leq \frac{a + b + 2c}{2} &\Leftrightarrow c \leq \frac{a + b}{2}; (1) \\
 a^2 + b^2 - 2c^2 &\stackrel{Power Mean \& (1)}{\geq} 2 \left(\frac{a + b}{2} \right)^2 - 2 \left(\frac{a + b}{2} \right)^2 = 0
 \end{aligned}$$

Similarly, we have: $a^3 + b^3 - 2c^3 \geq 0$ and $a^4 + b^4 - 2c^4 \geq 0$.



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Also, by Chebyshev's inequality, we have:

$$3(a^3 + b^3 + c^3) \geq (a^2 + b^2 + c^2)(a + b + c)$$

It's suffices to prove that:

$$a^3 + b^3 - 2c^3 \leq (a^2 + b^2 - 2c^2)(a + b + c); (2)$$

$$(2) \Leftrightarrow ab(a + b) + c(a^2 + b^2) - 2c(ca + bc) \geq 0 \Leftrightarrow$$

$$ab(a + b) + c(a + b)^2 - 2abc - 2c(ca + bc) \geq 0$$

$$(a + b)(ab + bc + ca) - 2c(ab + bc + ca) \geq 0 \Leftrightarrow (a + b - 2c)(ab + bc + ca) \geq 0$$

Which is true from (1), then (*) is true.

Similarly, since $3(a^4 + b^4 + c^4) \stackrel{\text{Chebyshev's}}{\geq} (a^3 + b^3 + c^3)(a + b + c)$

It's suffices to prove that: $a^4 + b^4 + c^4 \leq (a + b + c)(a^3 + b^3 - 2c^3)$

$$\Leftrightarrow ab(a^2 + b^2) + c(a^3 + b^3) \geq 2(a + b)c^3; (3)$$

From (2) we have $2(a + b)c^2 \leq ab(a + b) + c(a^2 + b^2)$

$$2(a + b)c^3 \stackrel{(1)}{\geq} (ab(a + b) + c(a^2 + b^2))\left(\frac{a + b}{2}\right) =$$

$$= ab \cdot \frac{(a + b)^2}{2} + c \cdot \frac{(a + b)(a^2 + b^2)}{2} \stackrel{\text{CBS \& Chebyshev}}{\leq}$$

$$\leq ab(a^2 + b^2) + c(a^3 + b^3) \Rightarrow (3) \text{ is true, then } (**) \text{ is true.}$$

Therefore,

$$\frac{1}{3} \left(\frac{a^4 + b^4 - 2c^4}{a^4 + b^4 + c^4} \right) \leq \frac{a^3 + b^3 - 2c^3}{a^3 + b^3 + c^3} \leq 3 \left(\frac{a^2 + b^2 - 2c^2}{a^2 + b^2 + c^2} \right)$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 ab + bc + ca &= 3c^2 \Rightarrow 3c^2 - c(a + b) - ab = 0 \Rightarrow c = \frac{a + b \pm \sqrt{(a + b)^2 + 12ab}}{6} \\
 &= \frac{a + b + \sqrt{(a + b)^2 + 12ab}}{6} \left(\because \frac{a + b - \sqrt{(a + b)^2 + 12ab}}{6} < 0 \text{ but } c > 0 \right) \\
 &\stackrel{4ab \leq (a+b)^2}{\leq} \frac{a + b + \sqrt{(a + b)^2 + 3(a + b)^2}}{6} = \frac{a + b}{2} \stackrel{(*)}{\geq} 2c
 \end{aligned}$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\text{Now, } \frac{1}{3} \left(\frac{a^4 + b^4 - 2c^4}{a^4 + b^4 + c^4} \right) - \frac{a^3 + b^3 - 2c^3}{a^3 + b^3 + c^3} = \frac{(a^4 + b^4 - 2c^4)(a^3 + b^3 + c^3) - 3(a^3 + b^3 - 2c^3)(a^4 + b^4 + c^4)}{3(a^4 + b^4 + c^4)(a^3 + b^3 + c^3)} \leq 0$$

$$\Leftrightarrow 3(a^3 + b^3 - 2c^3)(a^4 + b^4 + c^4) - (a^4 + b^4 - 2c^4)(a^3 + b^3 + c^3) \stackrel{(1)}{\geq} 0$$

$$\begin{aligned} \text{Now, } a^3 + b^3 - 2c^3 &\stackrel{\because 8c^3 \leq (a+b)^3 \text{ via (*)}}{\geq} (a+b)(a^2 + b^2 - ab) - \frac{(a+b)^3}{4} \\ &= (a+b) \left(a^2 + b^2 - ab - \frac{(a+b)^2}{4} \right) \\ &= \frac{(a+b)(4a^2 + 4b^2 - 4ab - a^2 - b^2 - 2ab)}{4} \\ &= \frac{3(a+b)(a-b)^2}{4} \geq 0 \Rightarrow a^3 + b^3 - 2c^3 \geq 0 \end{aligned}$$

$$\begin{aligned} \therefore \text{LHS of (1)} &\stackrel{\text{Chebyshev}}{\geq} 3(a^3 + b^3 - 2c^3) \frac{(a^3 + b^3 + c^3)(a + b + c)}{3} \\ &\quad - (a^4 + b^4 - 2c^4)(a^3 + b^3 + c^3) \\ &= (a^3 + b^3 + c^3) \left((a^3 + b^3 - 2c^3)(a + b + c) - (a^4 + b^4 - 2c^4) \right) \\ &= (a^3 + b^3 + c^3) \left(ab(a^2 + b^2) + c(a^3 + b^3) - 2c^3(a + b) \right) \\ &\stackrel{a^2 + b^2 \geq \frac{(a+b)^2}{2}}{\geq} (a^3 + b^3 + c^3) \left(ab \cdot \frac{(a+b)(a+b)}{2} + c(a^3 + b^3) - 2c^3(a + b) \right) \stackrel{\text{via (*)}}{\geq} (a^3 + b^3 + c^3) \left(ab \cdot \frac{(a+b)(2c)}{2} + c(a+b)(a^2 + b^2 - ab) - 2c^3(a + b) \right) \\ &\stackrel{\because 4c^2 \leq (a+b)^2 \text{ via (*)}}{\geq} c(a+b)(a^3 + b^3 + c^3) \left(ab + a^2 + b^2 - ab - \frac{(a+b)^2}{2} \right) \\ &= c(a+b)(a^3 + b^3 + c^3)(ab + a^2 + b^2 - ab - 2c^2) \stackrel{\geq}{\geq} c(a+b)(a^3 + b^3 + c^3)(a^2 + b^2 - 2ab) \\ &= \frac{c(a+b)(a^3 + b^3 + c^3)(2a^2 + 2b^2 - a^2 - b^2 - 2ab)}{2} = \frac{c(a+b)(a^3 + b^3 + c^3)(a-b)^2}{2} \\ &\geq 0 \Rightarrow (1) \text{ is true} \Rightarrow \frac{1}{3} \left(\frac{a^4 + b^4 - 2c^4}{a^4 + b^4 + c^4} \right) - \frac{a^3 + b^3 - 2c^3}{a^3 + b^3 + c^3} \leq 0 \end{aligned}$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\Rightarrow \boxed{\frac{1}{3} \left(\frac{a^4 + b^4 - 2c^4}{a^4 + b^4 + c^4} \right) \stackrel{(*)}{\geq} \frac{a^3 + b^3 - 2c^3}{a^3 + b^3 + c^3}}$$

$$\begin{aligned} \text{Again, } 3 \left(\frac{a^2 + b^2 - 2c^2}{a^2 + b^2 + c^2} \right) - \frac{a^3 + b^3 - 2c^3}{a^3 + b^3 + c^3} \\ = \frac{3(a^2 + b^2 - 2c^2)(a^3 + b^3 + c^3) - (a^3 + b^3 - 2c^3)(a^2 + b^2 + c^2)}{(a^2 + b^2 + c^2)(a^3 + b^3 + c^3)} \geq 0 \end{aligned}$$

$$\Leftrightarrow 3(a^2 + b^2 - 2c^2)(a^3 + b^3 + c^3) - (a^3 + b^3 - 2c^3)(a^2 + b^2 + c^2) \stackrel{(2)}{\geq} 0 \text{ and}$$

$$\begin{aligned} \because a^2 + b^2 - 2c^2 &\stackrel{a^2+b^2 \geq \frac{(a+b)^2}{2}}{\geq} \frac{(a+b)^2}{2} - 2c^2 \\ &= \frac{(a+b-2c)(a+b+2c)}{2} \stackrel{\text{via } (*)}{\geq} 0 \end{aligned}$$

$$\begin{aligned} \therefore \text{LHS of (2)} &\stackrel{\text{Chebyshev}}{\geq} 3(a^2 + b^2 - 2c^2) \frac{(a^2 + b^2 + c^2)(a + b + c)}{3} \\ &\quad - (a^3 + b^3 - 2c^3)(a^2 + b^2 + c^2) \\ &= (a^2 + b^2 + c^2) \left((a^2 + b^2 - 2c^2)(a + b + c) - (a^3 + b^3 - 2c^3) \right) \end{aligned}$$

$$\begin{aligned} &= (a^2 + b^2 + c^2) \left(ab(a + b) + c(a^2 + b^2) - 2c^2(a + b) \right) \stackrel{ab = 3c^2 - c(a+b)}{\cong} (a^2 + b^2 \\ &\quad + c^2) \left((3c^2 - c(a+b))(a + b) + c(a^2 + b^2) - 2c^2(a + b) \right) \end{aligned}$$

$$\begin{aligned} &= c(a^2 + b^2 + c^2)(c(a + b) - (a + b)^2 + a^2 + b^2) \\ &\stackrel{\text{via } (*)}{=} c(a^2 + b^2 + c^2)(c(a + b) - 2ab) \stackrel{\text{via } (*)}{\geq} 2c(a^2 + b^2 + c^2)(c^2 - ab) \\ &= 2c(a^2 + b^2 + c^2)(c(a + b) - 2c^2) \end{aligned}$$

$$\begin{aligned} (\because ab + c(a+b) = 3c^2 \Rightarrow c^2 - ab = c(a+b) - 2c^2) &\stackrel{\text{via } (*)}{\geq} 2c(a^2 + b^2 + c^2)(c(2c) - 2c^2) \\ &= 0 \Rightarrow 3 \left(\frac{a^2 + b^2 - 2c^2}{a^2 + b^2 + c^2} \right) - \frac{a^3 + b^3 - 2c^3}{a^3 + b^3 + c^3} \geq 0 \end{aligned}$$

$$\begin{aligned} \Rightarrow \boxed{\frac{a^3 + b^3 - 2c^3}{a^3 + b^3 + c^3} \stackrel{(**)}{\geq} 3 \left(\frac{a^2 + b^2 - 2c^2}{a^2 + b^2 + c^2} \right)} &\therefore (\bullet), (\bullet\bullet) \Rightarrow \frac{1}{3} \left(\frac{a^4 + b^4 - 2c^4}{a^4 + b^4 + c^4} \right) \leq \frac{a^3 + b^3 - 2c^3}{a^3 + b^3 + c^3} \\ &\leq 3 \left(\frac{a^2 + b^2 - 2c^2}{a^2 + b^2 + c^2} \right) \text{ (QED)} \end{aligned}$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

756. If $a, b \in \mathbb{R}_+$ such that $a^2 + b^2 = 2$ then:

$$3 - ab \geq (a + b)\sqrt{ab} + (a - b)^2 \geq 2ab$$

Proposed by Dang Le Gia Khanh-Vietnam

Solution 1 by Sanong Huayrerai-Nakon Pathom-Thailand

$$3 - ab \geq (a + b)\sqrt{ab} + (a - b)^2$$

$$3 - ab \geq (a + b) + a^2 + b^2 - 2ab$$

$$1 + ab \geq (a + b)\sqrt{ab}$$

$$\frac{1}{\sqrt{ab}} + \sqrt{ab} \geq a + b \text{ true from } a + b \leq 2 \text{ and}$$

$$(a + b)\sqrt{ab} + (a - b)^2 \geq 2ab$$

$$(a + b)\sqrt{ab} + a^2 - 2ab + b^2 \geq 2ab$$

$$(a + b)\sqrt{ab} + 2 \geq 4ab$$

$$\left(\frac{1}{b} + \frac{1}{a}\right)\sqrt{ab} + \frac{2}{ab} \geq 4$$

$$\frac{\sqrt{a}}{\sqrt{b}} + \frac{\sqrt{b}}{\sqrt{a}} + \frac{2}{ab} \geq 4$$

$$\text{Because: } ab \leq 1 \Rightarrow \frac{2}{ab} \geq 2, \frac{\sqrt{a}}{\sqrt{b}} + \frac{\sqrt{b}}{\sqrt{a}} \geq 2.$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$3 - ab \geq (a + b)\sqrt{ab} + (a - b)^2 \Leftrightarrow 3(2) - 2ab \geq 2(a + b)\sqrt{ab} + 2(a^2 + b^2 - 2ab)$$

$$\Leftrightarrow 3(a^2 + b^2) - 2ab \geq 2(a + b)\sqrt{ab} + 2(a^2 + b^2 - 2ab)$$

$$\Leftrightarrow a^2 + b^2 + 2ab \geq 2(a + b)\sqrt{ab} \Leftrightarrow (a + b)^2 \geq 2(a + b)\sqrt{ab} \Leftrightarrow a + b \geq 2\sqrt{ab}$$

$$\rightarrow \text{true via AM - GM} \therefore 3 - ab \stackrel{(*)}{\geq} (a + b)\sqrt{ab} + (a - b)^2$$

$$(a + b)\sqrt{ab} + (a - b)^2 \stackrel{\text{G-H}}{\geq} (a + b) \frac{2ab}{a + b} + (a - b)^2 = 2ab + (a - b)^2 \geq 2ab$$

$$\therefore (a - b)^2 \geq 0 \therefore (a + b)\sqrt{ab} + (a - b)^2 \stackrel{(**)}{\geq} 2ab$$

$$\therefore (*), (**) \Rightarrow 3 - ab \geq (a + b)\sqrt{ab} + (a - b)^2 \geq 2ab \text{ (QED)}$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Solution 3 by Hikmat Mammadov-Azerbaijan

$$a^2 + b^2 = 2. \text{ Note } \begin{cases} a = \sqrt{2} \cos \alpha \\ b = \sqrt{2} \sin \alpha \end{cases}; a, b > 0$$

$$\begin{aligned} & (a+b)\sqrt{ab} + (a-b)^2 = \\ &= \sqrt{2}(\cos \alpha + \sin \alpha)\sqrt{2 \sin \alpha \cos \alpha} + 2(\cos \alpha - \sin \alpha)^2 = \\ &= (\cos \alpha + \sin \alpha) \cdot 2\sqrt{\sin \alpha \cos \alpha} + 2 - 4 \sin \alpha \cos \alpha \leq \\ &\leq (\sin \alpha + \cos \alpha)(\sin \alpha + \cos \alpha) + 2 - 4 \sin \alpha \cos \alpha = \\ &= 1 + 2 \sin \alpha \cos \alpha + 2 - 4 \sin \alpha \cos \alpha = 3 - 2 \sin \alpha \cos \alpha = 2ab \end{aligned}$$

$$\begin{aligned} & (a+b)\sqrt{ab} + (a-b)^2 = \\ &= \sqrt{2}(\cos \alpha + \sin \alpha)\sqrt{2 \sin \alpha \cos \alpha} + 2(\cos \alpha - \sin \alpha)^2 = \\ &= (\cos \alpha + \sin \alpha) \cdot 2\sqrt{\sin \alpha \cos \alpha} + 2 - 4 \sin \alpha \cos \alpha \geq \\ &\geq 4 \sin \alpha \cos \alpha + 2 - 4 \sin \alpha \cos \alpha = 2ab \end{aligned}$$

Therefore,

$$3 - ab \geq (a+b)\sqrt{ab} + (a-b)^2 \geq 2ab$$

757.

Let $a, b \in R$ such that $\left| x \cdot \int_{-1}^x (at + b) dt \right| \sqrt{1-x^2} \leq 1, \forall |x| \leq 1$.

Prove that : $|a| \leq 16$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{Let } f(x) = x \cdot \int_{-1}^x (at + b) dt = x \left[\frac{a}{2} t^2 + bt \right]_{-1}^x = \frac{a}{2} (x^3 - x) + b(x^2 + x), x \in [-1, 1]$$

$$\begin{aligned} \text{We have : } f\left(\frac{\sqrt{2}}{2}\right) &= -\frac{\sqrt{2}}{8} \cdot a + \frac{1+\sqrt{2}}{2} \cdot b \text{ and } f\left(-\frac{\sqrt{2}}{2}\right) \\ &= \frac{\sqrt{2}}{8} \cdot a + \frac{1-\sqrt{2}}{2} \cdot b, \text{ with } \left| f\left(\pm \frac{\sqrt{2}}{2}\right) \right| \leq \sqrt{2} \end{aligned}$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 & \rightarrow a = 2(2 + \sqrt{2}) \cdot f\left(-\frac{\sqrt{2}}{2}\right) + 2(2 - \sqrt{2}) \cdot f\left(\frac{\sqrt{2}}{2}\right) \\
 \rightarrow |a| &= \left| 2(2 + \sqrt{2}) \cdot f\left(-\frac{\sqrt{2}}{2}\right) + 2(2 - \sqrt{2}) \cdot f\left(\frac{\sqrt{2}}{2}\right) \right| \stackrel{\Delta}{\leq} 2(2 + \sqrt{2}) \left| f\left(-\frac{\sqrt{2}}{2}\right) \right| \\
 &+ 2(2 - \sqrt{2}) \left| f\left(\frac{\sqrt{2}}{2}\right) \right| \leq \\
 &\leq 2(2 + \sqrt{2}) \cdot \sqrt{2} + 2(2 - \sqrt{2}) \cdot \sqrt{2} = 8\sqrt{2} \leq 16. \quad \text{Therefore,} \quad |a| \leq 16.
 \end{aligned}$$

758. Let $\min\{|x|, |y|\} \geq 1$. Prove that:

$$\sqrt{x^2 - 1} + \sqrt{y^2 - 1} \leq |xy| \leq \frac{x^2 + y^2}{2}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution 1 by Ravi Prakash-New Delhi-India

$$\min\{|x|, |y|\} \geq 1 \Rightarrow |x| \geq 1, |y| \geq 1.$$

Put $x = \sec \theta$ and $y = \sec \phi ; 0 \leq \theta, \phi \leq \pi, \theta, \phi \neq \frac{\pi}{2}$, then

$$\begin{aligned}
 \sqrt{x^2 - 1} + \sqrt{y^2 - 1} &= |\tan \theta| + |\tan \phi| = \frac{\sin \theta}{|\cos \theta|} + \frac{\sin \phi}{|\cos \phi|} = \pm \frac{\sin(\theta + \phi)}{|\cos \theta||\cos \phi|} \leq \\
 &\leq \frac{1}{|\cos \theta||\cos \phi|} \leq |xy|
 \end{aligned}$$

Also, from $(|x| - |y|)^2 \geq 0 \Rightarrow x^2 + y^2 - 2|xy| \geq 0 \Rightarrow |xy| \leq \frac{1}{2}(x^2 + y^2)$

Thus, for $|x|, |y| \geq 1$, we have:

$$\sqrt{x^2 - 1} + \sqrt{y^2 - 1} \leq |xy| \leq \frac{x^2 + y^2}{2}$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\begin{aligned}
 \sqrt{x^2 - 1} + \sqrt{y^2 - 1} &\leq |xy| \leq \frac{x^2 + y^2}{2} \\
 \Leftrightarrow \sqrt{\frac{x^2 - 1}{(xy)^2}} + \sqrt{\frac{y^2 - 1}{(xy)^2}} &\leq 1
 \end{aligned}$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 &\Leftrightarrow \sqrt{\frac{1}{y^2} - \frac{1}{(xy)^2}} + \sqrt{\frac{1}{x^2} - \frac{1}{(xy)^2}} \leq 1 \\
 &\Leftrightarrow \sqrt{\frac{1}{y^2} \left(1 - \frac{1}{x^2}\right)} + \sqrt{\frac{1}{x^2} \left(1 - \frac{1}{y^2}\right)} \leq 1 \\
 &\Leftrightarrow \frac{1}{2} \cdot \left(\frac{1}{y^2} + 1 - \frac{1}{x^2}\right) + \frac{1}{2} \cdot \left(\frac{1}{x^2} + 1 - \frac{1}{y^2}\right) \leq 1 \\
 &\frac{1}{2} \left(1 + 1 + \frac{1}{x^2} + \frac{1}{y^2} - \frac{1}{x^2} - \frac{1}{y^2}\right) = \frac{2}{2} = 1
 \end{aligned}$$

Solution 3 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sqrt{x^2 - 1} + \sqrt{y^2 - 1} \stackrel{(1)}{\leq} |xy| \stackrel{(2)}{\leq} \frac{x^2 + y^2}{2}$$

$$\begin{aligned}
 (1) &\Leftrightarrow (x^2 - 1) + (y^2 - 1) + 2\sqrt{(x^2 - 1)(y^2 - 1)} \leq x^2 y^2 \\
 &\Leftrightarrow (x^2 - 1)(y^2 - 1) - 2\sqrt{(x^2 - 1)(y^2 - 1)} + 1 \geq 0 \\
 &\Leftrightarrow \left(\sqrt{(x^2 - 1)(y^2 - 1)} - 1\right)^2 \geq 0 \text{ which is true } \rightarrow (1) \text{ is true.}
 \end{aligned}$$

$$\frac{x^2 + y^2}{2} \stackrel{AM-GM}{\geq} \sqrt{(xy)^2} = |xy| \rightarrow (2) \text{ is true.}$$

$$\text{Therefore, } \sqrt{x^2 - 1} + \sqrt{y^2 - 1} \leq |xy| \leq \frac{x^2 + y^2}{2}.$$

759. $f: I \in \mathbb{R}_+ \rightarrow \mathbb{R}$, f –decreasing, convexe. Prove that:

$$6f\left(\frac{a+b}{2}\right) \leq f(\sqrt{ab}) + 2f\left(\sqrt{\frac{a^2 + b^2}{2}}\right) + 3f\left(\frac{2ab}{a+b}\right), \forall a, b \in I$$

Proposed by Seyran Ibrahimov-Maasilli-Azerbaijan

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\begin{aligned}
 \text{Since } \frac{2ab}{a+b} &\stackrel{HM \leq GM}{\leq} \sqrt{ab} \stackrel{GM \leq AM}{\leq} \frac{a+b}{2} \stackrel{f-\text{decreasing}}{\Rightarrow} \\
 f\left(\frac{a+b}{2}\right) &\leq f(\sqrt{ab}) \leq f\left(\frac{2ab}{a+b}\right) \quad (1)
 \end{aligned}$$

By Cauchy – Schwarz's inequality, we have :

$$\sqrt{\frac{a^2 + b^2}{2}} + \sqrt{ab} \leq \sqrt{(1+1)\left(\frac{a^2 + b^2}{2} + ab\right)} = a + b \quad (2)$$

$$\rightarrow f(\sqrt{ab}) + 2f\left(\sqrt{\frac{a^2 + b^2}{2}}\right)$$

$$+ f\left(\frac{2ab}{a+b}\right) \stackrel{(1)}{\geq} 2 \left[f(\sqrt{ab}) + f\left(\sqrt{\frac{a^2 + b^2}{2}}\right) \right] \stackrel{Jensen}{\geq} 2 \cdot 2f\left(\frac{\sqrt{ab} + \sqrt{\frac{a^2 + b^2}{2}}}{2}\right) \geq$$

$$\stackrel{(2) \text{-decreasing}}{\geq} 4f\left(\frac{a+b}{2}\right) \rightarrow f(\sqrt{ab}) + 2f\left(\sqrt{\frac{a^2 + b^2}{2}}\right) + f\left(\frac{2ab}{a+b}\right) \geq 4f\left(\frac{a+b}{2}\right) \quad (3)$$

$$\begin{aligned} & \rightarrow f(\sqrt{ab}) + 2f\left(\sqrt{\frac{a^2 + b^2}{2}}\right) + 3f\left(\frac{2ab}{a+b}\right) \\ &= \left[f(\sqrt{ab}) + 2f\left(\sqrt{\frac{a^2 + b^2}{2}}\right) + f\left(\frac{2ab}{a+b}\right) \right] + 2f\left(\frac{2ab}{a+b}\right) \geq \\ & \stackrel{(1),(3)}{\geq} 4f\left(\frac{a+b}{2}\right) + 2f\left(\frac{a+b}{2}\right) = 6f\left(\frac{a+b}{2}\right) \quad (\text{Proved}) \end{aligned}$$

760. If $a, b, c > 0$ and $2a + b + c = 2$ then

$$(a+b)^4 + (b+c)^4 + (c+a)^4 + 8a \geq 7$$

Proposed by Marin Chirciu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By Hölder inequality, we have :

$$(a+b)^4 + (c+a)^4 \geq \frac{[(a+b) + (c+a)]^4}{(1+1)^3} = \frac{2^4}{2^3} = 2$$

→ It's suffices to prove that : $2 + (b+c)^4 + 8a \geq 7$ ()*

$$\begin{aligned} (*) &\leftrightarrow (2 - 2a)^4 + 8a - 5 \geq 0 \leftrightarrow 16a^4 - 64a^3 + 96a^2 - 56a + 11 \geq 0 \\ &\leftrightarrow (2a - 1)^2(4a^2 - 12a + 11) \geq 0 \end{aligned}$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\leftrightarrow (2a - 1)^2[(2a - 3)^2 + 2] \geq 0 \text{ which is true, with equality if } a = \frac{1}{2} \text{ and } a + b$$

$$= c + a \rightarrow a = b = c = \frac{1}{2}.$$

$$\text{Therefore, } (a + b)^4 + (b + c)^4 + (c + a)^4 + 8a \geq 7.$$

Solution 2 by George Florin Șerban-Romania

$$2a + b + c = 2 \Rightarrow 4a + 2b + 2c = 4$$

$$2a + 2b + 2c = 4 - 2a$$

$$\begin{aligned} (a + b)^4 + (b + c)^4 + (c + a)^4 &\stackrel{\text{Holder}}{\geq} \frac{(a + b + b + c + c + a)^4}{3^3} = \\ &= \frac{(2a + 2b + 2c)^4}{27} = \frac{(4 - 2a)^4}{27} = \frac{(2a - 4)^4}{27} \\ (a + b)^4 + (b + c)^4 + (c + a)^4 + 8a &\geq \frac{(2a - 4)^4}{27} + 8a \stackrel{(*)}{\geq} 7 \end{aligned}$$

$(*) \Leftrightarrow (2a - 4)^4 + 216a \geq 189 \Leftrightarrow (2a - 1)^2(4a^2 - 28a + 67) \geq 0$, which is true from

$$(2a - 1)^2 \geq 0 \text{ and } 4a^2 - 28a + 67 > 0; \forall a > 0.$$

Equality holds if and only if $a = b = c = \frac{1}{2}$.

