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ROMANIAN MATHEMATICAL MAGAZINE

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If $|q| < 1$, then prove that:

$$|(\omega q)_\infty|^2 = \frac{(q)_\infty {(q^6)_\infty}^2 (q^9)_\infty}{(q^2)_\infty (q^{18})_\infty}$$

where $(q)_\infty = (1-q)(1-q^2)(1-q^3)\ldots$ and ω is the complex cube root of unity.

Proposed by Angad Singh-India

Solution by proposer

We know that if $|q| < 1$, the $(q)_\infty$ is defined as

$$(q)_\infty = \prod_{k=1}^{\infty}(1-q^k), \textit{thus}$$

$$(\omega q)_\infty = \prod_{k=1}^{\infty}(1-\omega^k q^k) = \prod_{k=1}^{\infty}(1-\omega^{3k-2}q^{3k-2})(1-\omega^{3k-1}q^{3k-1})(1-\omega^{3k}q^{3k})$$

Using properties of ω, we have

$$(\omega q)_\infty = \prod_{k=1}^{\infty}(1-\omega q^{3k-2})(1-\omega^2 q^{3k-1})(1-q^{3k}) =$$

$$= (q^3)_\infty \prod_{k=1}^{\infty}(1-\omega q^{3k-2})(1-\omega^2 q^{3k-1})$$

Observe that:

$$\left|1-\omega q^{3k-2}\right|^2 = 1-q^{3k-2}+q^{6k-4} = \frac{1+q^{9k-6}}{1+q^{3k-2}} \text{ and}$$

$$\left|1-\omega^2 q^{3k-1}\right|^2 = 1-q^{3k-1}+q^{6k-2} = \frac{1+q^{9k-3}}{1+q^{3k-1}}, \text{therefore}$$

$$|(\omega q)_\infty|^2 = (q^3)_\infty^2 \prod_{k=1}^{\infty} \frac{1+q^{9k-6}}{1+q^{3k-2}} \cdot \frac{1+q^{9k-3}}{1+q^{3k-1}},$$

above equation can be written as

$$|(\omega q)_\infty|^2 =$$

$$= (q^3)_\infty^2 \prod_{k=1}^{\infty} \left(\frac{1}{(1+q^{3k-2})(1+q^{3k-1})(1+q^{3k})}\right) \cdot \left((1+q^{9k-6})(1+q^{9k-3})(1+q^{9k})\right) \cdot \left(\frac{1+q^{3k}}{1+q^{9k}}\right)$$

which can be simplified further,

$$|(\omega q)_\infty|^2 = (q^3)_\infty^2 \prod_{k=1}^{\infty} \frac{1}{1+q^k} \cdot \prod_{k=1}^{\infty} (1+q^{3k}) \cdot \prod_{k=1}^{\infty} \frac{1+q^{3k}}{1+q^{9k}}$$

$$\text{It is known that } \prod_{k=1}^{\infty} (1+q^k) = \prod_{k=1}^{\infty} \frac{1-q^{2k}}{1-q^k} = \frac{(q^2)_\infty}{(q)_\infty}$$

Using the above formula, we obtain:

$$|(\omega q)_\infty|^2 = (q^3)_\infty^2 \frac{(q)_\infty (q^6)_\infty (q^6)_\infty (q^9)_\infty}{(q^2)_\infty (q^3)_\infty (q^3)_\infty (q^{18})_\infty} = \frac{(q)_\infty {(q^6)_\infty}^2 (q^9)_\infty}{(q^2)_\infty (q^{18})_\infty}$$