761. Let $\alpha, \beta, \gamma \in R$ such that $|\alpha x^3 + \beta x^2 + \gamma x| \leq 1, \forall |x| \leq 1$. Prove that:

- a) $|\alpha|^3 |\beta - \gamma| + |\beta|^3 |\gamma - \alpha| + |\gamma|^3 |\alpha - \beta| \geq |\alpha - \beta| |\beta - \gamma| |\gamma - \alpha| |\alpha + \beta + \gamma|$
- b) $|3\alpha + 2\beta + \gamma| \leq 9$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\begin{aligned} a) |\alpha - \beta| |\beta - \gamma| |\gamma - \alpha| |\alpha + \beta + \gamma| &= |(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)(\alpha + \beta + \gamma)| \\ &= \left| \left(\sum \alpha \beta^2 - \sum \alpha^2 \beta \right) (\alpha + \beta + \gamma) \right| = \\ &= \left| \left(\sum \alpha^2 \beta^2 + \sum \alpha \beta^3 + \alpha \beta \gamma \sum \alpha \right) - \left(\sum \alpha^3 \beta + \sum \alpha^2 \beta^2 + \alpha \beta \gamma \sum \alpha \right) \right| \\ &= \left| - \sum \alpha^3 (\beta - \gamma) \right| \stackrel{\Delta}{\leq} \sum |\alpha|^3 |\beta - \gamma|. \end{aligned}$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned} \text{Therefore, } & |\alpha|^3|\beta - \gamma| + |\beta|^3|\gamma - \alpha| + |\gamma|^3|\alpha - \beta| \\ & \geq |\alpha - \beta||\beta - \gamma||\gamma - \alpha||\alpha + \beta + \gamma|. \end{aligned}$$

b) Let $f(x) = \alpha x^3 + \beta x^2 + \gamma x, x \in [-1, 1] \rightarrow |f(x)| \leq 1, \forall |x| \leq 1.$

We have : $f(1) = \alpha + \beta + \gamma, f(-1) = -\alpha + \beta - \gamma$ and $f\left(\frac{1}{2}\right) = \frac{\alpha}{8} + \frac{\beta}{4} + \frac{\gamma}{2}$

$$\rightarrow \alpha = f(1) - \frac{1}{3}f(-1) - \frac{8}{3}f\left(\frac{1}{2}\right), \beta = \frac{1}{2}f(1) + \frac{1}{2}f(-1) \text{ and}$$

$$\gamma = -\frac{1}{2}f(1) - \frac{1}{6}f(-1) + \frac{8}{3}f\left(\frac{1}{2}\right)$$

$$\rightarrow |3\alpha + 2\beta + \gamma|$$

$$= \left| 3\left(f(1) - \frac{1}{3}f(-1) - \frac{8}{3}f\left(\frac{1}{2}\right)\right) + 2\left(\frac{1}{2}f(1) + \frac{1}{2}f(-1)\right) \right|$$

$$+ \left(-\frac{1}{2}f(1) - \frac{1}{6}f(-1) + \frac{8}{3}f\left(\frac{1}{2}\right) \right) \Big| =$$

$$= \left| \frac{7}{2}f(1) - \frac{1}{6}f(-1) - \frac{16}{3}f\left(\frac{1}{2}\right) \right| \stackrel{\Delta}{\leq} \frac{7}{2}|f(1)| + \frac{1}{6}|f(-1)| + \frac{16}{3}\left|f\left(\frac{1}{2}\right)\right| \stackrel{|f(x)| \leq 1, \forall |x| \leq 1}{\leq} \frac{7}{2} \cdot 1 + \frac{1}{6} \cdot 1 + \frac{16}{3} \cdot 1 = 9.$$

$$\text{Therefore, } |3\alpha + 2\beta + \gamma| \leq 9.$$

762. Let $a, b, c, d \in R$ such that $|ax^3 + bx^2 + cx + d| \leq 1, \forall |x| \leq 1$. Prove

that:

$$|3a + b| \leq 18.$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $f(x) = ax^3 + bx^2 + cx + d, x \in [-1, 1] \rightarrow |f(x)| \leq 1, \forall x \in [-1, 1].$

We have : $f(1) = a + b + c + d, f(-1) = -a + b - c + d, f\left(\frac{1}{2}\right)$

$$= \frac{a}{8} + \frac{b}{4} + \frac{c}{2} + d, f\left(-\frac{1}{2}\right) = -\frac{a}{8} + \frac{b}{4} - \frac{c}{2} + d$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 \rightarrow a &= \frac{2}{3} \left[f(1) - f(-1) - 2f\left(\frac{1}{2}\right) + 2f\left(-\frac{1}{2}\right) \right] \text{ and } b \\
 &= \frac{2}{3} \left[f(1) + f(-1) - f\left(\frac{1}{2}\right) - f\left(-\frac{1}{2}\right) \right]. \\
 \rightarrow 3a + b &= \frac{8}{3}f(1) - \frac{4}{3}f(-1) - \frac{14}{3}f\left(\frac{1}{2}\right) + \frac{10}{3}f\left(-\frac{1}{2}\right) \\
 \rightarrow |3a + b| &= \left| \frac{8}{3}f(1) - \frac{4}{3}f(-1) - \frac{14}{3}f\left(\frac{1}{2}\right) + \frac{10}{3}f\left(-\frac{1}{2}\right) \right| \stackrel{\Delta}{\leq} \frac{8}{3}|f(1)| + \frac{4}{3}|f(-1)| \\
 &\quad + \frac{14}{3}|f\left(\frac{1}{2}\right)| + \frac{10}{3}|f\left(-\frac{1}{2}\right)| \leq \\
 &\stackrel{|f(x)| \leq 1, \forall x \in [-1, 1]}{\leq} \frac{8}{3} \cdot 1 + \frac{4}{3} \cdot 1 + \frac{14}{3} \cdot 1 + \frac{10}{3} \cdot 1 = 12 \leq 18. \text{ Therefore,} \\
 |3a + b| &\leq 18.
 \end{aligned}$$

763. Let $1 < \alpha < \beta < \gamma, x \in \mathbb{R}$. Prove that:

$$\begin{aligned}
 a) \quad &\frac{|(x-\beta)(x-\gamma)|}{|(\alpha-\beta)(\alpha-\gamma)|} + \frac{|(x-\alpha)(x-\beta)|}{|(\gamma-\alpha)(\gamma-\beta)|} + \frac{|(x-\alpha)(x-\gamma)|}{|(\beta-\alpha)(\beta-\gamma)|} \geq \frac{\sin(\alpha^2 + \beta^2 + \gamma^2)}{\alpha^2 + \beta^2 + \gamma^2} \\
 b) \quad &\alpha \cdot \log \alpha + \beta \cdot \log \beta + \gamma \cdot \log \gamma \geq \frac{2\alpha + \beta}{3} \cdot \log \frac{2\alpha + \gamma}{3}.
 \end{aligned}$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\begin{aligned}
 a) \quad &\frac{|(x-\beta)(x-\gamma)|}{|(\alpha-\beta)(\alpha-\gamma)|} + \frac{|(x-\alpha)(x-\beta)|}{|(\gamma-\alpha)(\gamma-\beta)|} \\
 &+ \frac{|(x-\alpha)(x-\gamma)|}{|(\beta-\alpha)(\beta-\gamma)|} \stackrel{\Delta}{\geq} \left| \frac{(x-\beta)(x-\gamma)}{(\alpha-\beta)(\alpha-\gamma)} + \frac{(x-\alpha)(x-\beta)}{(\gamma-\alpha)(\gamma-\beta)} \right. \\
 &\left. + \frac{(x-\alpha)(x-\gamma)}{(\beta-\alpha)(\beta-\gamma)} \right|
 \end{aligned}$$

$$\text{Let } f(x) = \frac{(x-\beta)(x-\gamma)}{(\alpha-\beta)(\alpha-\gamma)} + \frac{(x-\alpha)(x-\beta)}{(\gamma-\alpha)(\gamma-\beta)} + \frac{(x-\alpha)(x-\gamma)}{(\beta-\alpha)(\beta-\gamma)}, x \in \mathbb{R}.$$

$f(x)$ is a quadratic polynomial with $f(\alpha) = f(\beta) = f(\gamma) = 1 \rightarrow f(x) = 1, \forall x \in \mathbb{R}$.



ROMANIAN MATHEMATICAL MAGAZINE

$$\begin{aligned}
 & \rightarrow \frac{|(x-\beta)(x-\gamma)|}{|(\alpha-\beta)(\alpha-\gamma)|} + \frac{|(x-\alpha)(x-\beta)|}{|(\gamma-\alpha)(\gamma-\beta)|} + \frac{|(x-\alpha)(x-\gamma)|}{|(\beta-\alpha)(\beta-\gamma)|} \\
 & \geq 1 \stackrel{\sin t \leq t, \forall t \geq 0}{\geq} \frac{\sin(\alpha^2 + \beta^2 + \gamma^2)}{\alpha^2 + \beta^2 + \gamma^2}.
 \end{aligned}$$

b) Let $g(x) = x \cdot \ln x, x \geq 1$. We have : $g'(x) = \ln x + 1$ and $g''(x) = \frac{1}{x} > 0$

$\rightarrow g - \text{convex on } [1, \infty[$

$$\begin{aligned}
 & \rightarrow \alpha \cdot \ln \alpha + \beta \cdot \ln \beta + \gamma \cdot \ln \gamma = g(\alpha) + g(\beta) + g(\gamma) \stackrel{\text{Jensen}}{\geq} 3g\left(\frac{\alpha + \beta + \gamma}{3}\right) \\
 & = 3 \cdot \frac{\alpha + \beta + \gamma}{3} \cdot \ln \frac{\alpha + \beta + \gamma}{3} \geq \\
 & \stackrel{\alpha < \beta < \gamma}{\geq} (2\alpha + \beta) \ln \frac{2\alpha + \gamma}{3} \stackrel{\alpha, \beta, \gamma > 1}{\geq} \frac{2\alpha + \beta}{3} \cdot \ln \frac{2\alpha + \gamma}{3}.
 \end{aligned}$$

Therefore, $\alpha \cdot \ln \alpha + \beta \cdot \ln \beta + \gamma \cdot \ln \gamma \geq \frac{2\alpha + \beta}{3} \cdot \ln \frac{2\alpha + \gamma}{3}$.

764. For $1 < a < b$ prove that:

$$a^{9ab\sqrt{b}} b^{9b^2\sqrt{a}} e^{8b^2\sqrt{b}} a^{12a^2\sqrt{b}} < a^{9ab\sqrt{a}} b^{9a^2\sqrt{b}} e^{8a^2\sqrt{b}} b^{12b^2\sqrt{a}}$$

Proposed by Nikos Ntorvas-Greece

Solution by proposer

We consider the real function $f(x) = \sqrt{x} \cdot \log x, x \in (0, \infty)$, which is concave on $[1, +\infty)$

as a continuous function on $[1, +\infty)$ with a strictly negative second derivative $\forall x \in (1, +\infty)$

$$(1, +\infty)$$

$$f''(x) = \frac{-\log x}{4\sqrt{x^5}} < 0; \forall x > 1$$

From Hermite-Hadamard inequality for concave function, we have that for $1 < a < b$ it

holds the inequality:

$$(f(a) + f(b))(b - a) < 2 \int_a^b f(x) dx \Leftrightarrow$$

$$(\log(a^{\sqrt{a}}) + \log(b^{\sqrt{b}}))(b - a) < 2 \left[\frac{2}{9} \sqrt{x^3} (3 \log x - 2) \right]_a^b \Leftrightarrow$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 & (\log(a^{\sqrt{a}}) + \log(b^{\sqrt{b}}))(b-a) < \frac{4}{9} [\sqrt{x^3}(3 \log x - \log(e^2))]_a^b \Leftrightarrow \\
 & (\log(a^{\sqrt{a}}) + \log(b^{\sqrt{b}}))(9b-9a) < 4 [\sqrt{x^3}(3 \log x - \log(e^2))]_a^b \Leftrightarrow \\
 & \log(a^{9b\sqrt{a}}) + \log(b^{9b\sqrt{b}}) - \log(a^{9a\sqrt{a}}) - \log(b^{9a\sqrt{b}}) < \\
 & < 4\sqrt{b^3}(3 \log b - \log e^2) - 4\sqrt{a^3}(3 \log a - \log e^2) \Leftrightarrow \\
 & \log(a^{9b\sqrt{a}}b^{9b\sqrt{b}}e^{8\sqrt{b^3}}a^{12\sqrt{a^3}}) < \log(a^{9a\sqrt{b}}b^{9a\sqrt{a}}e^{8\sqrt{a^3}}b^{12\sqrt{b^3}})
 \end{aligned}$$

The real function $f(x) = \log x$ is strictly increasing on $(0, \infty)$, hence

$$a^{9b\sqrt{a}}b^{9b\sqrt{b}}e^{8\sqrt{b^3}}a^{12\sqrt{a^3}} < a^{9a\sqrt{b}}b^{9a\sqrt{a}}e^{8\sqrt{a^3}}b^{12\sqrt{b^3}}$$

The real function $h(x) = x^{\sqrt{ab}}$ is strictly increasing on $(0, \infty)$, therefore,

$$a^{9ab\sqrt{b}}b^{9b^2\sqrt{a}}e^{8b^2\sqrt{b}}a^{12a^2\sqrt{b}} < a^{9ab\sqrt{a}}b^{9a^2\sqrt{b}}e^{8a^2\sqrt{b}}b^{12b^2\sqrt{a}}$$

765. If $a, b, c, d > 0, a + b + c + d + e = 1$, then

$$\frac{a}{2a+1} + \frac{b}{3b+1} + \frac{c}{12c+1} + \frac{d}{18d+1} + \frac{3}{36e+1} \leq \frac{1}{2}$$

Proposed by Seyran Ibrahimov-Maasilli-Azerbaijan

Solution 1 by Nguyen Van Canh-BenTre-Vietnam

$$\text{Let } f(x) = \frac{x}{x+1}, (x > 0)$$

$$\rightarrow f'(x) = \frac{1}{(x+1)^2} \rightarrow f''(x) = -\frac{2}{(x+1)^3} < 0, \forall x > 0$$

$$\begin{aligned}
 & \frac{a}{2a+1} + \frac{b}{3b+1} + \frac{c}{12c+1} + \frac{d}{18d+1} + \frac{e}{36e+1} \\
 & = \frac{1}{2}f(2a) + \frac{1}{3}f(3b) + \frac{1}{12}f(12c) + \frac{1}{18}f(18d) + \frac{1}{36}f(36e)
 \end{aligned}$$

$$\begin{aligned}
 & \stackrel{\text{Jensen}}{\geq} \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{12} + \frac{1}{18} + \frac{1}{36} \right) f\left(\frac{\frac{1}{2} \cdot 2a + \frac{1}{3} \cdot 3b + \frac{1}{12} \cdot 12c + \frac{1}{18} \cdot 18d + \frac{1}{36} \cdot 36e}{\frac{1}{2} + \frac{1}{3} + \frac{1}{12} + \frac{1}{18} + \frac{1}{36}} \right) \\
 & = 1f\left(\frac{a+b+c+d+e}{1} \right) = f(1) = \frac{1}{1+1} = \frac{1}{2}.
 \end{aligned}$$

Equality if and only if $\leftrightarrow \begin{cases} a+b+c+d+e=1 \\ 2a=3b=12c=18d=36e \end{cases} \leftrightarrow$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$a = \frac{1}{2}, b = \frac{1}{3}, c = \frac{1}{12}, d = \frac{1}{18}, e = \frac{1}{36}$$

Solution 2 by Michael Sterghiou-Greece

$$\frac{a}{2a+1} + \frac{b}{3b+1} + \frac{c}{12c+1} + \frac{d}{18d+1} + \frac{3}{36e+1} \leq \frac{1}{2}; \quad (1)$$

$$(1) \Leftrightarrow \frac{1}{2 + \frac{1}{a}} + \frac{1}{3 + \frac{1}{b}} + \frac{1}{12 + \frac{1}{c}} + \frac{1}{18 + \frac{1}{d}} + \frac{1}{36 + \frac{1}{e}} \leq \frac{1}{2}$$

$$\sqrt{\frac{1}{2} \cdot a} \stackrel{AGM}{\leq} \frac{2}{2 + \frac{1}{a}}, \quad \sqrt{\frac{1}{3} \cdot b} \leq \frac{2}{3 + \frac{1}{b}}, \quad (\text{and analogs})$$

We can write as: $\sqrt{\frac{1}{2} \cdot a} = \frac{1}{2} \sqrt{2a}$ (and analogs). So, we obtain:

$$\frac{1}{2} \left(\frac{1}{2} \sqrt{2a} + \frac{1}{3} \sqrt{3b} + \frac{1}{12} \sqrt{12c} + \frac{1}{18} \sqrt{18d} + \frac{1}{36} \sqrt{36e} \right) \leq \frac{1}{2}; \quad (2)$$

We have:

$$\begin{aligned} & \frac{1}{2} \sqrt{2a} + \frac{1}{3} \sqrt{3b} + \frac{1}{12} \sqrt{12c} + \frac{1}{18} \sqrt{18d} + \frac{1}{36} \sqrt{36e} \leq \\ & \leq \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{12} + \frac{1}{18} + \frac{1}{36} \right) \sqrt{\frac{\frac{1}{2} \cdot 2a + \frac{1}{3} \cdot 3b + \frac{1}{12} \cdot 12c + \frac{1}{18} \cdot 18d}{1}} = 1 \Rightarrow (2) \text{ true.} \end{aligned}$$

Equality holds for $a = \frac{1}{2}, b = \frac{1}{3}, c = \frac{1}{12}, d = \frac{1}{18}, e = \frac{1}{36}$.

Solution 3 by Sergey Primazon-Russia

$$\begin{aligned} & \frac{a}{2a+1} + \frac{b}{3b+1} + \frac{c}{12c+1} + \frac{d}{18d+1} + \frac{3}{36e+1} \leq \\ & \leq a \cdot \frac{1}{4} \left(\frac{1}{2a} + 1 \right) + b \cdot \frac{1}{4} \left(\frac{1}{3b} + 1 \right) + c \cdot \frac{1}{4} \left(\frac{1}{12c} + 1 \right) + d \cdot \frac{1}{4} \left(\frac{1}{18d} + 1 \right) + e \\ & \quad \cdot \frac{1}{4} \left(\frac{1}{36e} + 1 \right) = \\ & = \frac{1}{4} (a + b + c + d) + \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{12} + \frac{1}{18} + \frac{1}{36} \right) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \end{aligned}$$

766. Let $a_i \in (1, \infty), i = \overline{1, n}, n \in \mathbb{N}, a_1 + a_2 + \dots + a_n = ne^4$, then:

$$\sqrt{\log a_1^{a_2} + \log a_1^{a_3} + \dots + \log a_1^{a_n} + \dots} + \sqrt{\log a_n^{a_1} + \log a_n^{a_2} + \dots + \log a_n^{a_{n-1}}} \leq 9n^2$$

Proposed by Florică Anastase-Romania



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Solution 1 by Adrian Popa-Romania

$$\begin{aligned}
 S &= \sqrt{\log a_1^{a_2} + \log a_1^{a_3} + \cdots + \log a_1^{a_n}} + \cdots + \sqrt{\log a_n^{a_1} + \log a_n^{a_2} + \cdots + \log a_n^{a_{n-1}}} \\
 &= \sqrt{(a_2 + a_3 + \cdots + a_n) \log a_1} + \cdots + \sqrt{(a_1 + a_2 + \cdots + a_{n-1}) \log a_n} = \\
 &= \sqrt{(ne^4 - a_1) \log a_1} + \sqrt{(ne^4 - a_2) \log a_2} + \cdots + \sqrt{(ne^4 - a_n) \log a_n} \stackrel{CBS}{\leq} \\
 &\leq \sqrt{(n^2 e^4 - ne^4) \cdot \log(a_1 a_2 \dots a_n)}; (1) \\
 \frac{a_1 + a_2 + \cdots + a_n}{n} &\stackrel{AGM}{\geq} \sqrt[n]{a_1 a_2 \dots a_n} \Rightarrow \frac{ne^4}{n} \geq \sqrt[n]{a_1 a_2 \dots a_n} \Rightarrow \\
 a_1 a_2 \dots a_n &\leq e^{4n}; (2)
 \end{aligned}$$

From (1) and (2), it follows that:

$$\begin{aligned}
 S &< \sqrt{ne^4(n-1) \cdot \log e^{4n}} = \sqrt{4n^2 e^4(n-1)} \stackrel{e<3}{<} \\
 &< \sqrt{4n^2 \cdot 3^4(n-1)} \stackrel{?}{<} 9n^2 \Leftrightarrow 2\sqrt{n-1} \leq n^2 \Leftrightarrow (n-2)^2 \geq 0 \text{ true.}
 \end{aligned}$$

Solution 2 by proposer

$$\begin{aligned}
 &\sqrt{\log a_1^{a_2} + \log a_1^{a_3} + \cdots + \log a_1^{a_n}} + \cdots + \sqrt{\log a_n^{a_1} + \log a_n^{a_2} + \cdots + \log a_n^{a_{n-1}}} = \\
 &= \sqrt{(a_2 + a_3 + \cdots + a_n) \log a_1} + \cdots + \sqrt{(a_1 + a_2 + \cdots + a_{n-1}) \log a_n} \stackrel{CBS}{\leq} \\
 &\stackrel{CBS}{\leq} \sqrt{(a_2 + a_3 + \cdots + a_n) + \cdots + (a_1 + a_2 + \cdots + a_{n-1})} \cdot \sqrt[n]{\sum_{k=1}^n \log a_k} = \\
 &= \sqrt{(n-1)S_n} \cdot \sqrt{\log(a_1 a_2 \cdots a_n)} \leq \\
 &\leq \sqrt{(n-1)S_n} \cdot \sqrt{\log \left(\frac{a_1 + a_2 + \cdots + a_n}{n} \right)^n} = \\
 &= \sqrt{(n-1)S_n} \cdot \sqrt{n \cdot \log \left(\frac{S_n}{n} \right)} = \sqrt{(n-1)ne^4} \cdot \sqrt{4n} = 2ne^2 \cdot \sqrt{n-1} \leq \\
 &\leq 2ne^2 \cdot \frac{n-1+1}{2} = n^2 e^2 < 9n^2
 \end{aligned}$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

767. For $n > 1$, prove that:

$$\frac{\tan^{-1}(e^{\frac{x}{\pi}})}{n} + e^{-\frac{x}{\pi}} > \frac{\pi}{2n}$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution 1 by Adrian Popa-Romania

Let be $f(x) = \frac{\tan^{-1}(e^{\frac{x}{\pi}})}{n} + e^{-\frac{x}{\pi}}$, then

$$f'(x) = \frac{1}{n} \cdot \frac{1}{1+e^{\frac{2x}{\pi}}} \cdot e^{\frac{x}{\pi}} \cdot \frac{1}{\pi} - \frac{1}{\pi} e^{-\frac{x}{\pi}} = \frac{1}{\pi} \left(\frac{e^{\frac{x}{\pi}}}{n(1+e^{\frac{2x}{\pi}})} - \frac{1}{e^{\frac{x}{\pi}}} \right) = \\ = \frac{1}{\pi} \cdot \frac{e^{\frac{2x}{\pi}} - n - ne^{\frac{2x}{\pi}}}{ne^{\frac{x}{\pi}}(1+e^{\frac{2x}{\pi}})} = 0 \Rightarrow e^{\frac{2x}{\pi}} = \frac{n}{1-n}; e^{\frac{2x}{\pi}} > 0, \forall x \in \mathbb{R} \Rightarrow f'(x) = 0 \Leftrightarrow n < 1$$

If $n > 1 \Rightarrow f'(x) < 0 \downarrow; \forall x \in \mathbb{R} \Rightarrow$

$$f(x) > \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \left(\frac{\tan^{-1}(e^{\frac{x}{\pi}})}{n} + e^{-\frac{x}{\pi}} \right) = \frac{\pi}{2n}$$

If $f(x) > \frac{\pi}{2n}$, then $n > 1, \forall x \in \mathbb{R}$.

Solution 2 by Ravi Prakash-New Delhi-India

$$\frac{\tan^{-1}(e^{\frac{x}{\pi}})}{n} + e^{-\frac{x}{\pi}} > \frac{\pi}{2n}; \quad (1)$$

$$(1) \Leftrightarrow \tan^{-1}(e^{\frac{x}{\pi}}) + \frac{n}{e^{\frac{x}{\pi}}} > \frac{\pi}{2}$$

$$\text{Let } f(x) = \tan^{-1} x + \frac{n}{x} - \frac{\pi}{2}, \forall x > 0; \quad (2)$$

$$f'(x) = \frac{1}{1+x^2} - \frac{n}{x^2} = \frac{(1-n)x^2 - n}{x^2(1+x^2)} < 0; \forall x > 0 \Rightarrow f \downarrow \text{on } (0, \infty)$$

We have:

$$\lim_{x \rightarrow 0^+} f(x) = +\infty, \lim_{x \rightarrow \infty} f(x) = 0 \Rightarrow f(x) > 0; \forall x > 0$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Replacing x by $e^{\frac{x}{\pi}}$ in (2), we get (1).

768. If $x, y > 0, a, b, c, d \in \mathbb{N}, a \geq b \geq c \geq d$ then:

$$x^{a+b+c+d} + y^{a+b+c+d} \geq \frac{1}{8}(x^a + y^a)(x^b + y^b)(x^c + y^c)(x^d + y^d)$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution by Ravi Prakash-New Delhi-India

Let $x_1 = x^a, x_2 = x^b, x_3 = x^c, x_4 = x^d, y_1 = y^a, y_2 = y^b, y_3 = y^c, y_4 = y^d$

$a \geq b \geq c \geq d \Rightarrow x_1 \geq x_2 \geq x_3 \geq x_4$ and $y_1 \geq y_2 \geq y_3 \geq y_4$. Inequality becomes:

$$\begin{aligned} 8 \left(\prod_{i=1}^4 x_i + \prod_{i=1}^4 y_i \right) &\geq \prod_{i=1}^4 (x_i + y_i) \Leftrightarrow \\ 8 \left(\prod_{i=1}^4 x_i + \prod_{i=1}^4 y_i \right) &\geq x_1 x_2 x_3 x_4 + x_1 x_2 x_3 y_4 + x_1 x_2 y_3 x_4 + x_1 x_2 y_3 y_4 + \\ &+ x_1 y_2 x_3 x_4 + x_1 y_2 x_3 y_4 + x_1 y_2 y_3 x_4 + x_1 y_2 y_3 y_4 + y_1 x_2 x_3 x_4 + y_1 x_2 x_3 y_4 + \\ &+ y_1 y_2 x_3 x_4 + y_1 y_2 x_3 y_4 + y_1 y_2 y_3 x_4 + y_1 y_2 y_3 y_4 \Leftrightarrow \end{aligned}$$

$$E = x_1 x_2 (x_3 - y_3) (x_4 - y_4) + x_1 x_3 (x_2 - y_2) (x_4 - y_4) + x_1 x_4 (x_2 - y_2) (x_3 - y_3) \geq 0$$

If $x \geq y \Rightarrow x^a \geq y^a, x^b \geq y^b, x^c \geq y^c, x^d \geq y^d \Rightarrow$

$(x_1 - y_1)(x_2 - y_2) \geq 0$ (and analogs). Similarly for $x < y$.

Thus, $E \geq 0$.

769. Let $a, b, c \in \mathbb{R}$ such that: $|a|x^3 + |b|x^2 + |c| \leq \pi^2; \forall |x| \leq 1$.

Prove that: $3|a| + 2|b| < 90$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Adrian Popa-Romania

Let $f(x) = mx^3 + nx^2 + t$, where $|a| = m, |b| = n, |c| = t; m, n, p > 0$

$$\begin{cases} f(1) = m + n + t \\ f(0) = t \\ f(-1) = -m + n + t \end{cases} \Rightarrow \begin{cases} |f(1)| \leq \pi^2 \\ |f(0)| \leq \pi^2 \\ |f(-1)| \leq \pi^2 \end{cases} \Rightarrow \begin{cases} -\pi^2 \leq f(1) \leq \pi^2 \\ -\pi^2 \leq f(0) \leq \pi^2 \\ -\pi^2 \leq f(-1) \leq \pi^2 \end{cases}$$

$$\Rightarrow f(1) + f(-1) = 2n + 2f(0) \Rightarrow n = \frac{f(1) + f(-1) - 2f(0)}{2}$$

$$m = f(1) - n - f(0) = \frac{f(1) - f(-1)}{2}$$

$$3|a| + 2|b| = 3m + 2n = \frac{5f(1) - f(-1) - 4f(0)}{2} \leq 5\pi^2 < 90$$

770. For $n > 1$ prove that:

$$\frac{\sin^{-1}(e^{\frac{x}{\pi}})}{n} + e^{-\frac{x}{\pi}} \geq \frac{\frac{\sqrt{n(\sqrt{n^2+4}+n)}}{\sqrt{2}} + \sin^{-1}\left(\frac{\sqrt{n(\sqrt{n^2+4}-n)}}{\sqrt{2}}\right)}{n}$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Ravi Prakash-New Delhi-India

$$\frac{\sin^{-1}(e^{\frac{x}{\pi}})}{n} + e^{-\frac{x}{\pi}} \geq \frac{\frac{\sqrt{n(\sqrt{n^2+4}+n)}}{\sqrt{2}} + \sin^{-1}\left(\frac{\sqrt{n(\sqrt{n^2+4}-n)}}{\sqrt{2}}\right)}{n}$$

$$\Leftrightarrow \sin^{-1}(e^{\frac{x}{\pi}}) + ne^{-\frac{x}{\pi}} \geq \sin^{-1}\frac{\sqrt{n(\sqrt{n^2+4}-n)}}{\sqrt{2}} + \frac{1}{\sqrt{2}}\sqrt{n\sqrt{n^2+4}+n}$$

As $e^{\frac{x}{\pi}} > 0; \forall x \in \mathbb{R}$, LHS is defined for $0 < e^{\frac{x}{\pi}} \leq 1 \Leftrightarrow \frac{x}{\pi} \leq 0 \Leftrightarrow x \leq 0$.

Let $f(t) = \sin^{-1} t + \frac{n}{t}, 0 < t \leq 1 \Rightarrow f'(t) = \frac{1}{\sqrt{1-t^2}} - \frac{n}{t^2}, 0 < t < 1$

$$f'(t) = 0 \Leftrightarrow t^4 = n^2(1-t^2) \Leftrightarrow t^4 + n^2t^2 = n^2 \Leftrightarrow \left(t^2 + \frac{n^2}{2}\right)^2 = n^2 + \frac{n^4}{4}$$

$$t^2 = \frac{n}{2}\sqrt{n^2+4} - \frac{n^2}{2} \Rightarrow t = \frac{1}{\sqrt{2}}\sqrt{n(\sqrt{n^2+4}-n)}, 0 < t < 1$$

Also, $f'(t) > 0, t_0 < t < 1$ and $f'(t) < 0, 0 < t < t_0$. Thus,

$$f(t) \geq f(t_0), \forall t \in (0, 1]$$

We have:



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\frac{n}{t_0} = \frac{\sqrt{2n}}{\sqrt{n(\sqrt{n^2+4}-n)}} = \frac{1}{\sqrt{2}} \cdot \sqrt{n} \cdot \sqrt{\sqrt{n^2+4}+n}$$

$$\Rightarrow \sin^{-1} t + \frac{n}{t} \geq \sin^{-1} \frac{\sqrt{n(\sqrt{n^2+4}-n)}}{\sqrt{2}} + \frac{1}{\sqrt{2}} \sqrt{n\sqrt{n^2+4}+n}, t \in (0, 1]$$

To obtain the inequality, put $t = e^{\frac{x}{\pi}}, x \leq 0$.

771.

If $a_k, b_k \in (1, \infty); k = \overline{1, n}$ such that $\sum_{k=1}^n (a_k + b_k) = 4n$, then :

$$\sum_{k=1}^n \sqrt{n(\log^2 a_k + \log^2 b_k) + (n^2 + 1) \log a_k \cdot \log b_k} \leq n(n + 1).$$

Proposed by Florică Anastase-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let's prove that : $\sqrt{n(x^2 + y^2) + (n^2 + 1)xy} \leq \frac{(n + 1)(x + y)}{2}, \forall x, y > 0$

$$\begin{aligned} \sqrt{n(x^2 + y^2) + (n^2 + 1)xy} &\leq \frac{(n + 1)(x + y)}{2} \leftrightarrow 4n(x^2 + y^2) + 4(n^2 + 1)xy \\ &\leq (n + 1)^2[(x^2 + y^2) + 2xy] \end{aligned}$$

$$\leftrightarrow 0 \leq (n - 1)^2(x - y)^2 \text{ which is true } \rightarrow \forall x, y > 0, \sqrt{n(x^2 + y^2) + (n^2 + 1)xy}$$

$$\leq \frac{(n + 1)(x + y)}{2}$$

For $x = \log a_k > 0, y = \log b_k > 0$,

$$\begin{aligned} \text{We have : } \sqrt{n(\log^2 a_k + \log^2 b_k) + (n^2 + 1) \log a_k \cdot \log b_k} \\ \leq \frac{(n + 1)(\log a_k + \log b_k)}{2}, \forall k = \overline{1, n} \end{aligned}$$

We know that : $\log x \leq x - 1, \forall x > 0$

$$\begin{aligned} \rightarrow \sqrt{n(\log^2 a_k + \log^2 b_k) + (n^2 + 1) \log a_k \cdot \log b_k} \\ \leq \frac{(n + 1)(a_k + b_k - 2)}{2}, \forall k = \overline{1, n} \end{aligned}$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned} \rightarrow \sum_{k=1}^n \sqrt{n(\log^2 a_k + \log^2 b_k) + (n^2 + 1) \log a_k \cdot \log b_k} &\leq \frac{n+1}{2} \sum_{k=1}^n (a_k + b_k - 2) \\ &= \frac{n+1}{2} \cdot (4n - 2n) = n(n+1) \end{aligned}$$

Therefore, $\sum_{k=1}^n \sqrt{n(\log^2 a_k + \log^2 b_k) + (n^2 + 1) \log a_k \cdot \log b_k} \leq n(n+1).$

Solution 2 by proposer

$$\begin{aligned} n(\log^2 a_k + \log^2 b_k) + (n^2 + 1) \log a_k \cdot \log b_k &= \\ &= (n \cdot \log a_k + \log b_k)(\log a_k + n \cdot \log b_k) \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{k=1}^n \sqrt{n(\log^2 a_k + \log^2 b_k) + (n^2 + 1) \log a_k \cdot \log b_k} &= \\ &= \sum_{k=1}^n \sqrt{(n \cdot \log a_k + \log b_k)(\log a_k + n \cdot \log b_k)} \stackrel{AGM}{\leq} \\ &\leq \sum_{k=1}^n \frac{(n \cdot \log a_k + \log b_k) + (\log a_k + n \cdot \log b_k)}{2} = \\ &= \frac{n+1}{2} \cdot \sum_{k=1}^n (\log a_k + \log b_k) \stackrel{\log t \leq t-1}{\leq} \frac{n+1}{2} \cdot \sum_{k=1}^n (a_k - 1 + b_k - 1) = \\ &= \frac{n+1}{2} \cdot \left[\sum_{k=1}^n (a_k + b_k) - 2n \right] = n(n+1) \end{aligned}$$

772. If $x, y, z, t > 0$ then

$$xyzt(x + y + z + t)^2 \leq 2(xy + zt)(xz + yt)(xt + yz)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$xyzt(x + y + z + t)^2 \leq 2(xy + zt)(xz + yt)(xt + yz); (*)$$

$$\begin{aligned} RHS_{(*)} &= 2(x^2yz + xy^2t + xz^2t + yzt^2)(xt + yz) \\ &= 2xyzt(x^2 + y^2 + z^2 + t^2) + 2(x^2y^2z^2 + x^2y^2t^2 + x^2z^2t^2 + y^2z^2t^2) \end{aligned}$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\rightarrow (*) \Leftrightarrow xyzt(x + y + z + t)^2$$

$$\leq 2xyzt(x^2 + y^2 + z^2 + t^2) + 2(x^2y^2z^2 + x^2y^2t^2 + x^2z^2t^2 + y^2z^2t^2)$$

$$\Leftrightarrow 2xyzt(xy + xz + xt + yz + yt + zt)$$

$$\leq xyzt(x^2 + y^2 + z^2 + t^2) + 2(x^2y^2z^2 + x^2y^2t^2 + x^2z^2t^2 + y^2z^2t^2)$$

$$\Leftrightarrow xyzt[(x - y)^2 + (z - t)^2] + x^2z^2(y - t)^2 + x^2t^2(y - z)^2 + y^2z^2(x - t)^2 \\ + y^2t^2(x - z)^2 \geq 0 \text{ which is true.}$$

$$\text{Therefore, } xyzt(x + y + z + t)^2$$

$$\leq 2(xy + zt)(xz + yt)(xt + yz) \text{ with equality iff } x = y = z = t.$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$2(xy + zt)(xz + yt)(xt + yz) - xyzt(x + y + z + t)^2 \geq 0$$

expanding and re-arranging

$$\Leftrightarrow 2(x^2y^2z^2 + x^2y^2t^2 + x^2z^2t^2 + y^2z^2t^2)$$

$$+ xyzt(x^2 + y^2 + z^2 + t^2) \stackrel{(i)}{\geq} 2xyzt(xy + xz + xt + yz + yt + zt)$$

We shall use $a^2 + b^2 + c^2 \geq ab + bc + ca$

$\geq ab + bc + ca$ in case of each of the following inequalities :

$$x^2y^2z^2 + x^2y^2t^2 + x^2z^2t^2 \geq xyz \cdot xy \cdot xt + xyz \cdot xt \cdot xz + xyz \cdot xz \cdot yt$$

$$\Rightarrow x^2y^2z^2 + x^2y^2t^2 + x^2z^2t^2 \stackrel{(*)}{\geq} xyzt(xy + xt + xz)$$

$$x^2y^2z^2 + x^2y^2t^2 + y^2z^2t^2 \geq xyz \cdot xy \cdot yt + xyz \cdot yt \cdot yz + xyz \cdot yz \cdot xt$$

$$\Rightarrow x^2y^2z^2 + x^2y^2t^2 + y^2z^2t^2 \stackrel{(**)}{\geq} xyzt(xy + yt + yz)$$

$$x^2y^2z^2 + x^2z^2t^2 + y^2z^2t^2 \geq xyz \cdot xz \cdot yt + xz \cdot yt \cdot yz + yz \cdot xt \cdot xyz$$

$$\Rightarrow x^2y^2z^2 + x^2z^2t^2 + y^2z^2t^2 \stackrel{(***)}{\geq} xyzt(xz + yt + yz)$$

$$x^2y^2t^2 + x^2z^2t^2 + y^2z^2t^2 \geq xy \cdot xt \cdot xz + xy \cdot yt \cdot yz + yt \cdot xt \cdot xyz$$

$$\Rightarrow x^2y^2t^2 + x^2z^2t^2 + y^2z^2t^2 \stackrel{****}{\geq} xyzt(xy + yt + zt)$$

$$\therefore (*) + (**) + (***) + (****) \Rightarrow 3(x^2y^2z^2 + x^2y^2t^2 + x^2z^2t^2 + y^2z^2t^2)$$

$$\geq 2xyzt(xy + xz + xt + yz + yt + zt)$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\Rightarrow 2(x^2y^2z^2 + x^2y^2t^2 + x^2z^2t^2 + y^2z^2t^2) \stackrel{(m)}{\geq} \frac{4}{3}xyzt(xy + xz + xt + yz + yt + zt)$$

$$\text{Similarly, } x^2 + y^2 + z^2 \stackrel{(\bullet)}{\geq} xy + yz + zx, x^2 + y^2 + t^2 \stackrel{(\bullet\bullet)}{\geq} xy + yt + xt, x^2 + z^2$$

$$+ t^2 \stackrel{(\bullet\bullet\bullet)}{\geq} xz + zt + xt, y^2 + z^2 + t^2 \stackrel{(\bullet\bullet\bullet\bullet)}{\geq} yz + zt + yt$$

$$\therefore (\bullet) + (\bullet\bullet) + (\bullet\bullet\bullet) + (\bullet\bullet\bullet\bullet) \Rightarrow 3xyzt(x^2 + y^2 + z^2 + t^2)$$

$$\geq 2xyzt(xy + xz + xt + yz + yt + zt)$$

$$\Rightarrow xyzt(x^2 + y^2 + z^2 + t^2) \stackrel{(n)}{\geq} \frac{2}{3}xyzt(xy + xz + xt + yz + yt + zt)$$

$$\therefore (m) + (n) \Rightarrow 2(x^2y^2z^2 + x^2y^2t^2 + x^2z^2t^2 + y^2z^2t^2) + xyzt(x^2 + y^2 + z^2 + t^2)$$

$$\geq \left(\frac{4}{3} + \frac{2}{3} \right) xyzt(xy + xz + xt + yz + yt + zt)$$

$$= 2xyzt(xy + xz + xt + yz + yt + zt)$$

$$\Rightarrow (i) \text{ is true} \Rightarrow xyzt(x + y + z + t)^2 \leq 2(xy + zt)(xz + yt)(xt + yz) \text{ (QED)}$$

773. If $a, b, c > 0$ such that $3a^2 + b^2 = 16c^2$ then

$$\frac{3}{a} + \frac{1}{b} \geq \frac{2}{c}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Adrian Popa-Romania

$$\begin{aligned} 3a^2 + b^2 &= \frac{a^2}{1} + \frac{a^2}{1} + \frac{a^2}{1} + \frac{b^2}{1} \stackrel{\text{Bergstrom}}{\geq} \frac{(3a + b)^2}{4} \Rightarrow \\ 16c^2 &\geq \frac{(3a + b)^2}{4} \Rightarrow 4c \geq \frac{3a + b}{2} \Rightarrow 2c \geq \frac{3a + b}{4} \stackrel{\text{AHM}}{\geq} \frac{4}{\frac{3}{a} + \frac{1}{b}} \\ &\Rightarrow c \left(\frac{3}{a} + \frac{1}{b} \right) \geq 2 \Rightarrow \frac{3}{a} + \frac{1}{b} \geq \frac{2}{c} \end{aligned}$$

Solution 2 by Ravi Prakash-New Delhi-India

$$3a^2 + b^2 = 16c^2, a, b, c > 0 \Leftrightarrow \left(\frac{\sqrt{3}a}{4c} \right)^2 + \left(\frac{b}{4c} \right)^2 = 1$$

$$\text{Put } \frac{\sqrt{3}a}{4c} = \cos \theta, \frac{b}{4c} = \sin \theta; 0 < \theta < \frac{\pi}{2} \Rightarrow a = \frac{4c}{\sqrt{3}} \cos \theta, b = 4c \sin \theta$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

We wish to show:

$$\frac{3}{a} + \frac{1}{b} \geq \frac{2}{c} \Rightarrow \frac{3\sqrt{3}}{\cos \theta} + \frac{1}{\sin \theta} \geq 8; (1)$$

$$\text{Let } f(\theta) = \frac{3\sqrt{3}}{\cos \theta} + \frac{1}{\sin \theta}; 0 < \theta < \frac{\pi}{2}$$

$$f'(\theta) = \frac{(\sqrt{3} \sin \theta)^3 - \cos^3 \theta}{\sin^2 \theta \cos^2 \theta}$$

$$f'(\theta) = 0 \Leftrightarrow \tan \theta = \sqrt{3} \text{ or } \theta = \frac{\pi}{6}.$$

Also, $f'(\theta) < 0$ if $0 < \theta < \frac{\pi}{6}$ and $f'(\theta) > 0$ if $\frac{\pi}{6} < \theta < \frac{\pi}{2}$. Thus,

$$f(\theta) \geq f\left(\frac{\pi}{6}\right) \text{ for } 0 < \theta < \frac{\pi}{2} \Leftrightarrow$$

$$\frac{3\sqrt{3}}{\cos \theta} + \frac{1}{\sin \theta} \geq \frac{3\sqrt{3}}{\frac{\sqrt{3}}{2}} + \frac{1}{\frac{1}{2}} = 8$$

Hence, (8) is true.

Solution 3 by Hikmat Mammadov-Azerbaijan

$$16c^2 = 3a^2 + b^2 = a^2 + a^2 + a^2 + b^2 \stackrel{AGM}{\geq} 4 \cdot \sqrt[4]{a^6 b^2}$$

$$\Rightarrow 4c^2 \geq \sqrt{a^3 b} \Rightarrow a^3 b \leq 16c^4$$

$$\frac{3}{a} + \frac{1}{b} \geq 4 \cdot \sqrt[4]{\frac{1}{a^3 b}} \geq 4 \cdot \sqrt[4]{\frac{1}{16c^4}} = \frac{2}{c}$$

$$\text{Hence, } \frac{3}{a} + \frac{1}{b} \geq \frac{2}{c}.$$

774. Let $k \in \mathbb{Z}^+$, $0 < \alpha \leq \beta \leq \gamma \leq \delta$, $\frac{1}{\alpha} + \frac{2}{\beta} + \frac{\delta}{\gamma} \geq 3$, $\frac{2}{\beta} + \frac{\delta}{\gamma} \geq 2$. Prove that:

$$\alpha^k + \beta^k + \gamma^k - \delta^k \leq 2^k + 1$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by proposer

Using Holder's Inequality, we have:

- $\left[\left(\frac{1}{\alpha} \right)^k + \left(\frac{2}{\beta} \right)^k + \left(\frac{\delta}{\gamma} \right)^k \right] 3^{k-1} \geq \left(\frac{1}{\alpha} + \frac{2}{\beta} + \frac{\delta}{\gamma} \right)^k \geq 3^k; \rightarrow \left(\frac{1}{\alpha} \right)^k + \left(\frac{2}{\beta} \right)^k + \left(\frac{\delta}{\gamma} \right)^k \geq 3;$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

- $\left[\left(\frac{2}{\beta} \right)^k + \left(\frac{\delta}{\gamma} \right)^k \right] 2^{k-1} \geq \left(\frac{2}{\beta} + \frac{\delta}{\gamma} \right)^k \geq 2^k; \rightarrow \left(\frac{2}{\beta} \right)^k + \left(\frac{\delta}{\gamma} \right)^k \geq 2;$

Then, we have:

$$\begin{aligned} \delta^k + 2^k + 1 &= (\gamma^k - \beta^k) \left(\frac{\delta}{\gamma} \right)^k + (\beta^k - \alpha^k) \left(\left(\frac{2}{\beta} \right)^k + \left(\frac{\delta}{\gamma} \right)^k \right) + \alpha^k \left(\left(\frac{1}{\alpha} \right)^k + \left(\frac{2}{\beta} \right)^k + \left(\frac{\delta}{\gamma} \right)^k \right) \\ &\geq (\gamma^k - \beta^k) \cdot 1 + (\beta^k - \alpha^k) \cdot 2 + \alpha^k \cdot 3 = \alpha^k + \beta^k + \gamma^k; \end{aligned}$$

Thus,

$$\alpha^k + \beta^k + \gamma^k - \delta^k \leq 2^k + 1$$

Proved. Equality if only if $\begin{cases} \alpha = 1 \\ \beta = 2 \\ \gamma = \delta \geq 2 \end{cases}$.

775.

Let $a \geq b \geq c > 0$ and $t = \frac{a+b}{2}$. Prove that:

$$\frac{1}{\sqrt{4a^2 + bc}} + \frac{1}{\sqrt{4b^2 + ca}} + \frac{1}{\sqrt{4c^2 + ab}} \geq \frac{2}{\sqrt{4t^2 + tc}} + \frac{1}{\sqrt{4c^2 + t^2}}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We have : ab \stackrel{AM-GM}{\leq} \left(\frac{a+b}{2} \right)^2 = t^2 \rightarrow \frac{1}{\sqrt{4c^2 + ab}} \geq \frac{1}{\sqrt{4c^2 + t^2}} \quad (1)$$

Also, we have

$$\begin{aligned} &: \frac{1}{\sqrt{4a^2 + bc}} + \frac{1}{\sqrt{4b^2 + ca}} \stackrel{AM-GM}{\leq} \frac{2}{\sqrt[4]{(4a^2 + bc)(4b^2 + ca)}} \stackrel{?}{\leq} \frac{2}{\sqrt{4t^2 + tc}} \\ &\leftrightarrow (4t^2 + tc)^2 \geq (4a^2 + bc)(4b^2 + ca) \leftrightarrow 16 \left(\frac{a+b}{2} \right)^4 + 8c \left(\frac{a+b}{2} \right)^3 + c^2 \left(\frac{a+b}{2} \right)^2 \\ &\geq 16a^2b^2 + 4c(a^3 + b^3) + abc^2 \\ &\leftrightarrow 4[(a+b)^4 - 16a^2b^2] - 4c[4(a^3 + b^3) - (a+b)^3] + c^2[(a+b)^2 - 4ab] \geq 0 \\ &\leftrightarrow 4(a^2 + b^2 + 6ab)(a-b)^2 - 12c(a+b)(a-b)^2 + c^2(a-b)^2 \geq 0 \\ &\leftrightarrow [4(a^2 + b^2) + 12a(b-c) + 12b(a-c) + c^2](a-b)^2 \geq 0 \end{aligned}$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 \text{Which is true because : } c = \min\{a, b, c\} \rightarrow & \frac{1}{\sqrt{4a^2 + bc}} + \frac{1}{\sqrt{4b^2 + ca}} \\
 & \geq \frac{2}{\sqrt{4t^2 + tc}} \quad (2)
 \end{aligned}$$

$$(1) + (2) \rightarrow \frac{1}{\sqrt{4a^2 + bc}} + \frac{1}{\sqrt{4b^2 + ca}} + \frac{1}{\sqrt{4c^2 + ab}} \geq \frac{2}{\sqrt{4t^2 + tc}} + \frac{1}{\sqrt{4c^2 + t^2}}.$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\begin{aligned}
 \text{We have : } ab & \stackrel{AM-GM}{\leq} \left(\frac{a+b}{2}\right)^2 = t^2 \rightarrow \frac{1}{\sqrt{4c^2 + ab}} \geq \frac{1}{\sqrt{4c^2 + t^2}} \\
 \rightarrow \text{It's suffices to prove : } & \frac{1}{\sqrt{4a^2 + bc}} + \frac{1}{\sqrt{4b^2 + ca}} \stackrel{(1)}{\geq} \frac{2}{\sqrt{4t^2 + tc}} \leftrightarrow \\
 \frac{1}{\sqrt{4a^2 + bc}} + \frac{1}{\sqrt{4b^2 + ca}} & \geq \frac{4}{\sqrt{4(a+b)^2 + 2(a+b)c}} \\
 \leftrightarrow & \frac{1}{\sqrt{4b^2 + ca}} - \frac{2}{\sqrt{4(a+b)^2 + 2(a+b)c}} \\
 & \geq \frac{2}{\sqrt{4(a+b)^2 + 2(a+b)c}} - \frac{1}{\sqrt{4a^2 + bc}} \\
 \leftrightarrow & \frac{\sqrt{4(a+b)^2 + 2(a+b)c} - 2\sqrt{4b^2 + ca}}{\sqrt{4b^2 + ca}} \geq \frac{2\sqrt{4a^2 + bc} - \sqrt{4(a+b)^2 + 2(a+b)c}}{\sqrt{4a^2 + bc}} \\
 \leftrightarrow & \frac{[4(a+b)^2 + 2(a+b)c] - 4(4b^2 + ca)}{\sqrt{4b^2 + ca} (\sqrt{4(a+b)^2 + 2(a+b)c} + 2\sqrt{4b^2 + ca})} \\
 & \geq \frac{4(4a^2 + bc) - [4(a+b)^2 + 2(a+b)c]}{\sqrt{4a^2 + bc} (2\sqrt{4a^2 + bc} + \sqrt{4(a+b)^2 + 2(a+b)c})} \\
 \leftrightarrow & \frac{(a-b)(2a+6b-c)}{\sqrt{4b^2 + ca} (\sqrt{4(a+b)^2 + 2(a+b)c} + 2\sqrt{4b^2 + ca})} \\
 & \geq \frac{(a-b)(6a+2b-c)}{\sqrt{4a^2 + bc} (2\sqrt{4a^2 + bc} + \sqrt{4(a+b)^2 + 2(a+b)c})}
 \end{aligned}$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 &\leftrightarrow (a-b) \left(\frac{2a+6b-c}{\sqrt{4b^2+ca} \cdot \sqrt{4(a+b)^2+2(a+b)c} + 8b^2+2ca} \right. \\
 &\quad \left. - \frac{6a+2b-c}{8a^2+2bc+\sqrt{4a^2+bc} \cdot \sqrt{4(a+b)^2+2(a+b)c}} \right) \geq 0 \\
 &\leftrightarrow (a-b) \left[(2a+6b-c) \left(8a^2+2bc+\sqrt{4a^2+bc} \cdot \sqrt{4(a+b)^2+2(a+b)c} \right) \right. \\
 &\quad \left. - (6a+2b-c) \left(\sqrt{4b^2+ca} \cdot \sqrt{4(a+b)^2+2(a+b)c} + 8b^2+2ca \right) \right] \\
 &\geq 0 \\
 &\leftrightarrow (a-b) \left[(a-b)(16a^2+64ab+16b^2-20bc-20ca+2c^2) \right. \\
 &\quad \left. + \sqrt{4(a+b)^2+2(a+b)c} \left[(2a+6b-c)\sqrt{4a^2+bc} \right. \right. \\
 &\quad \left. \left. - (6a+2b-c)\sqrt{4b^2+ca} \right] \right] \geq 0
 \end{aligned}$$

$$\begin{aligned}
 &\text{Since : } 16a^2+64ab+16b^2-20bc-20ca+2c^2 \\
 &\quad = 16a^2+16b^2+2c^2+24ab+20a(b-c)+20b(a-c) > 0
 \end{aligned}$$

$$\rightarrow \text{It's suffices to prove : } (2a+6b-c)\sqrt{4a^2+bc} - (6a+2b-c)\sqrt{4b^2+ca} \geq 0$$

$$\begin{aligned}
 &\leftrightarrow (2a+6b-c)^2(4a^2+bc) \geq (6a+2b-c)^2(4b^2+ca) \\
 &\leftrightarrow (4a^2+36b^2+c^2+24ab-12bc-4ca)(4a^2+bc) \\
 &\quad \geq (36a^2+4b^2+c^2+24ab-4bc-12ca)(4b^2+ca)
 \end{aligned}$$

$$\begin{aligned}
 &\leftrightarrow 16(a^4-b^4) + 96(a^3b-ab^3) - 68(a^2bc-ab^2c) - 52(a^3c-b^3c) - (ac^3-bc^3) \\
 &\quad + 16(c^2a^2-b^2c^2) \geq 0
 \end{aligned}$$

$$\begin{aligned}
 &\leftrightarrow (a-b)(16a^3+16b^3+112a^2b+112ab^2-120abc-52ca^2-52b^2c-c^3) \\
 &\quad + 16c^2a+16bc^2) \geq 0
 \end{aligned}$$

$$\begin{aligned}
 &\leftrightarrow (a-b)[16a^3+16b^3+60ab(a+b-2c)+52a^2(b-c)+52b^2(a-c) \\
 &\quad + c^2(a-c)+15c^2a+16bc^2] \geq 0
 \end{aligned}$$

Which is true from $a \geq b \geq c \rightarrow (1)$ is true. Therefore,

$$\frac{1}{\sqrt{4a^2+bc}} + \frac{1}{\sqrt{4b^2+ca}} + \frac{1}{\sqrt{4c^2+ab}} \geq \frac{2}{\sqrt{4t^2+tc}} + \frac{1}{\sqrt{4c^2+t^2}}$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

776.

If $a, b, c > 0, u \geq 0$, such that : $\min\{a^2, b^2, c^2\} \geq \frac{u}{1+3u} \sum_{cyc} a^2$, then prove that :

$$(1+3u) \sum_{cyc} \frac{a^2}{b} \geq u \left(\sum_{cyc} a^2 \right) \left(\sum_{cyc} \frac{1}{a} \right) + \sqrt{3 \sum_{cyc} a^2}.$$

Proposed by Marius Drăgan, Neculai Stanciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\begin{aligned} WLOG \text{ we may assume that } c = \min\{a, b, c\} \rightarrow (1+3u).c^2 \geq u(a^2 + b^2 + c^2) \\ \xrightarrow{AM-GM} (1+2u).c^2 \geq u(a^2 + b^2) \stackrel{\geq}{\geq} 2u.ab \end{aligned}$$

$$\begin{aligned} \text{Since } \max\{a, b, c\} \in \{a, b\} \rightarrow ab \geq \max\{a, b, c\}.c \rightarrow (1+2u)c \geq 2u.\max\{a, b, c\} \\ \leftrightarrow \frac{1+2u}{2} \cdot \min\{a, b, c\} \geq u \cdot \max\{a, b, c\} \end{aligned}$$

$$\text{Therefore, } \forall x, y \in \{a, b, c\}, \frac{1+2u}{2} \cdot x \geq u \cdot y \text{ or } \frac{1+2u}{2y} \geq \frac{u}{x} \quad (1)$$

$$\begin{aligned} \text{Now, we have : } (1+3u) \sum_{cyc} \frac{a^2}{b} &\geq u \left(\sum_{cyc} a^2 \right) \left(\sum_{cyc} \frac{1}{a} \right) + \sqrt{3 \sum_{cyc} a^2} \leftrightarrow (1+2u) \sum_{cyc} \frac{a^2}{b} \\ &\geq u \left(\sum_{cyc} a + \sum_{cyc} \frac{a^2}{c} \right) + \sqrt{3 \sum_{cyc} a^2} \\ &\leftrightarrow (1+2u) \left(\sum_{cyc} \frac{a^2}{b} - \sum_{cyc} a \right) \geq u \left(\sum_{cyc} \frac{a^2}{c} - \sum_{cyc} a \right) + \left(\sqrt{3 \sum_{cyc} a^2} - \sum_{cyc} a \right) \\ &\leftrightarrow (1+2u) \sum_{cyc} \frac{(a-b)^2}{b} \geq u \sum_{cyc} \frac{(c-a)^2}{c} + \frac{3(a^2+b^2+c^2)-(a+b+c)^2}{\sqrt{3(a^2+b^2+c^2)}+(a+b+c)} \\ &\leftrightarrow (1+2u) \sum_{cyc} \frac{(b-c)^2}{c} \geq u \sum_{cyc} \frac{(b-c)^2}{b} + \frac{\sum_{cyc} (b-c)^2}{\sqrt{3 \sum_{cyc} a^2} + \sum_{cyc} a} \\ &\leftrightarrow \sum_{cyc} \left[\left(\frac{1+2u}{c} - \frac{u}{b} \right) \left(\sqrt{3 \sum_{cyc} a^2} + \sum_{cyc} a \right) - 1 \right] (b-c)^2 \geq 0 \end{aligned}$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

From (1), we have : $\frac{1+2u}{2c} \geq \frac{u}{b}$ (And analogs), also, we have :

$$\sqrt{3 \sum_{cyc} a^2} \geq \sum_{cyc} a \quad (\text{Cauchy - Schwarz Inequality})$$

Therefore, it suffices to prove :

$$\sum_{cyc} \left[\frac{(1+2u)}{2c} \cdot 2 \sum_{cyc} a - 1 \right] (b-c)^2 \geq 0$$

$$\leftrightarrow \sum_{cyc} \left[\frac{(1+2u)(a+b+c)}{c} - 1 \right] (b-c)^2 \geq 0$$

Which is true because : $\frac{(1+2u)(a+b+c)}{c} \stackrel{a,b,u \geq 0}{\geq} \frac{1 \cdot c}{c} = 1$ (And analogs)

Therefore,

$$(1+3u) \sum_{cyc} \frac{a^2}{b} \geq u \left(\sum_{cyc} a^2 \right) \left(\sum_{cyc} \frac{1}{a} \right) + \sqrt{3 \sum_{cyc} a^2}$$

777.

Let $n \in N^*$, $a_1, a_2, \dots, a_n \geq 1$ and $S = \sum_{i=1}^n a_i$.

Prove that : $(nS^{n-1})^{(S-n)^2} \cdot \prod_{i=1}^n (a_i + n - 1)^{(a_i-1)(n-1+a_i-S)} \geq 1$.

Proposed by Pavlos Trifon-Greece

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$(*) : (nS^{n-1})^{(S-n)^2} \cdot \prod_{i=1}^n (a_i + n - 1)^{(a_i-1)(n-1+a_i-S)} \geq 1.$$

Let $b_i = a_i - 1 \geq 0, \forall i = \overline{1, n}$ and $T = \sum_{i=1}^n b_i = \sum_{i=1}^n (a_i - 1) = S - n$.

$$\rightarrow (*) \leftrightarrow \prod_{i=1}^n (b_i + n)^{b_i \cdot (T-b_i)} \leq [n(T+n)^{n-1}]^{T^2} \leftrightarrow$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\prod_{i=1}^n \left[n^{b_i \cdot (T-b_i)} \cdot \left(\frac{b_i}{n} + 1 \right)^{b_i \cdot (T-b_i)} \right] \leq \left[n^n \left(\frac{T}{n} + 1 \right)^{n-1} \right]^{T^2} \Leftrightarrow \\ \Leftrightarrow n^{T^2 - (b_1^2 + \dots + b_n^2)} \cdot \prod_{i=1}^n \left(\frac{b_i}{n} + 1 \right)^{b_i \cdot (T-b_i)} \leq n^{n \cdot T^2} \left(\frac{T}{n} + 1 \right)^{(n-1) \cdot T^2} \quad (**)$$

We have : $\frac{b_i}{n} + 1 \leq \frac{T}{n} + 1, b_i \cdot (T-b_i) \geq 0, \forall i = \overline{1, n} \rightarrow \prod_{i=1}^n \left(\frac{b_i}{n} + 1 \right)^{b_i \cdot (T-b_i)}$

$$\leq \prod_{i=1}^n \left(\frac{T}{n} + 1 \right)^{b_i \cdot (T-b_i)} = \left(\frac{T}{n} + 1 \right)^{T^2 - (b_1^2 + \dots + b_n^2)} \quad (1)$$

$$\text{Also, } \sum_{i=1}^n b_i^2 \stackrel{CBS}{\geq} \frac{1}{n} \left(\sum_{i=1}^n b_i \right)^2 = \frac{T^2}{n} \rightarrow T^2 - \sum_{i=1}^n b_i^2 \leq T^2 - \frac{T^2}{n}$$

$$= \frac{n-1}{n} \cdot T^2 \stackrel{n \geq 1}{\leq} (n-1) \cdot T^2 (2)$$

Since : $n \cdot \frac{b_i}{n} + 1 \geq 1$, and from (1) and (2), we get :

$$LHS_{(**)} \leq n^{(n-1) \cdot T^2} \cdot \left(\frac{T}{n} + 1 \right)^{(n-1) \cdot T^2} \stackrel{n \geq 1}{\leq} n^{n \cdot T^2} \left(\frac{T}{n} + 1 \right)^{(n-1) \cdot T^2} = RHS_{(**)}$$

$\rightarrow (**)$ is true.

$$\text{Therefore, } (nS^{n-1})^{(S-n)^2} \cdot \prod_{i=1}^n (a_i + n - 1)^{(a_i-1)(n-1+a_i-S)} \geq 1,$$

with equality iff $b_i = T, \forall i = \overline{1, n} \leftrightarrow b_i = 0, \forall i = \overline{1, n} \leftrightarrow a_i = 1, \forall i = \overline{1, n}$ or $n = 1$.

Solution 2 by proposer

$$f(x) = \log x \nearrow [1, \infty) \text{ by Vornicu-Schur, for } (a, b, c) = (S, a_1 + n - 1, n) \Rightarrow \\ (S - a_1 - n + 1) \cdot (S - n) \cdot \log S + (a_1 - 1) \cdot (n - 1 + a_1 - S) \cdot \log(a_1 + n - 1) + \\ +(n - S) \cdot (1 - a_1) \cdot \log n \geq 0 \\ S^{(S-a_1-n+1)(S-n)} \cdot (a_1 + n - 1)^{(a_1-1)(n-1+a_1-S)} \cdot n^{(n-S)(1-a_1)} \geq 1; (1)$$

Equality holds for $S = a_1 + n - 1 = n$

$$f(x) = \log x \nearrow [1, \infty) \text{ by Vornicu-Schur, for } (a, b, c) = (S, a_2 + n - 1, n) \Rightarrow$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$(S - a_2 - n + 1) \cdot (S - n) \cdot \log S + (a_2 - 1) \cdot (n - 1 + a_2 - S) \cdot \log(a_2 + n - 1) + \\ + (n - S) \cdot (1 - a_2) \cdot \log n \geq 0$$

$$S^{(S-a_2-n+1)(S-n)} \cdot (a_2 + n - 1)^{(a_2-1)(n-1+a_2-S)} \cdot n^{(n-S)(1-a_2)} \geq 1; (2)$$

Equality holds for $S = a_2 + n - 1 = n$

$$f(x) = \log x \nearrow [1, \infty) \text{ by Vornicu-Schur, for } (a, b, c) = (S, a_n + n - 1, n) \Rightarrow \\ (S - a_n - n + 1) \cdot (S - n) \cdot \log S + (a_n - 1) \cdot (n - 1 + a_n - S) \cdot \log(a_n + n - 1) + \\ + (n - S) \cdot (1 - a_n) \cdot \log n \geq 0$$

$$S^{(S-a_n-n+1)(S-n)} \cdot (a_n + n - 1)^{(a_n-1)(n-1+a_n-S)} \cdot n^{(n-S)(1-a_n)} \geq 1; (n)$$

Equality holds for $S = a_n + n - 1 = n$

By multiplying (1),(2),...,(n), it follows

$$(nS^{n-1})^{(S-n)^2} \cdot \prod_{i=1}^n (a_i + n - 1)^{(a_i-1)(n-1+a_i-S)} \geq 1$$

Equality holds for $a_1 = a_2 = \dots = a_n = 1$.

778. Let $a, b, c, d \in \mathbb{R}$ such that $|ax^4 + bx^3 + cx^2 + dx| \leq 2, \forall |x| \leq 1$.

Prove that:

$$|4a + 3b + 2c + d| \leq 32$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Lazaros Zachariadis-Thessaloniki-Greece

$$f(x) = ax^4 + bx^3 + cx^2 + dx; |x| \leq 1 \text{ and } |f(x)| \leq 2$$

$$A = f\left(-\frac{1}{2}\right) = \frac{a}{16} - \frac{b}{8} + \frac{c}{4} - \frac{d}{2} \Leftrightarrow 16A = a - 2b + 4c - 8d; (1)$$

$$B = f\left(\frac{1}{2}\right) = \frac{a}{16} + \frac{b}{8} + \frac{c}{4} + \frac{d}{2} \Leftrightarrow 16B = a + 2b + 4c + 8d; (2)$$

$$C = f(-1) = a - b + c - d; (3)$$

$$D = f(1) = a + b + c + d$$

$$(1) - (2) \Rightarrow 16A - 16B = -4b - 16d \Leftrightarrow 4A - 4B = -b - 4d$$

$$(3) - (4) \Rightarrow C - D = -2b - 2d \Leftrightarrow \frac{C}{2} - \frac{D}{2} = -b - d$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$4A - 4B - \frac{C}{2} + \frac{D}{2} = -3d \Leftrightarrow d = -\frac{4}{3}A + \frac{4}{3}B + \frac{C}{6} - \frac{D}{6}$$

So,

$$\frac{C}{2} - \frac{D}{2} = -b + \frac{4}{3}A - \frac{4}{3}B - \frac{C}{6} + \frac{D}{6} \Leftrightarrow \frac{2}{3}C - \frac{2}{3}D - \frac{4}{3}A + \frac{4}{3}B = -b$$

$$3b = -2C + 2D + 4A - 4B$$

$$(1) + (2) \Rightarrow 16A + 16B = 2a + 8c \Leftrightarrow 8A + 8B = a + 4c$$

$$(3) + (4) \Rightarrow C + D = 2a + 2c \Leftrightarrow \frac{C}{2} + \frac{D}{2} = a + c; (*)$$

$$\Rightarrow 8A + 8B - \frac{C}{2} - \frac{D}{2} = 3c \Leftrightarrow \frac{8}{3}A + \frac{8}{3}B - \frac{C}{6} - \frac{D}{6} = c$$

$$2c = \frac{16}{3}A + \frac{16}{3}B - \frac{C}{3} - \frac{D}{3}$$

$$(*) \Rightarrow 2C + 2D = 4a + \frac{32}{3}A + \frac{32}{3}B - \frac{2}{3}C - \frac{2}{3}D \Leftrightarrow$$

$$4a = \frac{8}{3}C + \frac{8}{3}D - \frac{32}{3}A - \frac{32}{3}B$$

So, $|4a + 3b + 2c + d| =$

$$\begin{aligned} &= \left| A \left(-\frac{32}{3} + \frac{12}{3} + \frac{16}{3} - \frac{4}{3} \right) + B \left(-\frac{32}{3} - \frac{12}{3} + \frac{16}{3} + \frac{4}{3} \right) + C \left(\frac{8}{3} - \frac{6}{3} - \frac{1}{3} + \frac{1}{6} \right) \right. \\ &\quad \left. + D \left(\frac{8}{3} + \frac{6}{3} - \frac{1}{3} - \frac{1}{6} \right) \right| = \left| -\frac{8}{3}A - \frac{24}{3}B + \frac{C}{2} - \frac{25}{6}D \right| \leq \\ &\leq \frac{8}{3} \cdot 2 + \frac{24}{3} \cdot 2 + \frac{1}{2} \cdot 2 + \frac{25}{6} \cdot 2 = 32 \end{aligned}$$

Therefore,

$$|4a + 3b + 2c + d| \leq 32$$

779. If $a, b, c, d, e > 0$ then:

$$c + \frac{ad}{b \cot^2 \frac{\pi}{20}} + \frac{bd}{c \cot^2 \frac{3\pi}{20}} + \frac{d^2}{e \cot^2 \frac{7\pi}{20}} + \frac{ed}{a \cot^2 \frac{9\pi}{20}} \geq 5d$$

Proposed by Daniel Sitaru-Romania

Solution 1 by George Titakis-Greece



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$c + \frac{ad}{b \cot^2 \frac{\pi}{20}} + \frac{bd}{c \cot^2 \frac{3\pi}{20}} + \frac{d^2}{e \cot^2 \frac{7\pi}{20}} + \frac{ed}{a \cot^2 \frac{9\pi}{20}} \geq$$

$$\geq 5 \cdot \sqrt[5]{c \cdot \frac{ad}{b \cot^2 \frac{\pi}{20}} \cdot \frac{bd}{c \cot^2 \frac{3\pi}{20}} \cdot \frac{d^2}{e \cot^2 \frac{7\pi}{20}} \cdot \frac{ed}{a \cot^2 \frac{9\pi}{20}}} = 5d$$

$$\text{Because: } \cot^2 \frac{\pi}{20} \cot^2 \frac{3\pi}{20} \cot^2 \frac{7\pi}{20} \cot^2 \frac{9\pi}{20} = 1$$

This is true, due to the trigonometric identity:

$$\cot(a + b)(\cot a + \cot b) = \cot a \cdot \cot b - 1; (1)$$

If $a = \frac{\pi}{20}$ and $b = \frac{9\pi}{20}$, then the left part of (1) is zero, due to $\cot \frac{\pi}{2} = 0$. So, from the right

$$\text{part we have that } \cot \frac{\pi}{20} \cot \frac{9\pi}{20} = 1; (2)$$

$$\text{With the same way, if } a = \frac{7\pi}{20} \text{ and } b = \frac{3\pi}{20};$$

$$\cot \frac{7\pi}{20} \cot \frac{3\pi}{20} = 1; (3)$$

From (2) and (3) it is obvious that: $\cot^2 \frac{\pi}{20} \cot^2 \frac{3\pi}{20} \cot^2 \frac{7\pi}{20} \cot^2 \frac{9\pi}{20} = 1$

By AM-GM inequality we have that:

$$c + \frac{ad}{b \cot^2 \frac{\pi}{20}} + \frac{bd}{c \cot^2 \frac{3\pi}{20}} + \frac{d^2}{e \cot^2 \frac{7\pi}{20}} + \frac{ed}{a \cot^2 \frac{9\pi}{20}} \geq$$

$$\geq 5 \cdot \sqrt[5]{c \cdot \frac{ad}{b \cot^2 \frac{\pi}{20}} \cdot \frac{bd}{c \cot^2 \frac{3\pi}{20}} \cdot \frac{d^2}{e \cot^2 \frac{7\pi}{20}} \cdot \frac{ed}{a \cot^2 \frac{9\pi}{20}}} = 5d$$

Solution 2 by Samar Das-India

$$\frac{c + \frac{ad}{b \cot^2 \frac{\pi}{20}} + \frac{bd}{c \cot^2 \frac{3\pi}{20}} + \frac{d^2}{e \cot^2 \frac{7\pi}{20}} + \frac{ed}{a \cot^2 \frac{9\pi}{20}}}{5} \geq$$

$$\geq \sqrt[5]{c \cdot \frac{ad}{b \cot^2 \frac{\pi}{20}} \cdot \frac{bd}{c \cot^2 \frac{3\pi}{20}} \cdot \frac{d^2}{e \cot^2 \frac{7\pi}{20}} \cdot \frac{ed}{a \cot^2 \frac{9\pi}{20}}}$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 & c + \frac{ad}{b \cot^2 \frac{\pi}{20}} + \frac{bd}{c \cot^2 \frac{3\pi}{20}} + \frac{d^2}{e \cot^2 \frac{7\pi}{20}} + \frac{ed}{a \cot^2 \frac{9\pi}{20}} \geq \\
 & \geq 5 \cdot \sqrt[5]{c \cdot \frac{ad}{b \cot^2 \frac{\pi}{20}} \cdot \frac{bd}{c \cot^2 \frac{3\pi}{20}} \cdot \frac{d^2}{e \cot^2 \frac{7\pi}{20}} \cdot \frac{ed}{a \cot^2 \frac{9\pi}{20}}} \\
 & c + \frac{ad}{b \cot^2 \frac{\pi}{20}} + \frac{bd}{c \cot^2 \frac{3\pi}{20}} + \frac{d^2}{e \cot^2 \frac{7\pi}{20}} + \frac{ed}{a \cot^2 \frac{9\pi}{20}} \\
 & \geq 5d \sqrt[5]{\cot^2 \frac{\pi}{20} \cot^2 \frac{3\pi}{20} \cot^2 \frac{7\pi}{20} \cot^2 \frac{9\pi}{20}}; \quad (1)
 \end{aligned}$$

$$\begin{aligned}
 & \tan \frac{\pi}{20} \tan \frac{3\pi}{20} \tan \frac{7\pi}{20} \tan \frac{9\pi}{20} = \tan \frac{\pi}{20} \tan \frac{3\pi}{20} \tan \frac{10\pi - 3\pi}{20} \tan \frac{10\pi - \pi}{20} = \\
 & = \tan \frac{\pi}{20} \tan \frac{3\pi}{20} \tan \left(\frac{\pi}{2} - \frac{3\pi}{20} \right) \tan \left(\frac{\pi}{2} - \frac{\pi}{20} \right) = \tan \frac{\pi}{20} \tan \frac{3\pi}{20} \cot \frac{3\pi}{20} \cot \frac{\pi}{20} = 1; \quad (2)
 \end{aligned}$$

From (1) and (2) we have:

$$c + \frac{ad}{b \cot^2 \frac{\pi}{20}} + \frac{bd}{c \cot^2 \frac{3\pi}{20}} + \frac{d^2}{e \cot^2 \frac{7\pi}{20}} + \frac{ed}{a \cot^2 \frac{9\pi}{20}} \geq 5d$$

780.

Let $a, b > 0, c \geq 0$. Prove that :

$$\frac{1}{b^2 + bc + c^2} + \frac{1}{c^2 + ca + a^2} \geq \frac{2(a+b)}{ab(a+b) + bc(b+c) + ca(c+a)} \geq \frac{2(a+b)}{a^3 + b^3 + c^3 + 3abc}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\begin{aligned}
 & \frac{1}{b^2 + bc + c^2} + \frac{1}{c^2 + ca + a^2} \stackrel{(1)}{\geq} \frac{2(a+b)}{ab(a+b) + bc(b+c) + ca(c+a)} \stackrel{(2)}{\geq} \frac{2(a+b)}{a^3 + b^3 + c^3 + 3abc} \\
 (1) \leftrightarrow & \frac{1}{b^2 + bc + c^2} - \frac{a+b}{ab(a+b) + bc(b+c) + ca(c+a)} \\
 & \geq \frac{a+b}{ab(a+b) + bc(b+c) + ca(c+a)} - \frac{1}{c^2 + ca + a^2}
 \end{aligned}$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 & \leftrightarrow \frac{a^2b + ca^2 - abc - b^3}{(b^2 + bc + c^2)[ab(a+b) + bc(b+c) + ca(c+a)]} \\
 & \geq \frac{abc + a^3 - b^2c - ab^2}{(c^2 + ca + a^2)[ab(a+b) + bc(b+c) + ca(c+a)]} \\
 & \leftrightarrow \frac{(a-b)(ab + b^2 + ca)}{b^2 + bc + c^2} \geq \frac{(a-b)(bc + a^2 + ab)}{c^2 + ca + a^2} \\
 & \leftrightarrow (a-b)[(ab + b^2 + ca)(c^2 + ca + a^2) \\
 & \quad - (bc + a^2 + ab)(b^2 + bc + c^2)] \geq 0 \\
 & \leftrightarrow (a-b)(a^3b + c^3a + ca^3 - b^3c - bc^3 - ab^3) \geq 0 \\
 & \leftrightarrow (a-b)^2(a^2b + ab^2 + c^3 + ca^2 + abc + b^2c) \geq 0
 \end{aligned}$$

Which is true.

$$(2) \leftrightarrow a^3 + b^3 + c^3 + 3abc$$

$\geq ab(a+b) + bc(b+c) + ca(c+a)$ which is Schur's inequality.

Therefore,

$$\begin{aligned}
 & \frac{1}{b^2 + bc + c^2} + \frac{1}{c^2 + ca + a^2} \geq \frac{2(a+b)}{ab(a+b) + bc(b+c) + ca(c+a)} \\
 & \geq \frac{2(a+b)}{a^3 + b^3 + c^3 + 3abc}.
 \end{aligned}$$

781. Let I be a convex subset in \mathbb{R}_+ , $f : I \rightarrow R^+$ concave function, $x_i, y_i \in I$,

$\forall i = \overline{1, n}, f(0) \geq 0, p \geq 1$, then:

$$\left(\sum_{i=1}^n f(x_i + y_i)^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^n f(x_i)^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^n f(y_i)^p \right)^{\frac{1}{p}}.$$

Proposed by Seyran Ibrahimov-Maasilli-Azerbaijan

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since : $f(x_i), f(y_i) \geq 0, \forall i = \overline{1, n}$, then from Minkowski's inequality, we have :

$$\left(\sum_{i=1}^n f(x_i)^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^n f(y_i)^p \right)^{\frac{1}{p}} \geq \left(\sum_{i=1}^n [f(x_i) + f(y_i)]^p \right)^{\frac{1}{p}} \quad (1)$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\text{Also, we have : } x_i = \frac{y_i}{x_i + y_i} \cdot \mathbf{0} + \frac{x_i}{x_i + y_i} \cdot (x_i + y_i), \forall i$$

$= \overline{1, n}$, then from Jensen's inequality, we have :

$$f(x_i) \geq \frac{y_i}{x_i + y_i} \cdot f(\mathbf{0}) + \frac{x_i}{x_i + y_i} \cdot f(x_i + y_i), \forall i = \overline{1, n}$$

$$\text{Similarly, we have : } f(y_i) \geq \frac{x_i}{x_i + y_i} \cdot f(\mathbf{0}) + \frac{y_i}{x_i + y_i} \cdot f(x_i + y_i), \forall i = \overline{1, n}$$

Summing up these inequalities, we obtain : $f(x_i) + f(y_i)$

$$\geq f(\mathbf{0}) + f(x_i + y_i) \stackrel{f(\mathbf{0}) \geq 0}{\geq} f(x_i + y_i), \forall i = \overline{1, n} \quad (2)$$

$$\text{From (1) and (2), we get : } \left(\sum_{i=1}^n f(x_i + y_i)^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^n f(x_i)^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^n f(y_i)^p \right)^{\frac{1}{p}}.$$

782. Let $0 \leq x, y, z \leq 1$. Find the maximum value of the expression:

$$Q = (x - y)(y - z)(z - x)(x + y + z)$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by proposer

Without loss of generality, assume that $x = \max\{x, y, z\} \rightarrow x \geq y \geq z$ or $x \geq z \geq y$.

Case 1: $x \geq y \geq z \rightarrow Q \leq 0$.

Case 2: $x \geq z \geq y \rightarrow 0 \leq x - z, 0 \leq z - y, 0 \leq x - y \leq 1$

Thus,

$$Q = (x - y)(y - z)(z - x)(x + y + z) \leq (z - y)(x - z)(x + y + z)$$

$$\leftrightarrow 4Q \leq [2(z - y)][(\sqrt{3} + 1)(x - z)][(\sqrt{3} - 1)(x + y + z)]$$

Other, by AM-GM Inequality we have:

$$[2(z - y)][(\sqrt{3} + 1)(x - z)][(\sqrt{3} - 1)(x + y + z)] \leq$$

$$\leq \frac{1}{27} [2(z - y) + (\sqrt{3} + 1)(x - z) + (\sqrt{3} - 1)(x + y + z)]^3$$

$$= \frac{1}{27} [2\sqrt{3}x - (3 - \sqrt{3})y]^3;$$

But:



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$0 \leq 2\sqrt{3}x + \sqrt{3}y - 3y = 2\sqrt{3}x - (3 - \sqrt{3})y \leq 2\sqrt{3}$$

Therefore,

$$4Q \leq \frac{8\sqrt{3}}{9}$$

$$\rightarrow Q \leq \frac{2\sqrt{3}}{9}$$

Equality if and only if : $\begin{cases} x - y = 1 \\ 2(z - y) = (\sqrt{3} + 1)(x - z) = (\sqrt{3} - 1)(x + y + z) \\ x = 1, y = 0 \end{cases}$

$$\Leftrightarrow \begin{cases} x = 1 \\ y = 0 \\ z = \frac{1}{\sqrt{3}} \end{cases}$$

783. For $0 < a < b < c$ prove that:

$$\frac{b + \ln \frac{c}{a}}{a + \ln \frac{c}{a}} < \frac{(b + \frac{b}{a} + \frac{c}{b} - 2)(c - \frac{a}{b} - \frac{b}{c} + 2)}{(c + \frac{b}{a} + \frac{c}{b} - 2)(a - \frac{a}{b} - \frac{b}{c} + 2)}.$$

Proposed by Pavlos Trifon-Greece

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $x = 2 - \frac{a}{b} - \frac{b}{c}$, $y = \ln \frac{c}{a}$, $z = \frac{b}{a} + \frac{c}{b} - 2$. We know that : If $\alpha > 0$, then : $1 - \frac{1}{\alpha} \leq \ln(\alpha) \leq \alpha - 1$. Equality holds iff $\alpha = 1$.

$$\begin{aligned} \rightarrow x &= \left(1 - \frac{a}{b}\right) + \left(1 - \frac{b}{c}\right) \stackrel{a < b < c}{\gtrless} \ln \frac{b}{a} + \ln \frac{c}{b} = \ln \frac{c}{a} = y \stackrel{a < b < c}{\gtrless} \left(\frac{b}{a} - 1\right) + \left(\frac{c}{b} - 1\right) = z \\ &\rightarrow 0 \stackrel{a < b < c}{\gtrless} x < y < z \end{aligned}$$

$$\begin{aligned} \text{We need to prove : } \frac{b+y}{a+y} &< \frac{(b+z)(c+x)}{(c+z)(a+x)} \leftrightarrow (c+z)(b+y)(a+x) \\ &< (c+x)(b+z)(a+y) \end{aligned}$$

$$\leftrightarrow \ln(c+z) + \ln(b+y) + \ln(a+x) < \ln(c+x) + \ln(b+z) + \ln(a+y) \quad (1)$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Now, we consider the function $f : (0, \infty) \rightarrow \mathbb{R}$ defined by $f(t) = \ln(t)$, which is strictly concave.

Next we will apply Karamata's inequality. Namely, if (x_1, x_2, x_3) majorizes (y_1, y_2, y_3) and f is strictly concave, then :

$f(x_1) + f(x_2) + f(x_3) < f(y_1) + f(y_2) + f(y_3)$. Setting $(x_1, x_2, x_3) = (c+z, b+y, a+x)$ and

$(y_1, y_2, y_3) = (\max(c+x, b+z, a+y), \text{mid}(c+x, b+z, a+y), \min(c+x, b+z, a+y))$

Since : $a < b < c$ and $x < y < z \rightarrow x_1 > x_2 > x_3, y_1 > y_2 > y_3, x_1 > y_1, x_1 + x_2 > y_1 + y_2$ and $x_1 + x_2 + x_3 = y_1 + y_2 + y_3$

$\rightarrow (x_1, x_2, x_3)$ majorizes (y_1, y_2, y_3) , then by Karamata's inequality, we get
 $\therefore f(x_1) + f(x_2) + f(x_3) < f(y_1) + f(y_2) + f(y_3)$

$\Leftrightarrow \ln(c+z) + \ln(b+y) + \ln(a+x) < \ln(c+x) + \ln(b+z) + \ln(a+y)$
 $\rightarrow (1) \text{ is true.}$

Therefore,

$$\frac{b + \ln \frac{c}{a}}{a + \ln \frac{c}{a}} < \frac{\left(b + \frac{b}{a} + \frac{c}{b} - 2\right)\left(c - \frac{a}{b} - \frac{b}{c} + 2\right)}{\left(c + \frac{b}{a} + \frac{c}{b} - 2\right)\left(a - \frac{a}{b} - \frac{b}{c} + 2\right)}.$$

Solution 2 by proposer

By MVT for $f(x) = \log x, \exists x_1 \in (a, b), x_2 \in (b, c)$

$$f'(x_1) = \frac{f(b) - f(a)}{b - a}, f'(x_2) = \frac{f(c) - f(b)}{c - b}$$

$a < x_1 < b; (f' \searrow) \Rightarrow f'(a) > f'(x_1) > f'(b) \Rightarrow$

$$(b - a)f'(a) > f(b) - f(a) > (b - a)f'(b)$$

$b < x_2 < c; (f' \searrow) \Rightarrow f'(b) > f'(x_2) > f'(c) \Rightarrow$

$$(c - b)f'(b) > f(c) - f(b) > (c - b)f'(c)$$

Hence,

$$(b - a)f'(b) + (c - b)f'(c) < f(c) - f(a) < (b - a)f'(a) + (c - b)f'(b)$$

$$\frac{b - a}{b} + \frac{c - b}{c} < \log \frac{c}{a} < \frac{b - a}{a} + \frac{c - b}{b} \Rightarrow -\frac{a}{b} - \frac{b}{c} + 2 < \log \frac{c}{a} < \frac{b}{a} + \frac{c}{b} - 2$$

Now,



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned} \frac{b + \log \frac{c}{a}}{a + \log \frac{c}{a}} &< \frac{\left(b + \frac{b}{a} + \frac{c}{b} - 2\right)\left(c - \frac{a}{b} - \frac{b}{c} + 2\right)}{\left(c + \frac{b}{a} + \frac{c}{b} - 2\right)\left(a - \frac{a}{b} - \frac{b}{c} + 2\right)} \Leftrightarrow \\ &\left(b + \log \frac{c}{a}\right)\left(c + \frac{b}{a} + \frac{c}{b} - 2\right)\left(a - \frac{a}{b} - \frac{b}{c} + 2\right) \leq \\ &\leq \left(a + \log \frac{c}{a}\right)\left(b + \frac{b}{a} + \frac{c}{b} - 2\right)\left(c - \frac{a}{b} - \frac{b}{c} + 2\right) \end{aligned}$$

True by rearrangement inequality for $a < b < c$, $\left(-\frac{a}{b} - \frac{b}{c} + 2 < \log \frac{c}{a} < \frac{b}{a} + \frac{c}{b} - 2\right)$

784. If $a, b, c, x, y > 0$ then:

$$\frac{a^{\sqrt{xy}} + b^{\sqrt{xy}} + c^{\sqrt{xy}}}{\frac{1}{\sqrt{a^{x+y}}} + \frac{1}{\sqrt{b^{x+y}}} + \frac{1}{\sqrt{c^{x+y}}}} \leq \frac{\sqrt{a^{x+y}} + \sqrt{b^{x+y}} + \sqrt{c^{x+y}}}{a^{\sqrt{xy}} + b^{\sqrt{xy}} + c^{\sqrt{xy}}}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

Let be the function: $f: (0, \infty) \rightarrow \mathbb{R}; f(x) = (a^x + b^x + c^x)(a^{-x} + b^{-x} + c^{-x})$

$$f(x) = 3 + \sum_{cyc} \left(\left(\frac{a}{b}\right)^x + \left(\frac{b}{a}\right)^x \right)$$

$$f'(x) = \sum_{cyc} \left(\left(\frac{a}{b}\right)^x \log \left(\frac{a}{b}\right) + \left(\frac{b}{a}\right)^x \log \left(\frac{b}{a}\right) \right)$$

$$f''(x) = \sum_{cyc} \left(\left(\frac{a}{b}\right)^x \log^2 \left(\frac{a}{b}\right) + \left(\frac{b}{a}\right)^x \log^2 \left(\frac{b}{a}\right) \right) > 0$$

$\Rightarrow f'$ -increasing $\Rightarrow \min f'(x) = \lim_{x \rightarrow 0^+} f'(x) = 0 \Rightarrow f'(x) > 0 \Rightarrow f$ -increasing

$$\sqrt{xy} \leq \frac{x+y}{2} \Rightarrow f(\sqrt{xy}) \leq f\left(\frac{x+y}{2}\right)$$

$$\begin{aligned} &\left(a^{\sqrt{xy}} + b^{\sqrt{xy}} + c^{\sqrt{xy}}\right) \left(\frac{1}{a^{\sqrt{xy}}} + \frac{1}{b^{\sqrt{xy}}} + \frac{1}{c^{\sqrt{xy}}}\right) \leq \\ &\leq (\sqrt{a^{x+y}} + \sqrt{b^{x+y}} + \sqrt{c^{x+y}}) \left(\frac{1}{\sqrt{a^{x+y}}} + \frac{1}{\sqrt{b^{x+y}}} + \frac{1}{\sqrt{c^{x+y}}}\right) \end{aligned}$$

Therefore,



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\frac{a^{\sqrt{xy}} + b^{\sqrt{xy}} + c^{\sqrt{xy}}}{\frac{1}{\sqrt{a^{x+y}}} + \frac{1}{\sqrt{b^{x+y}}} + \frac{1}{\sqrt{c^{x+y}}}} \leq \frac{\sqrt{a^{x+y}} + \sqrt{b^{x+y}} + \sqrt{c^{x+y}}}{\frac{1}{a^{\sqrt{xy}}} + \frac{1}{b^{\sqrt{xy}}} + \frac{1}{c^{\sqrt{xy}}}}$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\frac{a^{\sqrt{xy}}}{b^{\sqrt{xy}}} + \frac{b^{\sqrt{xy}}}{a^{\sqrt{xy}}} \leq \frac{a^{\frac{x+y}{2}}}{b^{\frac{x+y}{2}}} + \frac{b^{\frac{x+y}{2}}}{a^{\frac{x+y}{2}}}$$

$$e^{\sqrt{xy}} + \frac{1}{e^{\sqrt{xy}}} \leq e^{\frac{x+y}{2}} + \frac{1}{e^{\frac{x+y}{2}}}; \text{ where } e = \frac{a}{b}$$

$$\text{If } e \geq 1 \Rightarrow \frac{1}{e^{\sqrt{xy}}} - \frac{1}{e^{\frac{x+y}{2}}} \leq e^{\frac{x+y}{2}} - e^{\sqrt{xy}}$$

$$\frac{1}{e^{\sqrt{xy}}} \left(1 - \frac{1}{e^{\frac{x+y}{2}-\sqrt{xy}}} \right) \leq e^{\sqrt{xy}} \left(e^{\frac{x+y}{2}-\sqrt{xy}} - 1 \right)$$

$$1 - \frac{1}{e^{\frac{x+y}{2}-\sqrt{xy}}} \leq e^{2\sqrt{xy}} \left(e^{\frac{x+y}{2}-\sqrt{xy}} - 1 \right)$$

$$2 \leq \frac{1}{e^{\frac{x+y}{2}-\sqrt{xy}}} + e^{\frac{x+y}{2}-\sqrt{xy}}, e \geq 2 \text{ true.}$$

$$\text{If } e < 1, e^{\sqrt{xy}} - e^{\frac{x+y}{2}} \leq \frac{1}{e^{\frac{x+y}{2}}} - \frac{1}{e^{\sqrt{xy}}}$$

$$e^{\sqrt{xy}} \left(1 - e^{\frac{x+y}{2}-\sqrt{xy}} \right) \leq \frac{1}{e^{\sqrt{xy}}} \left(\frac{1}{e^{\frac{x+y}{2}-\sqrt{xy}}} - 1 \right)$$

$$1 - e^{\frac{x+y}{2}-\sqrt{xy}} \leq \frac{1}{e^{2\sqrt{xy}}} \left(\frac{1}{e^{\frac{x+y}{2}-\sqrt{xy}}} - 1 \right)$$

$$2 \leq e^{\frac{x+y}{2}-\sqrt{xy}} + \frac{1}{\frac{x+y}{2} - \sqrt{xy}}$$

$$\left(\frac{b}{c}\right)^{\sqrt{xy}} + \left(\frac{c}{b}\right)^{\sqrt{xy}} \leq \left(\frac{b}{c}\right)^{\frac{x+y}{2}} + \left(\frac{c}{b}\right)^{\frac{x+y}{2}} \text{ (and analogs)}$$

Hence,

$$\frac{a^{\sqrt{xy}} + b^{\sqrt{xy}} + c^{\sqrt{xy}}}{\frac{1}{\sqrt{a^{x+y}}} + \frac{1}{\sqrt{b^{x+y}}} + \frac{1}{\sqrt{c^{x+y}}}} \leq \frac{\sqrt{a^{x+y}} + \sqrt{b^{x+y}} + \sqrt{c^{x+y}}}{\frac{1}{a^{\sqrt{xy}}} + \frac{1}{b^{\sqrt{xy}}} + \frac{1}{c^{\sqrt{xy}}}}$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Solution 3 by Samar Das-India

$$\begin{aligned}
 \frac{a^{\sqrt{xy}} + b^{\sqrt{xy}} + c^{\sqrt{xy}}}{\frac{1}{\sqrt{a^{x+y}}} + \frac{1}{\sqrt{b^{x+y}}} + \frac{1}{\sqrt{c^{x+y}}}} &\leq \frac{\sqrt{a^{x+y}} + \sqrt{b^{x+y}} + \sqrt{c^{x+y}}}{\frac{1}{a^{\sqrt{xy}}} + \frac{1}{b^{\sqrt{xy}}} + \frac{1}{c^{\sqrt{xy}}}} \Leftrightarrow \\
 (a^{\sqrt{xy}} + b^{\sqrt{xy}} + c^{\sqrt{xy}}) \left(\frac{1}{a^{\sqrt{xy}}} + \frac{1}{b^{\sqrt{xy}}} + \frac{1}{c^{\sqrt{xy}}} \right) &\leq \\
 (\sqrt{a^{x+y}} + \sqrt{b^{x+y}} + \sqrt{c^{x+y}}) \left(\frac{1}{\sqrt{a^{x+y}}} + \frac{1}{\sqrt{b^{x+y}}} + \frac{1}{\sqrt{c^{x+y}}} \right); (1)
 \end{aligned}$$

$$3 + \sum_{cyc} \left(\frac{a}{b}\right)^{\sqrt{xy}} - 3 - \sum_{cyc} \left(\frac{a}{b}\right)^{\frac{x+y}{2}} \stackrel{(?)}{<} 0 \Leftrightarrow \sum_{cyc} \left(\frac{a}{b}\right)^{\sqrt{xy}} - \sum_{cyc} \left(\frac{a}{b}\right)^{\frac{x+y}{2}} \stackrel{(?)}{<} 0$$

Since $\sqrt{xy} \leq \frac{x+y}{2}$, when $a > b \Rightarrow \frac{a}{b} > 1 \Rightarrow \left(\frac{a}{b}\right)^{\sqrt{xy}} \leq \left(\frac{a}{b}\right)^{\frac{x+y}{2}}$

$$\Rightarrow \sum_{cyc} \left(\frac{a}{b}\right)^{\sqrt{xy}} - \sum_{cyc} \left(\frac{a}{b}\right)^{\frac{x+y}{2}} \leq 0$$

Equality holds for $a = b$. Thus, (1) is true.

Solution 4 by Samar Das-India

$$\begin{aligned}
 \frac{a^{\sqrt{xy}} + b^{\sqrt{xy}} + c^{\sqrt{xy}}}{\frac{1}{\sqrt{a^{x+y}}} + \frac{1}{\sqrt{b^{x+y}}} + \frac{1}{\sqrt{c^{x+y}}}} &\leq \frac{\sqrt{a^{x+y}} + \sqrt{b^{x+y}} + \sqrt{c^{x+y}}}{\frac{1}{a^{\sqrt{xy}}} + \frac{1}{b^{\sqrt{xy}}} + \frac{1}{c^{\sqrt{xy}}}} \Leftrightarrow \\
 (a^{\sqrt{xy}} + b^{\sqrt{xy}} + c^{\sqrt{xy}}) \left(\frac{1}{a^{\sqrt{xy}}} + \frac{1}{b^{\sqrt{xy}}} + \frac{1}{c^{\sqrt{xy}}} \right) &\leq \\
 (\sqrt{a^{x+y}} + \sqrt{b^{x+y}} + \sqrt{c^{x+y}}) \left(\frac{1}{\sqrt{a^{x+y}}} + \frac{1}{\sqrt{b^{x+y}}} + \frac{1}{\sqrt{c^{x+y}}} \right); (*)
 \end{aligned}$$

Let: $\frac{a}{b} = m$.

$$\begin{aligned}
 \text{If } m > 1 \Rightarrow A = m^{\sqrt{xy}} + \frac{1}{m^{\sqrt{xy}}} - m^{\frac{x+y}{2}} - \frac{1}{m^{\frac{x+y}{2}}} &= \\
 = \left(m^{\sqrt{xy}} - m^{\frac{x+y}{2}} \right) - \frac{m^{\sqrt{xy}} - m^{\frac{x+y}{2}}}{m^{\sqrt{xy} + \frac{x+y}{2}}}; (1)
 \end{aligned}$$

$$m > 1 \Rightarrow m^{\sqrt{xy}} - m^{\frac{x+y}{2}} < 0; (2) \text{ and } m^{\sqrt{xy} + \frac{x+y}{2}} > 1$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\frac{1}{m^{\sqrt{xy}+\frac{x+y}{2}}} < 1 \Rightarrow \left(1 - \frac{1}{m^{\sqrt{xy}+\frac{x+y}{2}}}\right) > 0; (3)$$

From (1),(2) and (3), we get $A < 0$; (4)

$$\text{If } m < 1 \Rightarrow 1 - \frac{1}{m^{\sqrt{xy}+\frac{x+y}{2}}} < 0; (5)$$

From (5) and (6), we get: $A < 0$; (6)

Therefore,

$$\sum_{cyc} \left(\left(\frac{a}{b}\right)^{\sqrt{xy}} + \left(\frac{b}{a}\right)^{\sqrt{xy}} \right) - \sum_{cyc} \left(\left(\frac{a}{b}\right)^{\frac{x+y}{2}} + \left(\frac{b}{a}\right)^{\frac{x+y}{2}} \right) \leq 0 \Rightarrow (*) \text{ true.}$$

Equality holds for $a = b = c$.

Solution 5 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \frac{a^{\sqrt{xy}} + b^{\sqrt{xy}} + c^{\sqrt{xy}}}{\sqrt{a^{x+y}} + \sqrt{b^{x+y}} + \sqrt{c^{x+y}}} &\leq \frac{\frac{1}{\sqrt{a^{x+y}}} + \frac{1}{\sqrt{b^{x+y}}} + \frac{1}{\sqrt{c^{x+y}}}}{\frac{1}{a^{\sqrt{xy}}} + \frac{1}{b^{\sqrt{xy}}} + \frac{1}{c^{\sqrt{xy}}}} \\ &\Leftrightarrow (a^{\sqrt{xy}} + b^{\sqrt{xy}} + c^{\sqrt{xy}}) \left(\frac{1}{a^{\sqrt{xy}}} + \frac{1}{b^{\sqrt{xy}}} + \frac{1}{c^{\sqrt{xy}}} \right) \\ &\leq (\sqrt{a^{x+y}} + \sqrt{b^{x+y}} + \sqrt{c^{x+y}}) \left(\frac{1}{\sqrt{a^{x+y}}} + \frac{1}{\sqrt{b^{x+y}}} + \frac{1}{\sqrt{c^{x+y}}} \right) \\ &\Leftrightarrow 3 + \sum_{cyc} \left(\left(\frac{a}{b}\right)^{\sqrt{xy}} + \left(\frac{b}{a}\right)^{\sqrt{xy}} \right) \leq 3 + \sum_{cyc} \left(\left(\frac{a}{b}\right)^{\frac{x+y}{2}} + \left(\frac{b}{a}\right)^{\frac{x+y}{2}} \right) \\ &\Leftrightarrow \sum_{cyc} \left(\left(\frac{a}{b}\right)^{\sqrt{xy}} + \left(\frac{b}{a}\right)^{\sqrt{xy}} \right) \stackrel{(l)}{\geq} \sum_{cyc} \left(\left(\frac{a}{b}\right)^{\frac{x+y}{2}} + \left(\frac{b}{a}\right)^{\frac{x+y}{2}} \right) \end{aligned}$$

Case 1 $a \geq b$ and $\left(\frac{a}{b}\right)^{\frac{x+y}{2}} + \left(\frac{b}{a}\right)^{\frac{x+y}{2}} \stackrel{?}{\geq} \left(\frac{a}{b}\right)^{\sqrt{xy}} + \left(\frac{b}{a}\right)^{\sqrt{xy}}$

$$\Leftrightarrow \left(t^{\frac{x+y}{2}} + \frac{1}{t^{\frac{x+y}{2}}}\right) - \left(t^{\sqrt{xy}} + \frac{1}{t^{\sqrt{xy}}}\right) \stackrel{?}{\geq} 0 \quad (t = \frac{a}{b})$$

Let $f(\theta) = t^\theta + \frac{1}{t^\theta}$ $\forall \theta \in [\sqrt{xy}, \frac{x+y}{2}]$ and let $t^\theta = P$ and $\frac{1}{t^\theta} = Q$

$$\text{Now, } t^\theta = P \Rightarrow \theta \cdot \ln P = \ln P \Rightarrow \frac{d}{d\theta}(\theta \cdot \ln P) = \left(\frac{d}{dP} \ln P\right) \cdot \frac{dP}{d\theta} \Rightarrow \ln P = \frac{1}{P} \left(\frac{d}{d\theta} t^\theta\right) \Rightarrow \frac{d}{d\theta} t^\theta \stackrel{(i)}{\cong} t^\theta \cdot \ln P$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\text{Also, } t^{-\theta} = Q \Rightarrow -\theta \cdot \ln t = \ln Q \Rightarrow \frac{d}{d\theta}(-\theta \cdot \ln t) = \left(\frac{d}{dQ} \ln Q\right) \cdot \frac{dQ}{d\theta} \Rightarrow -\ln t = \frac{1}{Q} \left(\frac{d}{d\theta} t^{-\theta}\right)$$

$$\Rightarrow \frac{d}{d\theta} t^{-\theta} \stackrel{\text{(ii)}}{\cong} -t^{-\theta} \cdot \ln t \therefore (\text{i}) + (\text{ii}) \Rightarrow f'(\theta) \stackrel{(\star)}{\cong} (\ln t) \left(t^\theta - \frac{1}{t^\theta}\right)$$

$$\text{Again, } \left(t^{\frac{x+y}{2}} + \frac{1}{t^{\frac{x+y}{2}}}\right) - \left(t^{\sqrt{xy}} + \frac{1}{t^{\sqrt{xy}}}\right) \stackrel{\text{via MVT}}{\cong} \left(\frac{x+y}{2} - \sqrt{xy}\right) f'(\theta) \Big|_{\theta=\xi} \text{ for some } \xi$$

$$\in \left(\sqrt{xy}, \frac{x+y}{2}\right) \stackrel{\text{via } (\star)}{\cong} \left(\frac{x+y}{2} - \sqrt{xy}\right) (\ln t) \left(t^\xi - \frac{1}{t^\xi}\right)$$

$$\therefore \left(t^{\frac{x+y}{2}} + \frac{1}{t^{\frac{x+y}{2}}}\right) - \left(t^{\sqrt{xy}} + \frac{1}{t^{\sqrt{xy}}}\right) \stackrel{(\star)}{\cong} \frac{\left(\frac{x+y}{2} - \sqrt{xy}\right) (\ln t) (t^\xi + 1)(t^\xi - 1)}{t^\xi} \text{ for some } \xi$$

$$\in \left(\sqrt{xy}, \frac{x+y}{2}\right)$$

$$\because \xi > \sqrt{xy} > 0 \text{ and } \ln t = \ln \frac{a}{b} \stackrel{a/b \geq 1}{\cong} \ln 1 \stackrel{(1)}{\cong} 0 \therefore \xi \cdot \ln t \geq 0 \Rightarrow \ln(t^\xi) \geq 0 \Rightarrow t^\xi \geq 1$$

$$\Rightarrow t^\xi - 1 \stackrel{(2)}{\cong} 0 \text{ and } \therefore \frac{x+y}{2} - \sqrt{xy} = \frac{(\sqrt{x} - \sqrt{y})^2}{2} \stackrel{(3)}{\cong} 0 \therefore (1), (2), (3) \Rightarrow$$

$$\frac{\left(\frac{x+y}{2} - \sqrt{xy}\right) (\ln t) (t^\xi + 1)(t^\xi - 1)}{t^\xi} \geq 0 \stackrel{\text{via } (\star)}{\cong} \left(t^{\frac{x+y}{2}} + \frac{1}{t^{\frac{x+y}{2}}}\right) - \left(t^{\sqrt{xy}} + \frac{1}{t^{\sqrt{xy}}}\right) \geq 0$$

$$\Rightarrow (*) \text{ is true} \Rightarrow \left(\frac{a}{b}\right)^{\frac{x+y}{2}} + \left(\frac{b}{a}\right)^{\frac{x+y}{2}} \geq \left(\frac{a}{b}\right)^{\sqrt{xy}} + \left(\frac{b}{a}\right)^{\sqrt{xy}}$$

$$\boxed{\text{Case 2}} \quad a < b \text{ and } \left(\frac{a}{b}\right)^{\frac{x+y}{2}} + \left(\frac{b}{a}\right)^{\frac{x+y}{2}} \stackrel{?}{\cong} \left(\frac{a}{b}\right)^{\sqrt{xy}} + \left(\frac{b}{a}\right)^{\sqrt{xy}}$$

$$\Leftrightarrow \left(t^{\frac{x+y}{2}} + \frac{1}{t^{\frac{x+y}{2}}}\right) - \left(t^{\sqrt{xy}} + \frac{1}{t^{\sqrt{xy}}}\right) \stackrel{?}{\cong} 0 \quad (t = \frac{a}{b})$$

$$\Leftrightarrow \frac{t^{\sqrt{xy}}(t^{x+y} + 1) - t^{\frac{x+y}{2}}(t^{xy} + 1)}{t^{\sqrt{xy}} \cdot t^{\frac{x+y}{2}}} \stackrel{?}{\cong} 0 \Leftrightarrow \beta(\alpha^2 + 1) - \alpha(\beta^2 + 1) \stackrel{?}{\cong} 0$$

$$\left(\alpha = t^{\frac{x+y}{2}}, \beta = t^{\sqrt{xy}}\right) \Leftrightarrow \alpha\beta(\alpha - \beta) - (\alpha - \beta) \stackrel{?}{\cong} 0 \Leftrightarrow (\alpha - \beta)(\alpha\beta - 1) \stackrel{?}{\cong} 0 \quad (**)$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Now, $\alpha \leq \beta \Leftrightarrow t^{\frac{x+y}{2}} \leq t^{\sqrt{xy}} \Leftrightarrow \frac{x+y}{2} \cdot \ln t \leq \sqrt{xy} \cdot \ln t \Leftrightarrow \frac{(\sqrt{x}-\sqrt{y})^2}{2} \cdot \ln t \leq 0 \rightarrow \text{true} : \ln t$

$$= \ln \frac{a}{b} < \ln 1 \Rightarrow \ln t < 0 \text{ and } \frac{(\sqrt{x}-\sqrt{y})^2}{2} \stackrel{(m)}{\geq} 0 \therefore \alpha - \beta \stackrel{(m)}{\geq} 0$$

Again, $\alpha\beta < 1 \Leftrightarrow t^{\frac{x+y}{2}} \cdot t^{\sqrt{xy}} < 1 \Leftrightarrow t^{\frac{x+y}{2} + \sqrt{xy}} < 1 \Leftrightarrow \left(\frac{x+y}{2} + \sqrt{xy}\right) \cdot \ln t < 0 \Leftrightarrow \ln t < 0$

$$\rightarrow \text{true} : \ln t = \ln \frac{a}{b} < \ln 1 = 0 \therefore \alpha\beta - 1 \stackrel{(n)}{\geq} 0 \therefore (m), (n) \Rightarrow (\alpha - \beta)(\alpha\beta - 1) \geq 0$$

$$\Rightarrow (** \text{ is true}) \Rightarrow \left[\left(\frac{a}{b} \right)^{\frac{x+y}{2}} + \left(\frac{b}{a} \right)^{\frac{x+y}{2}} \geq \left(\frac{a}{b} \right)^{\sqrt{xy}} + \left(\frac{b}{a} \right)^{\sqrt{xy}} \right]$$

$\therefore \text{combining cases 1 and 2}, \left(\frac{a}{b} \right)^{\frac{x+y}{2}} + \left(\frac{b}{a} \right)^{\frac{x+y}{2}} \geq \left(\frac{a}{b} \right)^{\sqrt{xy}} + \left(\frac{b}{a} \right)^{\sqrt{xy}} \forall a, b, x, y > 0$

$> 0 \text{ and analogs}$

$$\stackrel{\text{summing up}}{\Rightarrow} \sum_{\text{cyc}} \left(\left(\frac{a}{b} \right)^{\frac{x+y}{2}} + \left(\frac{b}{a} \right)^{\frac{x+y}{2}} \right) \geq \sum_{\text{cyc}} \left(\left(\frac{a}{b} \right)^{\sqrt{xy}} + \left(\frac{b}{a} \right)^{\sqrt{xy}} \right) \Rightarrow (l) \text{ is true}$$

$$\therefore \frac{a^{\sqrt{xy}} + b^{\sqrt{xy}} + c^{\sqrt{xy}}}{\sqrt{a^{x+y}} + \sqrt{b^{x+y}} + \sqrt{c^{x+y}}} \leq \frac{\frac{1}{\sqrt{a^{x+y}}} + \frac{1}{\sqrt{b^{x+y}}} + \frac{1}{\sqrt{c^{x+y}}}}{\frac{1}{a^{\sqrt{xy}}} + \frac{1}{b^{\sqrt{xy}}} + \frac{1}{c^{\sqrt{xy}}}} \quad \forall a, b, c, x, y > 0 \quad (QED)$$

785. If $x > 0, r = pq, 1 \leq p \leq q$ then prove:

$$1 + rx \leq (1 + qx)^p \leq (1 + px)^q \leq (1 + x)^r$$

Proposed by Seyran Ibrahimov-Maasilli-Azerbaijan

Solution 1 by Ravi Prakash-New Delhi-India

For $x > 0, r = pq, 1 \leq p \leq q$ let $f(x) = (1 + qx)^p - (1 - pqx); x > 0$

$$f'(x) = pq(1 + qx)^{p-1} - pq$$

As $p - 1 \geq 0, (1 + qx)^{p-1} \geq 1$ then $f'(x) \geq 0 \Rightarrow f$ – is a non-decreasing function on

$$[0, \infty) \Rightarrow (1 + qx)^p - (1 + rx) \geq 0 \Rightarrow 1 + rx \geq (1 + qx)^p; (1)$$

$$\text{Let } g(x) = (1 + x)^{\frac{1}{x}}; x > 0 \Rightarrow \log g(x) = \frac{1}{x} \log(1 + x)$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\frac{g'(x)}{g(x)} = \frac{1}{x(1+x)} - \frac{1}{x^2} \log(1+x) = \frac{x - (x+1)\log(1+x)}{x^2(1+x)}; x > 0$$

For $x \geq 0$, let $h(x) = x - (1+x)\log(1+x)$, then $h'(x) = -\log(1+x) < 0; \forall x > 0$
 $h(x) < h(0) = 0; \forall x > 0 \Rightarrow g'(x) < 0; \forall x > 0 \Rightarrow g$ is strictly decreasing on $(0, \infty)$.

As $1 \leq p \leq q, px \leq qx; x > 0 \Rightarrow (1+px)^{\frac{1}{px}} \geq (1+qx)^{\frac{1}{qx}}$

$$(1+px)^q \geq (1+qx)^p; (2)$$

Also, $1+px \leq (1+x)^p \Rightarrow (1+px)^q \leq (1+x)^{pq}$ or $(1+px)^q \leq (1+x)^r; (3)$

From (1),(2) and (3) it follows the proposed inequality.

Solution 2 by Hikmat Mammadov-Azerbaijan

By Bernoulli's inequality: $(1+x)^r = ((1+x)^p)^q \geq (1+px)^q$

$$(1+qx)^p \geq 1 + pqx = 1 + rx$$

$$f(s) = \frac{\log(1+s)}{s} \Rightarrow px \leq qx \Rightarrow \frac{\log(1+px)}{p} \leq \frac{\log(1+qx)}{q} \Rightarrow$$

$$(1+px)^q \geq (1+px)^p$$

Hence,

$$1 + rx \leq (1+qx)^p \leq (1+px)^q \leq (1+x)^r$$

786. Let a, b be real numbers such that $a^2 + b^2 = 1$.

Prove that for each nonnegative integer n ,

$$(a+b)^{2^{n+1}} + (a-b)^{2^{n+1}} + n \geq (a^2 - b^2)^{2^n} + (a^2 - b^2)^{2^{n-1}} + \dots + (a^2 - b^2)^2 + 2$$

When equality holds?

Proposed by Kunihiko Chikaya-Tokyo-Japan

Solution 1 by Ravi Prakash-New Delhi-India

Let $a = \cos \theta, b = \sin \theta, 0 \leq \theta \leq 2\pi$, then:

$$(a+b)^{2^{n+1}} + (a-b)^{2^{n+1}} + n = (\cos \theta + \sin \theta)^{2^{n+1}} + (\cos \theta - \sin \theta)^{2^{n+1}} + n =$$

$$= \left(\sqrt{2} \sin\left(\theta + \frac{\pi}{4}\right)\right)^{2^{n+1}} + \left(\sqrt{2} \cos\left(\theta + \frac{\pi}{4}\right)\right)^{2^{n+1}} + n =$$

$$= \left(2 \sin^2\left(\theta + \frac{\pi}{4}\right)\right)^{2^n} + \left(2 \cos^2\left(\theta + \frac{\pi}{4}\right)\right)^{2^n} + n \geq$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\geq \frac{2}{2^{2n}} \left[2 \sin^2 \left(\theta + \frac{\pi}{4} \right) + 2 \cos^2 \left(\theta + \frac{\pi}{4} \right) \right]^{2^n} + n = 2 + n$$

$$\text{Also, } a^2 - b^2 = \cos 2\theta$$

$$(a^2 - b^2)^{2^n} + (a^2 - b^2)^{2^{n-1}} + \cdots + (a^2 - b^2)^2 + 2 =$$

$$= (\cos 2\theta)^{2^n} + (\cos 2\theta)^{2^{n-1}} + \cdots + (\cos 2\theta)^2 + n \leq \underbrace{1 + 1 + \cdots + 1}_{n\text{-times}} + 2 = n + 2$$

Thus, $LHS \geq n + 2 \geq RHS$. Equality holds when $\cos 2\theta = \pm 1 \Leftrightarrow$

$$\theta \in \left\{ 0; \frac{\pi}{2}; \pi; \frac{3\pi}{2} \right\} \Leftrightarrow (a, b) \in \{(1, 0); (0, 1); (-1, 0); (0, -1)\}.$$

Solution 2 by Michael Sterghiou-Greece

$$(a+b)^{2^{n+1}} + (a-b)^{2^{n+1}} + n \geq (a^2 - b^2)^{2^n} + (a^2 - b^2)^{2^{n-1}} + \cdots + (a^2 - b^2)^2 + 2; \quad (1)$$

Let $x = (a+b)^2 > 0; y = (a-b)^2 > 0$, then $x + y = 2$ and (1) it can be written as

$$x^{2^n} + y^{2^n} + n \geq (xy)^{2^{n-1}} + (xy)^{2^{n-2}} + \cdots + (xy)^{2^0} + 2; \quad (2)$$

By AM-GM: $\sqrt{xy} \leq \frac{x+y}{2} = 1 \Rightarrow xy \leq 1$ and all n powers of xy in RHS of (2) are ≤ 1 so,

$$RHS_{(2)} \leq n + 2.$$

It suffices to prove that $LHS_{(2)} \leq n + 2$

$$LHS_{(3)} = x^{2^n} + y^{2^n} + n \geq n + 2 \Leftrightarrow x^{2^n} + y^{2^n} \geq 2 \Leftrightarrow x^{2^n} + (2-x)^{2^n} \geq 2.$$

Let $2^n = p \in \mathbb{N}; p \geq 1$, so, $-2 + x^p + (2-x)^p = f(x)$

$$f'(x) = [x^{p-1} - (2-x)^{p-1}]p$$

WLOG, let $x \geq 1$ then $\frac{f'(x)}{p} = [x - (2-x)]\theta = 2(x-1)\theta \geq 0; (\theta \geq 0)$.

So, $f \nearrow$ and $f(x) \geq f(1) = 1^p + (2-1)^p - 2 = 0$, hence $f(x) \geq 0$ and

$$LHS_{(2)} \geq n + 2 \geq RHS_{(2)}$$

Equality holds for

$$(a, b) \in \{(1, 0); (0, 1); (-1, 0); (0, -1)\}$$

787. If $m, n, p, q \in \mathbb{N}; m, n, p, q \geq 2; mq \geq np$ then:

$$n(p-q)(\sqrt[n]{2^m} - 1) \geq q(m-n)(\sqrt[q]{2^p} - 1)$$

Proposed by Daniel Sitaru-Romania



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Solution by Khaled Abd Imouti-Damascus-Syria

$$n(p-q)(\sqrt[n]{2^m} - 1) \geq q(m-n)(\sqrt[q]{2^p} - 1)$$

$$n(p-q)\left(2^{\frac{m}{n}} - 1\right) \geq q(m-n)\left(2^{\frac{p}{q}} - 1\right)$$

$$nq\left(\frac{p}{q}-1\right)\left(2^{\frac{m}{n}} - 1\right) \geq qn\left(\frac{m}{n}-1\right)\left(2^{\frac{p}{q}} - 1\right)$$

$$\left(\frac{p}{q}-1\right)\left(2^{\frac{m}{n}} - 1\right) \geq \left(\frac{m}{n}-1\right)\left(2^{\frac{p}{q}} - 1\right)$$

$$\text{Let } \frac{m}{n} = x, \frac{p}{q} = y \text{ and } x > y \Rightarrow (y-1)(2^x - 1) \geq (x-1)(2^y - 1)$$

$$y \cdot 2^x - x \cdot 2^y \geq y - x \Leftrightarrow \frac{2^x}{x} - \frac{2^y}{y} \geq \frac{1}{x} - \frac{1}{y} \Leftrightarrow \frac{2^x - 1}{x} \geq \frac{2^y - 1}{y}; (*)$$

Let be the function $f: (0, \infty) \rightarrow \mathbb{R}$, $f(x) = \frac{2^x - 1}{x}$

$$\lim_{x \rightarrow 0} f(x) = 1; \lim_{x \rightarrow \infty} f(x) = +\infty$$

$$f'(x) = \frac{x \cdot 2^x \log 2 - 2^x + 1}{x^2}$$

$$f''(x) = \frac{(\log^2 2 \cdot x^2 - 2x \log 2 + 2)2^x + 2}{x^4} > 0, \Delta < 0$$

$$f'(0) = \lim_{x \rightarrow 0} f'(x) = \frac{\log^2 2}{2} > 0$$

So, f –increasing function and $x \geq y \Rightarrow f(x) \geq f(y) \Rightarrow (*)$ is true.

Therefore,

$$n(p-q)(\sqrt[n]{2^m} - 1) \geq q(m-n)(\sqrt[q]{2^p} - 1)$$

788. If f defined for reals x and y :

$$f(x, y) = \frac{xy}{x^2 + xy + y^2} + \frac{x-y}{x+y}, \text{then:}$$

$$1) f\left(\sqrt{xy}, \frac{x+y}{2}\right) \leq \frac{1}{3}. \quad 2) -1 < f\left(\frac{x+y}{x-y}, \frac{x-y}{x+y}\right) < 1.$$

Proposed by Srinivasa Raghava-AIRMC-India



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$1. \text{ Let } a = \sqrt{xy}, b = \frac{x+y}{2}.$$

$$\text{We have : } \frac{ab}{a^2 + ab + b^2} \stackrel{?}{\leq} \frac{1}{3} \stackrel{a^2 + ab + b^2 > 0}{\Leftrightarrow} 3ab \stackrel{?}{\leq} a^2 + ab + b^2 \\ \leftrightarrow 0 \stackrel{?}{\leq} (a - b)^2 \text{ which is true.}$$

$$\rightarrow \frac{ab}{a^2 + ab + b^2} \leq \frac{1}{3}. \text{ Equality holds iff } a = b \text{ or } x = y > 0.$$

Since $xy \geq 0 \rightarrow x$ and y have the same sign.

$$\text{If } x, y \geq 0 \rightarrow b \stackrel{AM-GM}{\geq} a \geq 0 \rightarrow \frac{a-b}{a+b} \leq 0. \text{ If } x, y \leq 0 \rightarrow a \geq 0 \geq b \text{ and } a+b \\ = -\frac{(\sqrt{-x} - \sqrt{-y})^2}{2} \leq 0 \rightarrow \frac{a-b}{a+b} \leq 0$$

$$\text{Therefore, } f\left(\sqrt{xy}, \frac{x+y}{2}\right) = f(a, b) = \frac{ab}{a^2 + ab + b^2} + \frac{a-b}{a+b} \\ \leq \frac{1}{3}. \text{ Equality holds iff } x = y > 0.$$

$$2. \text{ Let } z = \frac{x+y}{x-y} \neq 0. \text{ We have : } f\left(\frac{x+y}{x-y}, \frac{x-y}{x+y}\right) = f\left(z, \frac{1}{z}\right) = \frac{1}{z^2 + 1 + \frac{1}{z^2}} + \frac{z - \frac{1}{z}}{z + \frac{1}{z}} \\ = \frac{z^2}{z^4 + z^2 + 1} + \frac{z^2 - 1}{z^2 + 1} = \frac{z^6 + z^4 + z^2 - 1}{z^6 + 2z^4 + 2z^2 + 1} \stackrel{?}{\leq} 1$$

$$\leftrightarrow z^4 + z^2 + 2 \stackrel{?}{>} 0 \text{ which is true for all real } z \rightarrow f\left(\frac{x+y}{x-y}, \frac{x-y}{x+y}\right) < 1.$$

$$\text{Also, } f\left(\frac{x+y}{x-y}, \frac{x-y}{x+y}\right) = f\left(z, \frac{1}{z}\right) = \frac{z^6 + z^4 + z^2 - 1}{z^6 + 2z^4 + 2z^2 + 1} \stackrel{?}{>} -1$$

$$\leftrightarrow 2z^6 + 3z^4 + 3z^2 \stackrel{?}{>} 0, \text{ which is true for all real } z \neq 0$$

$$\text{Therefore, } -1 < f\left(\frac{x+y}{x-y}, \frac{x-y}{x+y}\right) < 1.$$

Solution 2 by Kamel Gandouli Rezgui-Tunisia

$$f(x, y) = \frac{xy}{x^2 + xy + y^2} + \frac{x-y}{x+y} \text{ if } y \geq x \text{ and } x \geq 0, y \geq 0.$$

$$f(x, y) \leq \frac{xy}{x^2 + xy + y^2} \text{ because } \frac{x-y}{x+y} \leq 0$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\frac{xy}{x^2 + xy + y^2} \leq \frac{1}{3} \text{ because by AGM: } x^2 + y^2 + xy \geq 3xy$$

$$\text{if } x \geq 0 \Rightarrow y \leq 0 \text{ and } -y \geq x \Rightarrow \frac{x-y}{x+y} \leq 0 \text{ and } \frac{xy}{x^2 + xy + y^2} \leq 0 \Rightarrow f(x,y) \leq 0 \leq \frac{1}{3}$$

$$\text{if } x = \frac{1}{y} \Rightarrow f(x,y) = \frac{1}{\left(\frac{1}{y}\right)^2 + y^2 + 1} + \frac{\frac{1}{y} - y}{\frac{1}{y} + y} =$$

$$= \frac{y^2}{1 + y^2 + y^4} + \frac{1 - y^2}{1 + y^2} = \frac{y^2(1 + y^2)}{(1 + y^2 y^4)(1 + y^2)} + \frac{(1 - y^2)(1 + y^2 + y^4)}{(1 + y^2)(1 + y^2 + y^4)} =$$

$$= \frac{1 + y^2 + y^4 - y^6}{(1 + y^2)(1 + y^2 + y^4)} = \frac{1}{1 + y^2} - \frac{y^6}{(1 + y^2)(1 + y^2 + y^4)}$$

$$-1 \leq \frac{y^6}{(1 + y^2)(1 + y^2 + y^4)} \leq \frac{1}{1 + y^2} - \frac{y^6}{(1 + y^2)(1 + y^2 + y^4)} \leq \frac{1}{1 + y^2} \leq 1$$

$$\Rightarrow -1 \leq f\left(x, \frac{1}{x}\right) \leq 1$$

$$\sqrt{xy} \leq \frac{x+y}{2}; \text{ if } x, y \geq 0 \text{ and if } x \leq 0, y \leq 0: -\frac{x+y}{2} \geq \sqrt{xy} \text{ because:}$$

$$-\frac{x+y}{2} = \frac{(-x) + (-y)}{2} = \frac{|x| + |y|}{2} \geq \sqrt{|x||y|} \Rightarrow$$

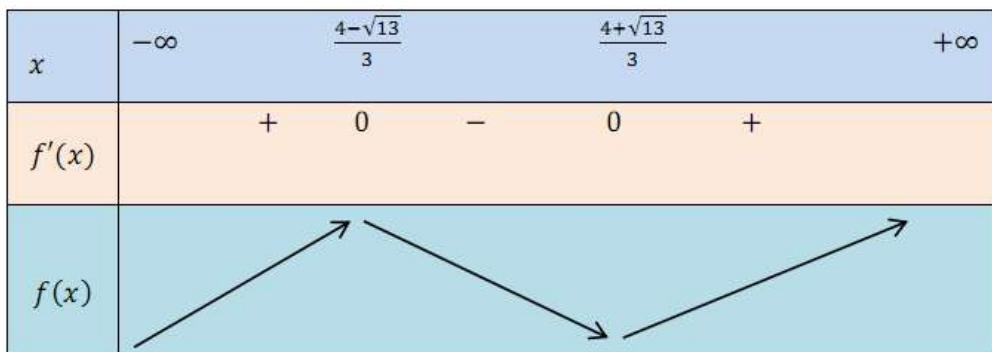
$$f\left(\sqrt{xy}, \frac{x+y}{2}\right) \leq \frac{1}{3}; \forall x, y \geq 0$$

Therefore,

$$-1 \leq f\left(\frac{x+y}{x-y}, \frac{x-y}{x+y}\right) \leq 1$$

789. Let $f(x) = ax^3 + bx^2 + cx + d$ ($a, b, c, d \in \mathbb{R}$) and

$$f(1) = 0, f(-1) = -4$$



Find all $\alpha > 0$ such that $\left|\frac{f'(x)}{f''(x)}\right| \leq \alpha, \forall x \in [-1, 1]$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Solution by proposer

We have:

$$\begin{cases} f(1) = 0 \\ f(-1) = -4 \\ f'\left(\frac{4-\sqrt{13}}{3}\right) = 0 \\ f'\left(\frac{4+\sqrt{13}}{3}\right) = 0 \end{cases} \Leftrightarrow \begin{cases} a + b + c + d = 0 \\ -a + b - c + d = -4 \\ 3a\left(\frac{4-\sqrt{13}}{3}\right)^2 + 2b\left(\frac{4-\sqrt{13}}{3}\right) + c = 0 \\ 3a\left(\frac{4+\sqrt{13}}{3}\right)^2 + 2b\left(\frac{4+\sqrt{13}}{3}\right) + c = 0 \end{cases} \Leftrightarrow \begin{cases} a = 1 \\ b = -4 \\ c = 1 \\ d = 2 \end{cases}$$

Therefore,

$$f(x) = x^3 - 4x^2 + x + 2$$

$$\rightarrow f'(x) = 3x^2 - 8x + 1, f''(x) = 6x - 8$$

Let us denote: $g(x) = \frac{f'(x)}{f''(x)} = \frac{3x^2 - 8x + 1}{6x - 8}, \forall x \in [-1, 1]$

$$\rightarrow g'(x) = \frac{9x^2 - 24x + 29}{2(3x - 4)^2} = \frac{(3x - 4)^2 + 13}{2(3x - 4)^2} > 0, \quad \forall x \in [-1, 1]$$

$$\rightarrow g(x) \uparrow \text{on } [-1, 1] \rightarrow g(-1) \leq g(x) \leq g(1) \rightarrow -\frac{6}{7} \leq g(x) \leq 2$$

$$\rightarrow |g(x)| \leq 2, \quad \forall x \in [-1, 1]$$

$$\rightarrow \alpha \geq 2$$

Hence: $\alpha \geq 2$ we have: $\left| \frac{f'(x)}{f''(x)} \right| \leq \alpha, \forall x \in [-1, 1]$

$$790. \text{ } a, b > 0, k \in \mathbb{N}, k \geq 1, Q = \sqrt{\frac{a^2 + b^2}{2}}, A = \frac{a+b}{2}, G = \sqrt{ab}, H = \frac{2ab}{a+b}$$

Prove that:

$$Q^{4k-2} \cdot H^k \geq A^k \cdot G^{4k-2}$$

Proposed by Seyran Ibrahimov-Maasilli-Azerbaijan

Solution by Ravi Prakash-New Delhi-India

$$\frac{Q}{G} = \sqrt{\frac{a^2 + b^2}{2ab}} \Rightarrow \left(\frac{Q}{G}\right)^{4k-2} = \left(\frac{a^2 + b^2}{2ab}\right)^{2k-1}$$

$$\frac{A}{H} = \frac{a+b}{2} \cdot \frac{a+b}{2ab} = \frac{(a+b)^2}{4ab}$$

$$\left(\frac{A}{H}\right)^k = \frac{(a+b)^{2k}}{(4ab)^k}$$

We must to prove that:

$$\left(\frac{a^2 + b^2}{2ab}\right)^{2k-1} \geq \frac{(a+b)^{2k}}{(4ab)^k} \Leftrightarrow \frac{(a^2 + b^2)^{2k-1}}{2^{2k-1}(ab)^{2k-1}} \geq \frac{(a+b)^{2k}}{2^{2k}(ab)^k}$$

$$2(a^2 + b^2)^{2k-1} \geq (ab)^{k-1}(a+b)^{2k}; \quad (1)$$

Put: $a = r \cos \theta, b = r \sin \theta, r > 0, 0 < \theta < \frac{\pi}{2}$. Now, (1) becomes:

$$2r^{4k-2} \geq r^{2k-2}(\sin \theta \cos \theta)^{k-1} \cdot r^{2k}(\sin \theta + \cos \theta)^{2k}$$

$$(\sin \theta + \cos \theta)^{2k}(\sin \theta \cos \theta)^{k-1} \leq 2 \Leftrightarrow$$

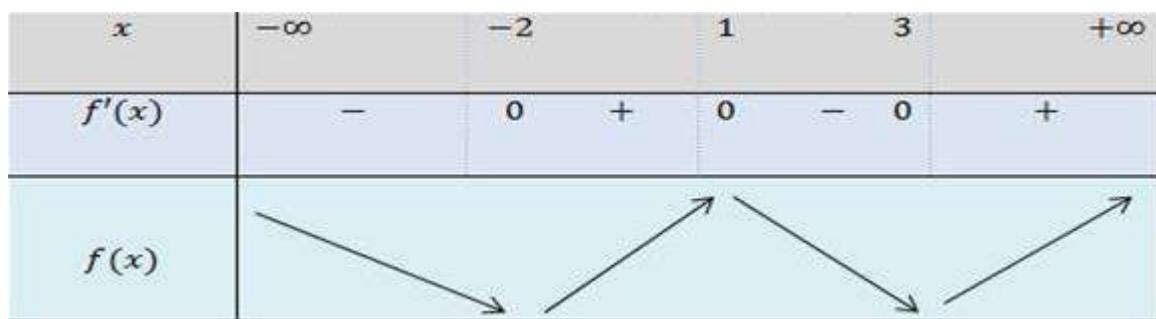
$$\left(\sqrt{2} \sin\left(\theta + \frac{\pi}{4}\right)\right)^{2k} \left(\frac{1}{2} \sin 2\theta\right)^{k-1} \leq 2 \Leftrightarrow$$

$$\left(\sin\left(\theta + \frac{\pi}{4}\right)\right)^{2k} \cdot \frac{2^k}{2^{k-1}} (\sin 2\theta)^{k-1} \leq 1 \text{ which is true.}$$

Equality holds for $\theta + \frac{\pi}{4} = \frac{\pi}{2}$ and $2\theta = \frac{\pi}{2} \Leftrightarrow \theta = \frac{\pi}{4} \Leftrightarrow a = b$.

791. Let $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ ($a, b, c, d, e \in \mathbb{R}$) such that:

$$f(1) = \frac{37}{12}, \quad f(3) = -\frac{9}{4}$$



Find all $\alpha > 0$ such that $|f(x)| \leq \alpha, \forall x \in [-1, 1]$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution 1 by proposer

We have:

$$f'(x) = A(x+2)(x-1)(x-3), \quad (A > 0)$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\rightarrow f(x) = \int A(x+2)(x-1)(x-3)dx = A\left(\frac{x^4}{4} - \frac{2x^3}{3} - \frac{5x^2}{2} + 6x\right) + C;$$

Other,

$$f(1) = \frac{37}{12}, f(3) = -\frac{9}{4} \rightarrow \begin{cases} \frac{37}{12}A + C = \frac{37}{12} \\ -\frac{9}{4}A + C = -\frac{9}{4} \end{cases} \rightarrow A = 1, C = 0;$$

$$\rightarrow f(x) = \frac{x^4}{4} - \frac{2x^3}{3} - \frac{5x^2}{2} + 6x;$$

$$\rightarrow f(-1) \leq f(x) \leq f(1), \quad \forall x \in [-1, 1]$$

$$\rightarrow -\frac{91}{12} \leq f(x) \leq \frac{37}{12} \rightarrow 0 \leq |f(x)| \leq \frac{91}{12}, \forall x \in [-1, 1]$$

$$\rightarrow \alpha \geq \frac{91}{12}$$

Hence, $\alpha \geq \frac{91}{12}$ we have: $|f(x)| \leq \alpha, \forall x \in [-1, 1]$

Solution 2 by Bedri Hajrizi-Mitrovica-Kosovo

$$f(x) = ax^4 + bx^3 + cx^2 + dx + e$$

$$f'(x) = k(x+2)(x-1)(x-3) = k(x^3 - 2x^2 - 5x + 6)$$

$$f(x) = k\left(\frac{x^4}{4} - \frac{2x^3}{3} - \frac{5x^2}{2} + 6x\right) + e$$

$$\begin{cases} f(1) = \frac{37}{12} \\ f(3) = -\frac{9}{4} \end{cases} \Rightarrow \begin{cases} 37k + 12e = 37 \\ 9k - 4e = 9 \end{cases} \Rightarrow \begin{cases} k = 1 \\ e = 0 \end{cases}$$

$$f(x) = \frac{x^4}{4} - \frac{2x^3}{3} - \frac{5x^2}{2} + 6x$$

For $x \in [-1, 1] \Rightarrow f(x) \in [f(-1), f(1)]$

It's clear that: $\alpha = \max\{|f(-1)|, |f(1)|\} = \max\left\{-\frac{91}{12}, \frac{37}{12}\right\}$

$$792. \ a, b > 0, k \in \mathbb{N}, k \geq 1, Q = \sqrt{\frac{a^2+b^2}{2}}, A = \frac{a+b}{2}, G = \sqrt{ab}, H = \frac{2ab}{a+b}$$

Prove that:

$$Q^k \cdot G^{k+1} \geq A^{k+1} \cdot H^k$$

Proposed by Seyran Ibrahimov-Maasilli-Azerbaijan



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Solution by Ravi Prakash-New Delhi-India

$$\frac{Q}{H} = \sqrt{\frac{a^2 + b^2}{2}} \cdot \frac{a+b}{2ab}$$

$$\left(\frac{Q}{H}\right)^k = \frac{1}{2^{\frac{3k}{2}}} (a^2 + b^2)^{\frac{k}{2}} \cdot \frac{(a+b)^k}{(ab)^k}$$

$$\left(\frac{A}{G}\right)^{k+1} = \left(\frac{a+b}{2\sqrt{ab}}\right)^{k+1} = \frac{(a+b)^{k+1}}{2^{k+1}(ab)^{\frac{k+1}{2}}}$$

$$Q^k G^{k+1} \geq A^{k+1} H^k \Leftrightarrow \left(\frac{Q}{H}\right)^k \geq \left(\frac{A}{G}\right)^{k+1} \Leftrightarrow \frac{(a^2 + b^2)^{\frac{k}{2}}}{2^{\frac{3k}{2}}} \left(\frac{a+b}{ab}\right)^k \geq \frac{(a+b)^{k+1}}{2^{k+1}(ab)^{\frac{k+1}{2}}}$$

$$(a^2 + b^2)^{\frac{k}{2}} \geq 2^{\frac{k-1}{2}} (ab)^{\frac{k-1}{2}} (a+b); (1)$$

Put: $a = r \cos \theta, b = r \sin \theta, r > 0, 0 < \theta < \frac{\pi}{2}$ so that (1) becomes:

$$r^k \geq 2^{\frac{k-1}{2}} r^{k-1} (\cos \theta \sin \theta)^{\frac{k-1}{2}} \cdot r (\cos \theta + \sin \theta) \Leftrightarrow$$

$$2^{\frac{k-1}{2}} \left(\frac{1}{2} \sin 2\theta\right)^{\frac{k-1}{2}} \cdot \sqrt{2} \sin\left(\theta + \frac{\pi}{4}\right) \leq 1 \Leftrightarrow$$

$$\frac{2^{\frac{k-1}{2} + \frac{1}{2}}}{2^{\frac{k-1}{2}}} (\sin 2\theta)^k \sin\left(\theta + \frac{\pi}{4}\right) \leq 1 \text{ which is true.}$$

Equality holds for $\theta = \frac{\pi}{4} \Leftrightarrow a = b$.

793.

$$n, k \geq 2, A_n = \frac{1}{n} \sum_{i=1}^n a_i, G_n = \sqrt[n]{\prod_{i=1}^n a_i}, a_i > 0, i \in \overline{1, n}. \text{ Prove that :}$$

$$A_n \geq G_n \cdot \sqrt[k]{1 + k \left(\frac{A_n}{G_n} - 1 \right)}.$$

Proposed by Seyran Ibrahimov-Maasilli-Azerbaijan



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since : $\frac{A_n}{G_n} - 1 > -1$ and $0 \leq \frac{1}{k} \leq 1$, from Bernoulli's inequality we have

$$\therefore \sqrt[k]{1 + k\left(\frac{A_n}{G_n} - 1\right)} \leq 1 + \frac{1}{k} \cdot k\left(\frac{A_n}{G_n} - 1\right) = \frac{A_n}{G_n}$$

Therefore, $A_n \geq G_n \cdot \sqrt[k]{1 + k\left(\frac{A_n}{G_n} - 1\right)}$. Equality holds iff $\frac{A_n}{G_n} - 1 = 0 \Leftrightarrow A_n = G_n \Leftrightarrow a_1 = a_2 = \dots = a_n$.

Solution 2 by Ravi Prakash-New Delhi-India

We have: $A_n \geq G_n > 0$. Now, $A_n \geq G_n \left[1 + k\left(\frac{A_n}{G_n} - 1\right)\right]^{\frac{1}{k}} \Leftrightarrow$

$$\left(\frac{A_n}{G_n}\right)^k \geq 1 + k\left(\frac{A_n}{G_n} - 1\right) \Leftrightarrow A_n^k \geq G_n^k + kG_n^{k-1}(A_n - G_n); (1)$$

By the binomial theorem:

$$\begin{aligned} A_n^k &= [G_n + (A_n - G_n)]^k = G_n^k + \binom{k}{1} G_n^{k-1}(A_n - G_n) + \sum_{j=2}^k \binom{k}{j} G_n^{k-j}(A_n - G_n)^j \geq \\ &\geq G_n^k + kG_n^{k-1}(A_n - G_n) > 0 \end{aligned}$$

Equality holds for $A_n = G_n \Leftrightarrow a_1 = a_2 = \dots = a_n$.

794. If $a, b, c > 0$ such that : $3 \sum a^3 \geq (\sum a^2)^2$. Prove that

$$\min \left\{ \sum \frac{a^2}{b}, \sum \frac{b^2}{a} \right\} \geq \sum a^2$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let's prove that : $\sum \frac{a^2}{b} \stackrel{(*)}{\geq} \frac{3 \sum a^3}{\sum a^2}, \forall a, b, c > 0$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 (*) &\leftrightarrow \sum \left(\frac{a^2}{b} - 2a + b \right) \geq \frac{3 \sum a^3}{\sum a^2} - \sum a \leftrightarrow \sum a^2 \cdot \sum \frac{(a-b)^2}{b} \\
 &\geq 3 \sum a^3 - (\sum a)(\sum a^2) = 2 \sum a^3 - \sum ab(a+b) \\
 &\leftrightarrow \sum \frac{a^2 + b^2 + c^2}{b} \cdot (a-b)^2 \geq \sum (a^3 + b^3 - ab(a+b)) = \sum (a+b)(a-b)^2 \\
 &\leftrightarrow \sum \left(\frac{a^2 + c^2}{b} - a \right) (a-b)^2 \geq 0 \\
 &\leftrightarrow S_c(a-b)^2 + S_a(b-c)^2 + S_b(c-a)^2 \stackrel{(**)}{\geq} 0 \text{ where} \\
 S_a &= \frac{b^2 + a^2}{c} - b, S_b = \frac{c^2 + b^2}{a} - c, S_c = \frac{a^2 + c^2}{b} - a.
 \end{aligned}$$

WLOG, we may assume that : $b = \text{mid}\{a, b, c\}$. If $a \leq b \leq c \rightarrow$

$$\begin{aligned}
 S_b &\geq \frac{ca + b^2}{a} - c = \frac{b^2}{a} \geq 0 \text{ and } S_c \geq \frac{a^2 + ab}{b} - a \geq 0 \\
 \rightarrow LHS_{(**)} &\stackrel{S_c \geq 0}{\geq} S_a(b-c)^2 + S_b[(c-b) + (b-a)]^2 \\
 &= (S_a + S_b)(b-c)^2 + S_b(a-b)^2 + 2(c-b)(b-a)S_b \\
 \rightarrow \text{It's suffices to prove : } S_a + S_b &\geq 0 \quad (\because (c-b)(b-a), S_b \geq 0) \\
 S_a + S_b &= \frac{b^2 + a^2}{c} - b + \frac{c^2 + b^2}{a} - c \geq \frac{b^2 + a^2}{c} - b + \frac{ca + ba}{a} - c = \frac{b^2 + a^2}{c} \geq 0 \\
 \rightarrow (***) \text{ is true.}
 \end{aligned}$$

$$\begin{aligned}
 \text{If } a \geq b \geq c \rightarrow S_a &\geq \frac{bc + a^2}{c} - b = \frac{a^2}{c} \geq 0 \text{ and } S_c \geq \frac{ab + c^2}{b} - a \geq 0, \text{ if } S_b \\
 &\geq 0 \text{ we are done.}
 \end{aligned}$$

$$\text{If } S_b = \frac{c^2 + b^2}{a} - c \leq 0 \rightarrow c \geq \frac{c^2 + b^2}{a} \stackrel{AM-GM}{\geq} \frac{2bc}{a} \rightarrow a \geq 2b \text{ and}$$

$$: 2[(a-b)^2 + (b-c)^2] \stackrel{CBS}{\geq} (a-c)^2$$

$$\begin{aligned}
 \rightarrow LHS_{(**)} &\stackrel{S_b \leq 0}{\geq} S_c(a-b)^2 + S_b \cdot 2[(a-b)^2 + (b-c)^2] + S_a(b-c)^2 \\
 &= (2S_b + S_c)(a-b)^2 + (2S_b + S_a)(b-c)^2 \\
 \rightarrow \text{It's suffices to prove : } 2S_b + S_c, 2S_b + S_a &\geq 0
 \end{aligned}$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$2S_b + S_c = 2\left(\frac{c^2 + b^2}{a} - c\right) + \frac{a^2 + c^2}{b} - a \stackrel{a \geq 2b}{\geq} -2c + \frac{2ab + c^2}{b} - a$$

$$= a - 2c + \frac{c^2}{b} \stackrel{a \geq 2b}{\geq} 2(b - c) \stackrel{b \geq c}{\geq} 0$$

$$2S_b + S_a = 2\left(\frac{c^2 + b^2}{a} - c\right) + \frac{b^2 + a^2}{c} - b \stackrel{a \geq 2b \text{ and } b \geq c}{\geq}$$

$$-2c + \frac{bc + 2ab}{c} - b \stackrel{a, b \geq c}{\geq} -2c + \frac{2c^2}{c} = 0$$

$\rightarrow (**)$ is true $\rightarrow (*)$ is true $\rightarrow \sum \frac{a^2}{b} \geq \frac{3 \sum a^3}{\sum a^2}$. Similarly, we have :

$$\sum \frac{b^2}{a} \geq \frac{3 \sum a^3}{\sum a^2}.$$

$$\text{Therefore, } \min \left\{ \sum \frac{a^2}{b}, \sum \frac{b^2}{a} \right\} \geq \frac{3 \sum a^3}{\sum a^2} = \sum a^2.$$

795. Prove that for $m, n \in \mathbb{R}_+$

$$m + n - (2 - \sqrt{2})\sqrt{mn} \leq \sqrt{m^2 + n^2}$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution 1 by Bedri Hajrizi-Mitrovica-Kosovo

$$\text{Let } m = a^2, n = b^2; a, b > 0 \Rightarrow a^2 + b^2 - (2 - \sqrt{2})ab = \sqrt{a^4 + b^4}$$

$$(2 - \sqrt{2})^2 a^2 b^2 + 2a^2 b^2 - 2(2 - \sqrt{2})a^3 b - 2(2 - \sqrt{2})ab^3 \leq 0$$

$$(6 - 4\sqrt{2} + 2)a^2 b^2 - 2(2 - \sqrt{2})a^3 b - 2(2 - \sqrt{2})ab^3 \leq 0$$

$$4(2 - \sqrt{2})a^2 b^2 - 2(2 - \sqrt{2})a^3 b - 2(2 - \sqrt{2})ab^3 \leq 0 | : [-2(2 - \sqrt{2})ab]$$

$$\Rightarrow a^2 - 2ab + b^2 \geq 0 \Leftrightarrow (a - b)^2 \geq 0$$

Equality holds for $a = b$.

Solution 2 by Bedri Hajrizi-Mitrovica-Kosovo

Let $m = nt^2; t > 0$. Then

$$t^2 + 1 - (2 - \sqrt{2})t \leq \sqrt{t^4 + 1}$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$(2 - \sqrt{2})^2 t^2 - 2(2 - \sqrt{2})t^3 - 2(2 - \sqrt{2})t \leq 0$$

$$(3 - 2\sqrt{2})t^2 + t^2 - (2 - \sqrt{2})t^3 - (2 - \sqrt{2})t \leq 0$$

$$2t^2 - t^3 - t \leq 0 \Leftrightarrow t^2 - 2t + 1 \geq 0 \Leftrightarrow (t - 1)^2 \geq 0$$

Equality holds for $a = b$.

796. If α is a real number and $\left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right)\left(\frac{y}{x} + \frac{z}{y} + \frac{x}{z}\right) = \alpha$ for any positive

real numbers x, y, z

then prove that $\left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right)^3 + \left(\frac{y}{x} + \frac{z}{y} + \frac{x}{z}\right)^3 \leq \frac{\alpha^2 + 18\alpha - 27}{4}$

Proposed by Marius Drăgan, Neculai Stanciu-Romania

Solution 1 by Nguyen Van Canh-Ben Tre-Vietnam

Because: $x, y, z > 0 \rightarrow \left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right)\left(\frac{y}{x} + \frac{z}{y} + \frac{x}{z}\right) = \alpha > 0$

Let us denote: $a = \frac{x}{y}; b = \frac{y}{z}; c = \frac{z}{x} \rightarrow a, b, c > 0, abc = 1$

Then, we have:

$$\begin{aligned} & \left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right)^3 + \left(\frac{y}{x} + \frac{z}{y} + \frac{x}{z}\right)^3 \leq \frac{\alpha^2 + 18\alpha - 27}{4}; \\ & \leftrightarrow \left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right)^3 + \left(\frac{y}{x} + \frac{z}{y} + \frac{x}{z}\right)^3 \\ & \leq \frac{\left(\left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right)\left(\frac{y}{x} + \frac{z}{y} + \frac{x}{z}\right)\right)^2 + 18\left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right)\left(\frac{y}{x} + \frac{z}{y} + \frac{x}{z}\right) - 27}{4}; \\ & \leftrightarrow (a + b + c)^3 + (ab + bc + ca)^3 \\ & \leq \frac{((a + b + c)(ab + bc + ca))^2 + 18(a + b + c)(ab + bc + ca) - 27}{4}; \end{aligned}$$

$abc=1$

$$\begin{aligned} & \Leftrightarrow 4(abc(a + b + c)^3 + (ab + bc + ca)^3) \\ & \leq ((a + b + c)(ab + bc + ca))^2 + 18abc(a + b + c)(ab + bc + ca) \\ & \quad - 27(abc)^2; \end{aligned}$$

$$\begin{aligned} & \leftrightarrow \sum a^2 b^2 (a^2 + b^2) + 2abc \left(\sum ab(a + b) \right) - 2 \sum a^3 b^3 - 2abc \sum a^3 - 6a^2 b^2 c^2 \geq 0; \\ & \leftrightarrow (a^2 - 2ab + b^2)(b^2 - 2bc + c^2)(c^2 - 2ca + a^2) \geq 0; \\ & \leftrightarrow (a - b)^2(b - c)^2(c - a)^2 \geq 0; (\therefore \text{true}) \end{aligned}$$

Proved. Equality $\Leftrightarrow a = b = c = 1 \Leftrightarrow x = y = z \Leftrightarrow \alpha = 9$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Solution 2 by Soumava Chakraborty-Kolkata-India

Substituting $y + z = a, z + x = b, x + y = c$, we see $a + b > c, b + c > a, c + a > b$
 $\Rightarrow a, b, c$ form sides of a triangle with semiperimeter, circumradius, inradius

$= s, R, r$ respectively (say) and we subsequently arrive at : $2 \sum x = \sum a = 2s \Rightarrow \sum x = s \therefore x = s - a, y = s - b, z = s - c$ and using such substitutions,

$$\begin{aligned} \sum x^3 y^3 &= \left(\sum xy \right)^3 - 3(xy + yz)(yz + zx)(zx + xy) \\ &= \left(\sum (s-a)(s-b) \right)^3 - 3 \left(\prod (s-a) \right) abc \\ &= (4Rr + r^2)^3 - 12Rr^3s^2 \stackrel{(i)}{\cong} \sum x^3 y^3 \text{ and} \end{aligned}$$

$$\begin{aligned} \sum x^3 &= \left(\sum x \right)^3 - 3(x+y)(y+z)(x+y) = s^3 - 12Rrs \stackrel{(ii)}{\cong} \sum x^3 \therefore \alpha \\ &= \frac{1}{x^2 y^2 z^2} \left(\sum_{\text{cyc}} x^2 y \right) \left(\sum_{\text{cyc}} x y^2 \right) = \frac{1}{r^4 s^2} \left(\sum x^3 y^3 + 3x^2 y^2 z^2 + xyz \sum x^3 \right) \\ &\stackrel{\text{via (i) and (ii)}}{\cong} \frac{(4Rr + r^2)^3 - 12Rr^3s^2 + 3r^4s^2 + r^2s(s^3 - 12Rrs)}{r^4s^2} \end{aligned}$$

$$= \frac{s^4 - (24Rr - 3r^2)s^2 + r(4R + r)^3}{r^2s^2} \stackrel{(*)}{\cong} \alpha \text{ and } \frac{x}{y} + \frac{y}{z} + \frac{z}{x} + \frac{y}{x} + \frac{x}{z} + \frac{z}{y}$$

$$= \sum_{\text{cyc}} \left(\frac{y}{z} + \frac{x}{z} \right) = \sum \frac{xy(x+y+z-z)}{xyz}$$

$$= \frac{1}{r^2 s} \left(s \sum (s-a)(s-b) - 3r^2 s \right) = \boxed{\frac{4R - 2r}{r} \stackrel{(**)}{\cong} \sum_{\text{cyc}} \frac{x}{y} + \sum_{\text{cyc}} \frac{y}{x}}$$

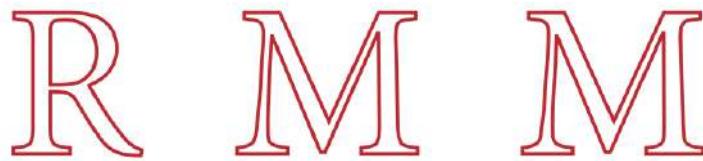
$$\text{Now, } \left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \right)^3 + \left(\frac{y}{x} + \frac{x}{z} + \frac{z}{y} \right)^3$$

$$= \left(\sum_{\text{cyc}} \frac{x}{y} + \sum_{\text{cyc}} \frac{y}{x} \right)^3$$

$$- 3 \left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \right) \left(\frac{y}{x} + \frac{x}{z} + \frac{z}{y} \right) \left(\sum_{\text{cyc}} \frac{x}{y} + \sum_{\text{cyc}} \frac{y}{x} \right) \stackrel{\text{via } (**)}{\cong} \frac{(4R - 2r)^3}{r^3} - 3\alpha \cdot \frac{4R - 2r}{r}$$

$$\stackrel{\text{via } (*)}{\cong} \frac{(4R - 2r)^3}{r^3} - 3 \cdot \frac{4R - 2r}{r} \cdot \frac{s^4 - (24Rr - 3r^2)s^2 + r(4R + r)^3}{r^2s^2}$$

$$= \boxed{\frac{s^2(4R - 2r)^3 - 3(4R - 2r)(s^4 - (24Rr - 3r^2)s^2 + r(4R + r)^3)}{r^3s^2} \stackrel{(*)}{\cong} \text{LHS}} \text{ and via (*),}$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\boxed{\text{RHS} \stackrel{(..)}{=} \frac{\left(\frac{s^4 - (24Rr - 3r^2)s^2 + r(4R + r)^3}{r^2 s^2} \right)^2 + 18 \left(\frac{s^4 - (24Rr - 3r^2)s^2 + r(4R + r)^3}{r^2 s^2} \right) - 27}{4}}$$

$$\begin{aligned}
& \therefore \text{via } (\bullet), (\bullet\bullet), \left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \right)^3 + \left(\frac{y}{x} + \frac{x}{z} + \frac{z}{y} \right)^3 \leq \frac{\alpha^2 + 18\alpha - 27}{4} \\
& \Leftrightarrow \frac{(s^4 - (24Rr - 3r^2)s^2 + r(4R + r)^3)^2 + 18r^2s^2(s^4 - (24Rr - 3r^2)s^2 + r(4R + r)^3) - 27r^4s^4}{s^2(4R - 2r)^3 - 3(4R - 2r)(s^4 - (24Rr - 3r^2)s^2 + r(4R + r)^3)} \\
& \geq \frac{4r^4s^4}{r^3s^2} \\
& \Leftrightarrow (s^4 - (24Rr - 3r^2)s^2 + r(4R + r)^3)^2 + 18r^2s^2(s^4 - (24Rr - 3r^2)s^2 + r(4R + r)^3) \\
& \quad - 27r^4s^4 - 4rs^4(4R - 2r)^3 \\
& \quad + 12rs^2(4R - 2r)(s^4 - (24Rr - 3r^2)s^2 + r(4R + r)^3) \geq 0 \\
& \text{expanding and re-arranging} \\
& \Leftrightarrow s^8 - 2rs^4(4R + r)^3 + r^2(4R + r)^6 \geq 0 \Leftrightarrow (s^4 - r(4R + r)^3)^2 \geq 0 \\
& \rightarrow \text{true} \Rightarrow \left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \right)^3 + \left(\frac{y}{x} + \frac{x}{z} + \frac{z}{y} \right)^3 \leq \frac{\alpha^2 + 18\alpha - 27}{4} \text{ (QED)}
\end{aligned}$$

797. If $a, b \geq 0$ then:

$$4\sqrt{3ab} \cdot e^{\sqrt{2}} + 2\sqrt{2}(a+b)e^{\sqrt{3}} \geq (\sqrt{a} + \sqrt{b})^2 (\sqrt{3}e^{\sqrt{2}} + \sqrt{2}e^{\sqrt{3}})$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\begin{aligned}
(*) & \Leftrightarrow \sqrt{2} \cdot e^{\sqrt{3}} \left[2(a+b) - (\sqrt{a} + \sqrt{b})^2 \right] \geq \sqrt{3} \cdot e^{\sqrt{2}} \left[(\sqrt{a} + \sqrt{b})^2 - 4\sqrt{ab} \right] \\
& \Leftrightarrow \sqrt{2} \cdot e^{\sqrt{3}} (\sqrt{a} - \sqrt{b})^2 \geq \sqrt{3} \cdot e^{\sqrt{2}} (\sqrt{a} - \sqrt{b})^2 \Leftrightarrow \left(\frac{e^{\sqrt{3}}}{\sqrt{3}} - \frac{e^{\sqrt{2}}}{\sqrt{2}} \right) (\sqrt{a} - \sqrt{b})^2 \geq 0 \quad (1)
\end{aligned}$$

$$\text{Let } f(x) = \frac{e^x}{x}, x > 1.$$

$$\text{We have : } f'(x) = \frac{(x-1)e^x}{x^2} > 0$$

$\rightarrow f$ is strictly increasing on $(1, \infty)$ $\rightarrow \frac{e^{\sqrt{3}}}{\sqrt{3}} > \frac{e^{\sqrt{2}}}{\sqrt{2}} \rightarrow (1) \text{ is true.}$

Therefore,

$$4\sqrt{3ab} \cdot e^{\sqrt{2}} + 2\sqrt{2}(a+b) \cdot e^{\sqrt{3}} \geq (\sqrt{a} + \sqrt{b})^2 (\sqrt{3} \cdot e^{\sqrt{2}} + \sqrt{2} \cdot e^{\sqrt{3}}).$$

Equality holds iff $a = b$.



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Solution 2 by Nguyen Van Canh-Ben Tre-Vietnam

Lemma: If $1 < y < x$ then:

$$\frac{x}{e^x} < \frac{y}{e^y}$$

Proof:

$$\text{Let } \varphi(t) = \frac{t}{e^t}, (t > 1) \rightarrow \varphi'(t) = e^{-t}(1-t) < 0 \rightarrow \varphi(t) \downarrow (1, +\infty)$$

$$\stackrel{1 < y \leq x}{\Rightarrow} \varphi(x) < \varphi(y) \rightarrow \frac{x}{e^x} < \frac{y}{e^y}$$

$$\text{Now, choosing: } x = \sqrt{3} > y = \sqrt{2} \rightarrow \frac{\sqrt{3}}{e^{\sqrt{3}}} < \frac{\sqrt{2}}{e^{\sqrt{2}}} \rightarrow \sqrt{3}e^{\sqrt{2}} < \sqrt{2}e^{\sqrt{3}}$$

$$\text{Other, we have: } 2\sqrt{ab} \leq a + b;$$

By Chebyshev's Inequality we have:

$$\begin{aligned} (2\sqrt{ab})(\sqrt{3}e^{\sqrt{2}}) + (a+b)(\sqrt{2}e^{\sqrt{3}}) &\geq \frac{1}{2}(2\sqrt{ab} + a+b)(\sqrt{3}e^{\sqrt{2}} + \sqrt{2}e^{\sqrt{3}}); \\ \Leftrightarrow 4\sqrt{3ab} \cdot e^{\sqrt{2}} + 2\sqrt{2}(a+b)e^{\sqrt{3}} &\geq (\sqrt{a} + \sqrt{b})^2 (\sqrt{3}e^{\sqrt{2}} + \sqrt{2}e^{\sqrt{3}}) \end{aligned}$$

Proved.

Solution 3 by Khaled Abd Imouti-Damascus-Syria

$$4\sqrt{3ab} \cdot e^{\sqrt{2}} + 2\sqrt{2}(a+b)e^{\sqrt{3}} \geq (\sqrt{a} + \sqrt{b})^2 (\sqrt{3}e^{\sqrt{2}} + \sqrt{2}e^{\sqrt{3}})$$

$$4\sqrt{3} \cdot \sqrt{ab} \cdot e^{\sqrt{2}} + 2\sqrt{2}(a+b) \cdot e^{\sqrt{3}} \geq (a+b+2\sqrt{ab})(\sqrt{3} \cdot e^{\sqrt{2}} + \sqrt{2} \cdot e^{\sqrt{3}})$$

Let $G = \sqrt{ab}$, $M = \frac{a+b}{2}$, then

$$4\sqrt{3}G \cdot e^{\sqrt{2}} + 4\sqrt{2} \cdot \sqrt{3} \cdot M \cdot e^{\sqrt{3}} \geq 2(M+G) \left(\sqrt{6} \cdot \frac{e^{\sqrt{2}}}{\sqrt{2}} + \sqrt{6} \cdot \frac{e^{\sqrt{3}}}{\sqrt{3}} \right)$$

$$4\sqrt{6}G \cdot \frac{e^{\sqrt{2}}}{\sqrt{2}} + 4\sqrt{6}M \cdot \frac{e^{\sqrt{3}}}{\sqrt{3}} \geq 2\sqrt{6}(M+G) \left(\frac{e^{\sqrt{2}}}{\sqrt{2}} + \frac{e^{\sqrt{3}}}{\sqrt{3}} \right)$$

$$2(\alpha G + \beta M) \stackrel{?}{\geq} (\alpha + \beta) \left(\frac{M}{G} + 1 \right), x = \frac{M}{G} \geq 1$$

$$2\alpha + 2\beta \cdot \frac{M}{G} \stackrel{?}{\geq} \alpha \cdot \frac{M}{G} + \alpha + \beta \cdot \frac{M}{G} + \beta$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\alpha + \beta \cdot \frac{M}{G} \geq \alpha \cdot \frac{M}{G} + \beta$$

$$\alpha - \beta \geq (\alpha - \beta) \cdot \frac{M}{G} \Leftrightarrow 1 \leq \frac{M}{G} \text{ true.}$$

Now, let $f(x) = \frac{e^x}{x}$, then $f'(x) = \frac{(x-1)e^x}{x^2}$

$$f'(x) = 0 \Leftrightarrow x = 1$$

x	0	1	$\sqrt{2}$	$\sqrt{3}$	∞
$f'(x)$	---	-0+	+++	+++	+++
$f(x)$	$+\infty \searrow e \nearrow$	$\frac{e^{\sqrt{2}}}{\sqrt{2}}$	$\nearrow \frac{e^{\sqrt{3}}}{\sqrt{3}}$	\nearrow	

$$\alpha < \beta.$$

798. If $a, b, c, x, y > 0$, then

$$\left(\sum a \right) \left(\sum \frac{1}{a} \right) - (x+y) \sum \frac{a+b}{xa+yb} \geq 3.$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$(*) : \left(\sum a \right) \left(\sum \frac{1}{a} \right) - (x+y) \sum \frac{a+b}{xa+yb} \geq 3.$$

$$\begin{aligned} \text{We have : } \left(\sum a \right) \left(\sum \frac{1}{a} \right) &= \sum \left(1 + \frac{a}{b} + \frac{b}{a} \right) = \sum \frac{a^2 + b^2 + ab}{ab} \rightarrow (*) \\ &\leftrightarrow \sum \left(\frac{a^2 + b^2 + ab}{ab} - \frac{(x+y)(a+b)}{xa+yb} \right) \geq 3 \\ &\leftrightarrow \sum \frac{xa^3 + yb^3}{ab(xa+yb)} \geq 3 \quad (1) \end{aligned}$$

$$\begin{aligned} \text{From Hölder's inequality, we have : } \sum \frac{a^3}{ab(xa+yb)} &\geq \frac{(a+b+c)^3}{(\sum ab)[\sum(xa+yb)]} \\ &\geq \frac{3(ab+bc+ca)(a+b+c)}{(ab+bc+ca)(x+y)(a+b+c)} = \frac{3}{x+y} \end{aligned}$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Similarly, we have : $\sum \frac{b^3}{ab(xa + yb)} \geq \frac{3}{x+y} \rightarrow \sum \frac{xa^3 + yb^3}{ab(xa + yb)}$
 $\geq x \cdot \frac{3}{x+y} + y \cdot \frac{3}{x+y} = 3 \rightarrow (1) \text{ is true.}$

Therefore,

$$\left(\sum a \right) \left(\sum \frac{1}{a} \right) - (x+y) \sum \frac{a+b}{xa+yb} \geq 3.$$

Equality holds iff $a = b = c$.

Solution 2 by Aggeliki Papaspyropoulou-Greece

$$\frac{a}{b} + \frac{b}{a} + \frac{b}{c} + \frac{c}{b} + \frac{c}{a} + \frac{a}{c} + 3 \geq 3 + \frac{(x+y)(a+b)}{xa+yb} + \frac{(x+y)(b+c)}{xb+yc} + \frac{(x+y)(c+a)}{xc+ya}$$

$$\frac{a}{b} + \frac{b}{a} + \frac{b}{c} + \frac{c}{b} + \frac{c}{a} + \frac{a}{c} \geq \frac{xa+yb}{xa+yb} + \frac{xb+yc}{xb+yc} + \frac{xc+ya}{xc+ya} + \frac{ay+bx}{xa+yb} + \frac{by+cx}{xb+yc} + \frac{cy+ax}{xc+ya}$$

$$\sum_{cyc} \left(\frac{a}{b} + \frac{b}{a} \right) \geq 3 + \frac{ay+bx}{xa+yb} + \frac{by+cx}{xb+yc} + \frac{cy+ax}{xc+ya}$$

$$\sum_{cyc} \left(\frac{a}{b} + \frac{b}{a} \right) - \frac{ay+bx}{xa+yb} - \frac{by+cx}{xb+yc} - \frac{cy+ax}{xc+ya} \geq 3; (*)$$

$$\frac{a}{b} + \frac{b}{a} - \frac{ay+bx}{xa+yb} = \frac{a^2 + b^2}{ab} - \frac{ay+bx}{xa+yb} = \frac{a^3x + b^3y}{ab(xa+yb)} = \frac{a^2x}{b(xa+yb)} + \frac{b^2y}{a(xa+yb)}$$

$$\frac{b}{c} + \frac{c}{b} - \frac{by+cx}{xb+yc} = \frac{b^2x}{c(xb+yc)} + \frac{c^2y}{b(xb+yc)}$$

$$\frac{c}{a} + \frac{a}{c} - \frac{by+cx}{xb+yc} = \frac{c^2x}{a(xc+ya)} + \frac{a^2y}{c(cx+ay)}$$

$$P_1 = \frac{a^2x}{b(xa+yb)} + \frac{b^2y}{c(xb+yc)} + \frac{c^2x}{a(xc+ya)}$$

$$P_2 = \frac{b^2y}{a(xa+yb)} + \frac{c^2y}{b(xb+yc)} + \frac{a^2y}{c(cx+ay)}$$

$$P_1 + P_2 \geq 3; (**)$$

From Holder's inequality:



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$P_1 \cdot (ab + bc + ca)(xa + yb + xc + yc + ay) \geq \left(\sqrt[3]{a^3x} + \sqrt[3]{b^3x} + \sqrt[3]{c^3x} \right)^3$$

$$P_1(ab + bc + ca)(a + b + c)(x + y) \geq (a + b + c)x$$

$$P_1 \geq \frac{(a + b + c)^2 x}{(ab + bc + ca)(x + y)}; \quad (1)$$

$$P_2(ab + bc + ca)(xa + yb + xb + yc + ay) \geq \left(\sqrt[3]{a^3y} + \sqrt[3]{b^3y} + \sqrt[3]{c^3y} \right)^3; \quad (2)$$

$$P_1 + P_2 \geq \frac{(a + b + c)^2(x + y)}{(ab + bc + ca)(x + y)} = \frac{(a + b + c)^2}{ab + bc + ca} \geq 3$$

$$(a + b + c)^3 \geq 3(ab + bc + ca)$$

Equality holds for $a = b = c$.

Solution 3 by Marian Dincă-Romania

Let $\frac{x}{x+y} = t, \frac{y}{x+y} = 1-t$. The inequality is equivalent to:

$$(a + b + c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) - \left(\frac{a+b}{at+b(1-t)} + \frac{b+c}{bt+c(1-t)} + \frac{c+a}{ct+a(1-t)} \right) \geq 3, \quad t \in [0, 1]$$

$$\text{Let } f(t) = \frac{a+b}{at+b(1-t)} + \frac{b+c}{bt+c(1-t)} + \frac{c+a}{ct+a(1-t)}$$

$$f'(t) = -\frac{(a+b)(a-b)}{(at+b(1-t))^2} - \frac{(b+c)(b-c)}{(bt+c(1-t))^2} - \frac{(c+a)(c-a)}{(ct+a(1-t))^2}$$

$$f''(t) = 2 \frac{(a+b)(a-b)^2}{(at+b(1-t))^3} + 2 \frac{(b+c)(b-c)^2}{(bt+c(1-t))^3} + 2 \frac{(c+a)(c-a)^2}{(ct+a(1-t))^3} \geq 0$$

$\Rightarrow f$ –convex, $f(t) \leq \max\{f(0), f(1)\}$ and $-f(t)$ is concave, then $-f(t) \geq$

$$\min\{-f(0), -f(1)\}$$

$$f(0) = \frac{a+b}{c} + \frac{b+c}{c} + \frac{c+a}{a}$$

$$(a + b + c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) - f(0) = (a + b + c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) - \left(\frac{a+b}{c} + \frac{b+c}{c} + \frac{c+a}{a} \right) = \\ = 3 + \frac{a}{b} + \frac{b}{a} + \frac{b}{c} + \frac{c}{b} + \frac{a}{c} + \frac{c}{a} - \frac{a}{b} - \frac{b}{c} - \frac{c}{a} - 3 =$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$= \frac{b}{a} + \frac{c}{b} + \frac{a}{c} \geq 3 \cdot \sqrt[3]{\frac{b}{a} \cdot \frac{c}{b} \cdot \frac{a}{c}} = 3$$

$$(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - f(1) = (a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - \left(\frac{a+b}{c} + \frac{b+c}{a} + \frac{c+a}{b}\right) =$$

$$= \frac{b}{a} + \frac{c}{b} + \frac{a}{c} \geq 3 \cdot \sqrt[3]{\frac{b}{a} \cdot \frac{c}{b} \cdot \frac{a}{c}} = 3$$

799.

Let $f(x) = \frac{1-x}{1+x^2}$. Solve in \mathbb{R} :

$$f(f(f(x))) \leq 0$$

Solution 1 by Vivek Kumar-India

$$\begin{aligned} f(f(f(x))) &= f\left(f\left(\frac{1-x}{1+x^2}\right)\right) = f\left(\frac{1 - \frac{1-x}{1+x^2}}{1 - \left(\frac{1-x}{1+x^2}\right)^2}\right) = \\ &= f\left(\frac{\frac{x^2+x}{1+x^2}}{\frac{(1+x^2)^2 + (1-x)^2}{(1+x^2)^2}}\right) = f\left(\frac{(x^2+x)(1+x^2)}{(1+x^2)^2 + (1-x)^2}\right) = f\left(\frac{x+x^2+x^3+x^4}{x^4+3x^2-2x+2}\right) = \\ &= \frac{1 - \frac{x+x^2+x^3+x^4}{x^4+3x^2-2x+2}}{1 + \left(\frac{x+x^2+x^3+x^4}{x^4+3x^2-2x+2}\right)^2} = \\ &= \frac{(x^4+3x^2-2x+2-x-x^2-x^3-x^4)(x^4+3x^2-2x+2)}{(x^4+3x^2-2x+2)+(x+x^2+x^3+x^4)^2} \\ f(f(f(x))) \leq 0 &\Rightarrow (-x^3+2x^2-3x+2)(x^4+3x^2-2x+2) \leq 0 \\ x^3-2x^2+3x-2 \geq 0 &\Leftrightarrow (x-1)\left(\left(x-\frac{1}{2}\right)^2 + \frac{1}{4}\right) \geq 0 \Rightarrow x \in [1, \infty) \end{aligned}$$

Solution 2 by Alex Szoros-Romania

$$f(f(f(x))) = \frac{1-f(f(x))}{1+(f(f(x)))^2} \Leftrightarrow 1+f^2(x) \leq 1-f(x) \Leftrightarrow$$

$$f^2(x) + f(x) \leq 0 \Leftrightarrow f(x)(1+f(x)) \leq 0$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$-1 - x^2 \leq 1 - x \leq 0 \Leftrightarrow \begin{cases} x^2 - x + 2 \geq 0 \\ 1 \leq x \end{cases} \Leftrightarrow x \in [1, \infty)$$

Solution 3 by Hikmat Mammadov-Azerbaijan

$$\text{Let } q = f(f(x)), f(x) \leq 0 \Rightarrow \frac{1-q}{1+q^2} \leq 0 \Rightarrow 1-q \leq 0 \Rightarrow q \geq 1.$$

$$\text{Let } q = f(p) \geq 1 \Rightarrow 1-p \geq 1+p^2 \Rightarrow p^2 + p \leq 0 \Rightarrow p(p+1) \geq 0$$

$$-1 \leq p \leq 0 \Rightarrow -1 \leq f(x) \leq 0 \Leftrightarrow -1 \leq \frac{1-x}{1+x^2} \leq 0 \Leftrightarrow -1 - x^2 \leq 1 - x \leq 0$$

$$x^2 - x + 2 \geq 0 \Leftrightarrow x \in [1, \infty).$$

800. If $n, p \in \mathbb{N}, n \geq 1, p \geq 2, 0 \leq a_k \leq 1, k \in \overline{1, n}$ then:

$$\prod_{k=1}^n (1 + a_k) \geq \sqrt[p]{\prod_{k=1}^n (1 + pa_k)} \geq 1 + \sum_{k=1}^n a_k \geq \sqrt[p]{1 + p \sum_{k=1}^n a_k}$$

Proposed by Seyran Ibrahimov-Azerbaijan

Solution by Ravi Prakash-New Delhi-India

$$(1 + a_k)^p \geq 1 + pa_k; \forall 1 \leq k \leq n \Rightarrow \prod_{k=1}^n (1 + a_k)^p \geq \prod_{k=1}^n (1 + pa_k)$$

$$\left[\prod_{k=1}^n (1 + a_k) \right]^p \geq \prod_{k=1}^n (1 + pa_k) \Rightarrow \prod_{k=1}^n (1 + a_k) \geq \sqrt[p]{\prod_{k=1}^n (1 + pa_k)}; (1)$$

$$\sqrt[p]{\prod_{k=1}^n (1 + pa_k)} \geq \prod_{k=1}^n \left(1 + \frac{p}{p} a_k \right) = \prod_{k=1}^n (1 + a_k) = 1 + \sum_{k=1}^n a_k + a \stackrel{a \geq 0}{\geq} 1 + \sum_{k=1}^n a_k; (2)$$

$$\left(1 + \sum_{k=1}^n a_k \right)^p \geq 1 + p \sum_{k=1}^n a_k \Rightarrow 1 + \sum_{k=1}^n a_k \geq \sqrt[p]{1 + p \sum_{k=1}^n a_k}; (3)$$

From (1), (2) and (3), we get

$$\prod_{k=1}^n (1 + a_k) \geq \sqrt[p]{\prod_{k=1}^n (1 + pa_k)} \geq 1 + \sum_{k=1}^n a_k \geq \sqrt[p]{1 + p \sum_{k=1}^n a_k}$$



ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru